A GENERALIZATION OF THE EPSTEIN-PENNER CONSTRUCTION TO PROJECTIVE MANIFOLDS.

D. COOPER, D. D. LONG

Abstract. We extend the canonical cell decomposition due to Epstein and Penner of a hyperbolic manifold with cusps to the strictly convex setting. It follows that a sufficiently small deformation of the holonomy of a finite volume strictly convex real projective manifold is the holonomy of some nearby projective structure with radial ends, provided the holonomy of each maximal cusp has a fixed point.

One of the powerful constructions in theory of cusped hyperbolic n-manifolds is a cellulation constructed by Epstein & Penner in [5], which in the particular case that the manifold has one cusp, gives rise to a canonical cell decomposition. In this note we extend their results to the case of strictly convex real projective manifolds.

The proof of [5] employs Minkowski space $\mathbb{R}^{n,1}$ and shows that if $p \in \mathbb{R}^{n,1}$ is a point on the lightcone that corresponds to a parabolic fixed point $\mathbb{P}(p) \in \mathbb{R}P^n$ then p has a *discrete* orbit. The convex hull of this orbit is an infinite sided polytope in Minkowski space that is preserved by the group. The boundary of the quotient of this polytope by the group gives the cell decomposition. This approach uses in an essential way the quadratic form $\beta = x_1^2 + \cdots + x_n^2 - x_{n+1}^2$ that defines $O(n,1)$ to identify Minkowski space with its dual. This gives a bijection between points on the light cone $t \cdot p$ with $t > 0$ in the direction of $\mathbb{P}(p)$ and horoballs $\mathcal{B}(t)$ centered on $\mathbb{P}(p)$. For t sufficiently large this horoball covers an *embedded* cusp, so the orbit consists of *disjoint* horoballs, which implies the orbit of p is discrete.

In this paper we use a *Vinberg hypersurface* to associate to p a horoball in the universal cover of the *dual* projective manifold that covers the dual cusp.

In the hyperbolic case in dimension 2 one obtains a cell decomposition of moduli space from the result of Epstein and Penner, [8]. For finite volume hyperbolic structures Mostow-Prasad rigidity implies that in dimension at least 3 the moduli space is a point. No similar result holds in the strictly convex setting: there are examples of one cusped 3-manifolds with families of finite volume strictly convex projective structure. This paper leads to a decomposition of the moduli space of such structures, but we do not know if the components of this decomposition are cells.

Background for theory of cusped projective manifolds can be found in [3]. We summarize the most important points here.

A subset $\Omega \subset \mathbb{R}P^n$ is *properly convex* if it is the interior of a compact convex set K that is disjoint from some codimension-1 projective hyperplane and *strictly convex* if in addition K contains no line segment of positive length in its boundary.

A *strictly convex real projective n-manifold* is $M = \Omega/\Gamma$ where $\Omega \subset \mathbb{R}P^n$ is strictly convex and $\Gamma \cong \pi_1 M$ is a discrete group of projective transformations that preserves Ω and acts freely on it. We may, and will, lift Γ to a subgroup of $SL(\Omega)$ which is the group of matrices of determinant ± 1 that preserve Ω. The Hilbert metric on Ω is invariant for the action of Γ and defines a metric and a notion of volume on M. All projective manifolds in this note will be assumed to have finite volume. An element of $SL(\Omega)$ is *parabolic* if all its eigenvalues have modulus 1 and it is not semisimple.

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A *maximal cusp* in $M = \Omega/\Gamma$ is a connected submanifold, B, such that $\partial B = \overline{M \setminus B} \cap B$ is compact and

C1 Every component $\tilde{B}$ of the pre-image of B in Ω has strictly convex interior.
C2 $p = cl(\tilde{B}) \cap \partial\Omega$ is a single point called a *parabolic fixed point*.
C3 The stabilizer $\Gamma_{\tilde{B}} \subset \Gamma$ of $\tilde{B}$ fixes p.
C4 There is a unique projective hyperplane $H \subset \mathbb{R}P^n$ with $p = H \cap \overline{\Omega}$.
C5 Every non-trivial element of $\Gamma_{\tilde{B}}$ is parabolic and preserves H.
C6 $\Gamma_{\tilde{B}}$ is conjugate into $PO(n,1)$ so contains $\mathbb{Z}^{n-1}$ as a subgroup of finite index.

It is proved in [3] that a strictly convex finite volume real projective manifold has finitely many ends and each is a maximal cusp.

We can identify the domain $\Omega \subset \mathbb{R}P^n$ with a subset Ω of some affine hyperplane in $\mathbb{R}^{n+1}$. Then $\mathcal{C}\Omega = (\mathbb{R}_{>0})\cdot\Omega \subset \mathbb{R}^{n+1}$ is an open cone based at 0 and $\mathbb{P}(\mathcal{C}\Omega) = \Omega \subset \mathbb{R}P^n$ where $\mathbb{P} : \mathbb{R}^{n+1}\backslash 0 \longrightarrow \mathbb{R}P^n$ denotes projectivization. The *(positive) lightcone* of Ω is the cone $\mathcal{L} = \mathcal{C}(\partial\Omega)$. It is the subset of the frontier of $\mathcal{C}\Omega$ obtained by deleting 0 and $\mathbb{P}(\mathcal{L}) = \partial\Omega$.

Theorem 0.1. *Suppose $M = \Omega/\Gamma$ is a strictly convex real projective n-manifold that contains a maximal cusp B, and that $p \in \mathcal{L}$ is a point in the lightcone of Ω such that $\mathbb{P}(p) \in \partial\Omega$ is the parabolic fixed point of $\pi_1 B$. Then the Γ-orbit of p is a discrete subset of $\mathbb{R}^{n+1}$.*

This has as an immediate consequence the existence of ideal cell decompositions:

Corollary 0.2. *Suppose $M = \Omega/\Gamma$ is a strictly convex real projective n-manifold of finite volume with at least one (maximal) cusp and $\mathcal{Q} \subset \partial\Omega$ is the set of fixed points of parabolics in Γ.*

Then there is a Γ-invariant tessellation of $\Omega \cup \mathcal{Q}$ by convex polytopes, each of which is the convex hull of a finite subset $V \subset \mathcal{Q}$. The interiors of the cells of dimension greater than zero project to a cell decomposition of M.

To prove this, one chooses a Γ-invariant collection $\mathcal{P} \subset \mathcal{L}$ of points in the light cone, one in the direction of each parabolic fixed point. If there are k cusps this amounts to choosing k positive reals. By 0.1, $\mathcal{P}$ is a discrete set. The closed convex hull of $\mathcal{P}$ is a Γ-invariant set C in $\mathbb{R}^{n+1}$. We show that C has polyhedral boundary. The image under projectivization decomposes $\Omega \cup \mathcal{Q}$ into convex cells.

Clearly this decomposition is unchanged by uniformly scaling $\mathcal{P}$, and so the k cusps result in a family of cell decompositions parameterized by a point in the interior of a simplex in $\mathbb{R}P^{k-1}$. In the case that the manifold has one cusp, this decomposition is canonical. Moreover a continuous choice of $\mathcal{P}$ may be made as Γ varies so that in this sense the cell decomposition *varies continuously*:

Corollary 0.3. *Suppose $M = \Omega/\Gamma$ is a strictly convex real projective n-manifold that contains a unique (maximal) cusp B. Then the cell decomposition above is canonical and $C(\Gamma)$ can be chosen to vary continuously with Γ. In particular, if all the cells are ideal simplices, then the combinatorics of the cell decomposition is locally constant.*

We refer the reader to [1] §1.7 for the (technical, but standard) details concerning the continuous variation of geometric structures. One can think of this as variations of the holonomy representation or of developing maps with an appropriate topology.

It follows immediately that the isometry group of a one-cusped strictly convex real projective n-manifold is finite. There is an extension of the continuity part of this statement to the multi-cusp case.

The following definition is essentially due to Choi [2]. A submanifold B of a projective n-manifold M is a *radial end* if $M = A \cup B$ with $\partial A = A \cap B = \partial B$ and B is foliated by rays oriented away from ∂B which develop into oriented lines in $\mathbb{R}P^n$ so that the limit of all these lines in the direction given by the orientation is a single point $x \in \mathbb{R}P^n$. A maximal cusp is a radial end.

There are interesting examples of cusped hyperbolic 3-manifolds for which the holonomy has many nearby deformations. The following ensures these correspond to *at least one* nearby projective structure with radial ends. The existence of some projective structure with deformed holonomy on a compact core is easy, and the content of the following is the nature of the ends. We mention that in general it is unknown if such a manifold admits an embedded ideal triangulation.

Theorem 0.4. *Suppose $\rho : [0,\delta) \longrightarrow \mathrm{Hom}(\pi_1 M, \mathrm{PGL}(n+1,\mathbb{R}))$ is continuous and $\rho(0)$ is the holonomy of a finite volume strictly convex real projective structure on the n-manifold M. Also assume that for all t that the restriction of $\rho(t)$ to each cusp of M has at least one fixed point in $\mathbb{R}P^n$. Then for some $\epsilon \in (0,\delta)$ and for all $t \in [0,\epsilon)$ there is a nearby (possibly not strictly convex) real projective structure with radial ends on M and with holonomy $\rho(t)$.*

Let V be a real vector space of dimension $(n+1)$ with dual V^* and $\mathbb{P} : V \setminus 0 \longrightarrow \mathbb{P}V$ the projectivization map. In the following discussion Ω is a properly convex set in $\mathbb{P}V$; we do not require the extra hypothesis of strictly convex.

The relation between a vector space and its dual gives rise to projective duality. Given a properly convex $\Omega \subset \mathbb{P}(V)$ the *dual cone* $\mathcal{C}\Omega^* \subset V^*$, is the set of linear functionals which take strictly positive values on $\mathcal{C}\Omega$. The *dual domain* $\Omega^* = \mathbb{P}(\mathcal{C}\Omega^*) \subset \mathbb{P}(V^*)$ is also properly convex. If Ω is strictly convex then so is Ω^*. The *dual lightcone* $\mathcal{L}^*$ of Ω is the lightcone of Ω^*.

The dual action of an element $\gamma \in PGL(V)$ on V^* is given by $\gamma^*(\phi) = \phi \circ \gamma^{-1}$. A choice of basis for V gives isomorphisms $V \cong \mathbb{R}^{n+1} \cong V^*$ and $PGL(V) \cong PGL(n+1,\mathbb{R}) \cong PGL(V^*)$. Using these identifications the dual action of $PGL(V)$ on V^* then corresponds to the Cartan involution $\theta(A) = (A^{-1})^t$ on $PGL(n+1,\mathbb{R})$. If $\Gamma \subset PGL(n+1,\mathbb{R})$ the *dual group* is $\Gamma^* = \theta(\Gamma)$.

The *dual manifold* of $M = \Omega/\Gamma$ is $M^* = \Omega^*/\Gamma^*$. If $p \in \partial\Omega$ is the parabolic fixed point of a maximal cusp $B \subset M$ by (C4) there is a unique supporting hyperplane H to Ω at p. The *dual parabolic fixed point* $[\phi] \in \partial\Omega^*$ is defined by $\mathbb{P}(\ker\phi) = H$. The dual action of $\pi_1 B$ fixes $[\phi]$ and there is a *dual cusp* B^*, well defined up to the equivalence relation generated by inclusion. Thus ϕ is a point on the dual light cone. Below we show that level sets of ϕ determine a type of horosphere in Ω centered at p.

The *hyperboloid model* of hyperbolic space is a certain level set of the quadratic form β. In general the holonomy of a strictly convex manifold does not preserve any non-degenerate quadratic form but it does preserve a certain convex function which has levels sets called *Vinberg hypersurfaces* [9] that provide a generalization of the hyperboloid. We briefly recall the construction here. Let $d\psi$ be a volume form on V^*. Then the *characteristic function* $f : \mathcal{C}\Omega \longrightarrow \mathbb{R}$ is defined by

$$f(x) = \int_{\mathcal{C}\Omega^*} e^{-\psi(x)} d\psi$$

This is real analytic, convex, and satisfies $f(tx) = t^{-n} f(x)$ for $t > 0$. For each $t > 0$ the level set $S_t = f^{-1}(t)$ is called a *Vinberg hypersurface* and is convex. These sets foliate $\mathcal{C}\Omega$ and are permuted by homotheties fixing the origin. For example, the hyperboloids $z^2 = x^2 + y^2 + t$ are Vinberg hypersurfaces in the cone $z^2 > x^2 + y^2$. The surfaces S_t are all preserved by $SL(\Omega)$, (we recall that this is the group of matrices of determinant ± 1 that preserve Ω) and in particular by Γ. Henceforth, we fix some choice $S := S_1$ which we refer to as *the* Vinberg surface for Ω. It is a substitute for the hyperboloid model of hyperbolic space.

Let $\pi : S \longrightarrow \Omega$ be the restriction of the projectivization map. A point, $\phi \in \mathcal{L}^*$, in the dual lightcone of Ω determines a *horofunction*

$$h_\phi = \phi \circ \pi^{-1} : \Omega \longrightarrow \mathbb{R}$$

Since $\phi \in \mathcal{L}^*$ it follows that $\ker\phi$ contains a ray $(0,\infty) \cdot v \subset \mathcal{L}$ in the lightcone.

We show below that the sublevel set $\mathcal{B}(\phi,t) = h_\phi^{-1}(0,t]$ is convex. This is called a *horoball* associated to ϕ. It is in general different from the *algebraic horoballs* defined in [3]. The boundary

$$\mathcal{S}(\phi,t) = \partial\mathcal{B}(\phi,t) = h_\phi^{-1}(t)$$

of a horoball is called a *horosphere*, and is analytic.

Lemma 0.5. *Suppose $\Omega \subset \mathbb{R}P^n$ is properly convex and ϕ is a point in the dual lightcone. Then the horofunction $h_\phi : \Omega \longrightarrow (0,\infty)$ is a smooth surjective submersion. Hence for all $t > 0$ the horoball $\mathcal{B}(\phi,t)$ is non-empty and convex.*

Proof. Here is an overview of the proof: The implicit function theorem and smoothness of f imply h_ϕ is smooth. Referring to the figure, notice the subset of $\mathbb{R}^{n+1}$ above S, which we denote by $W = f^{-1}(0,1]$, is convex. It follows that $X_t = W \cap \phi^{-1}(0,t]$ is convex, since it is the intersection of convex sets. Hence $\mathcal{B}(\phi,t) = \pi(X_t) \subset \Omega$ is convex.

We claim that for $t > 0$ it is not empty. This follows from the fact that if v is a point in the lightcone of Ω that is also in $\ker\phi$ then the vertical distance $\delta(t\cdot v)$ between S and $t\cdot v$ goes to zero as $t \to \infty$. This is shown by direct computation in the particular case that Ω is an open simplex, and the general result follows by a comparison argument.

Here are the details:

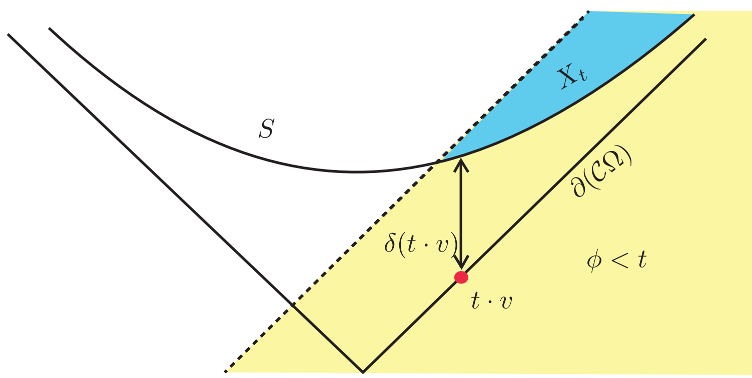

Figure 1. Vinberg hypersurface inside the lightcone

To begin with, consider the special case that $\mathcal{C}\Omega$ is the positive orthant in $\mathbb{R}^{n+1}$, which is the cone on an n-simplex $\Omega = \sigma$. In an appropriate basis, the group $SL(\sigma)$ contains positive diagonal matrices of determinant 1. The orbits are the Vinberg hypersurfaces $x_0 \cdot x_1 \cdots x_n = c$ for each $c > 0$. Each hypersurface is asymptotic to the ray $(0,\infty)\cdot v$ which proves the result in this special case.

In the general case there is a simplex σ with interior in Ω and v as a vertex. Then $\mathcal{C}(\sigma) \subset \mathcal{C}\Omega$, and it follows from the definition that $f_{\mathcal{C}(\sigma)} \geq f_{\mathcal{C}\Omega}|_{\mathcal{C}(\sigma)}$, which implies that the Vinberg surface for Ω lies below that for σ along the direction given by v. The claim now follows from the special case.

It follows from this, and the convexity of S, that S is never tangent to a level set of $\ker\phi$, hence h_ϕ is a submersion. □

It is immediate from the definition of horoball that if $\gamma \in SL(\Omega)$ and ϕ is in the dual light cone then

$$\gamma(\mathcal{B}(\phi,t)) = \mathcal{B}(\gamma^*\phi,t)$$

Thus if $\gamma^*\phi = \phi$ then the horoball $\mathcal{B}(\phi,t)$ is preserved by γ. If B is a maximal cusp of M then $\pi_1 B$ preserves horoballs corresponding to the unique supporting hyperplane at the parabolic fixed point. The next result is that a sufficiently small such horoball projects to an (embedded) cusp in

M called a *horocusp.* Recall that a subset $U \subset \Omega$ is *precisely invariant* under a subgroup $G \subset \pi_1 M$ if every element of G preserves U and every element of $\pi_1 M \setminus G$ sends U to a subset disjoint form U.

Corollary 0.6. *Suppose $M = \Omega/\Gamma$ is a properly convex manifold and $B \subset M$ is a maximal cusp with holonomy $\Gamma_B \subset \Gamma$ and with dual Γ_B^* that fixes the dual parabolic fixed point $[\phi] \in \partial\Omega^*$.*

Then there is $t > 0$ such that the horoball $\mathcal{B}(\phi, t)$ is precisely invariant under $\Gamma_B \subset \Gamma$. For sufficiently small $t > 0$ the manifold $B' = \mathcal{B}(\phi, t)/\Gamma_B$ projects injectively onto a cusp $B' \subset B \subset M$ and $B \setminus B'$ is bounded. The horofunction h_ϕ covers a proper smooth submersion $h : B' \longrightarrow (0, t]$. The sublevel sets are convex and the level sets, called horomanifolds*, are compact and give a product foliation.*

Proof. Let H be the supporting hyperplane to Ω at the parabolic fixed point for Γ_B. By (C3) H is preserved by Γ_B. There is a codimension-1 subspace $V \subset \mathbb{R}^{n+1}$ with $\mathbb{P}(V) = H$. Then $V = \ker\phi$ iff ϕ is a point on the dual lightcone $\mathcal{L}^*$ such that $[\phi] \in \partial\Omega^*$ is the parabolic fixed point for Γ_B^*.

The dual of a parabolic matrix is also parabolic, thus $\phi \in V^*$ is an eigenvector with eigenvalue 1 for every element of Γ_B^*. This means ϕ is a Γ_B-invariant function on V, thus h_ϕ covers a well defined function $h : \Omega/\Gamma_B \longrightarrow (0, \infty)$. This is a submersion because h_ϕ is, hence the level sets $\mathcal{H}_t = h^{-1}(t)$ are a foliation of B'.

Clearly $\pi_1\mathcal{H}_t \cong \pi_1 B$ and, since $\pi_1 B$ is a maximal cusp, by (C6) $\pi_1 B$ has (virtual) cohomological dimension $(n-1)$, and it follows that each horomanifold is compact. There is a transverse foliation by lines going out into the cusp, and these foliations give a product structure on B'. Convexity of sublevel sets follows from convexity of horoballs. □

Proof of 0.1. We will prove theorem for the dual manifold $M^* = \Omega^*/\Gamma^*$. By (0.15) and (6.7) of [3] M^* is a strictly convex projective manifold with finite volume. The result follows using the canonical isomorphism between a finite dimensional vector space and its double dual. As before we write $V = \mathbb{R}^{n+1}$.

Recall that B is a maximal cusp, and let $\phi \in \mathcal{L}^*$ be a point on the dual light cone fixed by the dual action of $\pi_1 B$. Let $\gamma_n^* \in \Gamma^*$ be a sequence such that $\gamma_n^*\phi = \phi \circ \gamma_n^{-1}$ converges to some $\psi \in V^*$. We must show this sequence is eventually constant. If not, we may assume $\gamma_n^*\phi$ are all distinct. Clearly $\psi \in \mathcal{L}^* \cup 0$.

By 0.6 there is $t > 0$ such that B contains the horocusp $B' = \mathcal{B}(\phi, t)/\pi_1 B$. Choose $w \in \mathcal{C}\Omega$ with $\psi(w) < t$. Since $\gamma_n^*\phi \to \psi$ then for all n sufficiently large $\gamma_n^*\phi(w) < t$. Since $\psi \in \mathcal{L}^*$ there is $v \in \mathcal{L}$ with $\psi(v) = 0$ thus $\psi(w + s \cdot v) = \psi(w)$. From the formula for the characteristic function f for Ω we see that $f(w + s \cdot v) \to 0$ as $s \to \infty$ (also see Figure 1), so for s sufficiently large $w + s \cdot v$ is above the Vinberg hypersurface, which implies $w + s \cdot v \in \gamma_n\mathcal{B}(\phi, t) \cap \gamma_{n+1}\mathcal{B}(\phi, t)$. Since B' is precisely invariant for $\pi_1 B \subset \pi_1 M$ it follows that $\gamma_{n+1}^{-1}\gamma_n \in \pi_1 B$. But ϕ is preserved by every element of this group thus $\gamma_n^*\phi = \gamma_{n+1}^*\phi$ which is a contradiction. □

The next result is well known, but we include it for the convenience of the reader.

Lemma 0.7 (orbits are dense)**.** *Suppose $M = \Omega/\Gamma$ is a strictly convex projective manifold with finite volume. Then every Γ orbit is dense in $\partial\Omega$.*

Proof. Given a point $b \in \partial\Omega$ define Ω^- to be the intersection with Ω of the closed convex hull of Γb. Since Ω is strictly convex, Γb is dense in $\partial\Omega$ iff $\Omega = \Omega^-$. The set Ω^- is convex and Γ invariant. The projection, $N = \Omega^-/\Gamma$, of Ω^- is a submanifold of M. Since M is finite volume and strictly convex, it is the union of a compact set and finitely many cusps, [3]. We replace Ω^- by a K-neighborhood with K so large that the complement of N is now a subset of the cusps of M.

Suppose R is a component of $\Omega \setminus \Omega^-$, with stabilizer $\Gamma_R \subset \Gamma$. Then R/Γ_R is mapped injectively into a cusp $B \subset M$ by the projection. Let $\tilde{B}$ be the component of the pre-image of B that contains

R. Observe that $\overline{R} \cap \partial\Omega$ contains more than one point, but $cl(\tilde{B}) \cap \partial\Omega$ is one point by (C2), a contradiction. □

The hypothesis of strictly convex in the above can not be weakened to properly convex because there is a properly convex projective torus that is the quotient of the interior of a triangle by a discrete group and each vertex of the triangle is an orbit.

Lemma 0.8. *Suppose $M = \Omega/\Gamma$ is strictly convex and has finite volume. If $x \in \mathcal{L}$ is a point in the light cone then 0 is an accumulation point of Γx iff $\mathbb{P}(x) \in \partial\Omega$ is not a parabolic fixed point.*

Proof. If $\mathbb{P}(x)$ is a parabolic fixed point the result follows from 0.1. We follow the proof of (3.2) in [5] with some adaptations for our situation to show the corresponding result for the projective dual $M^* = \Omega^*/\Gamma^*$. The result then follows by duality. The proof is broken down into several steps.

Let $H = \mathbb{P}(V)$ be the unique supporting hyperplane to Ω at $\mathbb{P}(x)$. Let $\phi \in \mathcal{L}^*$ be dual to H thus $V = \ker\phi$. This only defines ϕ up to scaling. The manifold M is the union of a compact thick part K and finitely many cusps. Choose a compact set $\tilde{K}$ in the Vinberg hypersurface S for Ω so that the projection of $\tilde{K}$ contains K. Given $t > 0$ there is a horoball $\mathcal{B} = \mathcal{B}(\phi, t) \subset \Omega$. If the Γ^* orbit of ϕ does not accumulate on 0 we may choose t so small ($\Rightarrow \mathcal{B}$ small) that the Γ orbit of $\mathcal{B}$ is disjoint from $\tilde{K}$. This implies the orbit of $\mathcal{B}$ projects into a cusp of M. It follows that ϕ is a parabolic fixed point of Γ^*. □

Proof of 0.2. Refer to the paragraph after 0.2. Regard $\mathbb{R}^{n+1}$ as an affine patch in $\mathbb{R}P^{n+1} = \mathbb{R}^{n+1} \sqcup \mathbb{R}P^n_\infty$. The image of Ω in $\mathbb{R}P^n_\infty$ under radial projection from $0 \in \mathbb{R}^{n+1}$ is a set $\Omega_\infty \subset \mathbb{R}P^n_\infty$ projectively equivalent to Ω. Define $K \subset \mathbb{R}P^{n+1}$ to be the cone that is the closure in $\mathbb{R}P^{n+1}$ of $\mathcal{C}\Omega$ then

$$K = \mathcal{C}\Omega \sqcup \mathcal{L} \sqcup \overline{\Omega}_\infty \sqcup 0$$

It is disjoint from a codimension-1 projective hyperplane H that is a small perturbation of $\mathbb{R}P^n_\infty$ and therefore $K \subset \mathbb{A}^{n+1} = \mathbb{R}P^{n+1} \setminus H$ is a properly convex set. Observe that $\mathcal{P} \subset \partial K$ and, since K is convex, $\overline{C} \subset K$ where $\overline{C}$ is the closure of C in $\mathbb{R}P^{n+1}$.

We claim that it follows from 0.7 that the set of accumulation points of $\mathcal{P}$ is $\partial\Omega_\infty$. The argument is the following: Every open set $U \subset \partial\Omega_\infty$ contains infinitely many parabolic fixed points. These correspond to an infinite subset of $\mathcal{P}$. By 0.1 this set is discrete in the light cone so that all but finitely many of these points are very high in the lightcone and in particular, very close to U.

This in turn implies $\overline{C}$ contains $\partial\Omega_\infty$ and therefore also contains $\overline{\Omega}_\infty$ thus

$$\overline{C} = C \sqcup \overline{\Omega}_\infty$$

It follows that that $\overline{C}$ is the closed convex hull in $\mathbb{A}^{n+1}$ of $\mathcal{P}$.

The holonomy, Γ, of M lies in $SL(n+1,\mathbb{R})$ and can be identified with a subgroup $\Gamma^+ \subset \mathrm{PGL}(n+2,\mathbb{R})$ that preserves K. In suitable coordinates Γ^+ is block diagonal with a trivial block of size 1 and the other block is Γ.

Closely following §3 of [5] we establish the following claims:

• The dimension of C is $n+1$ because $\overline{C}$ contains the n-dimensional set $\Omega_\infty \subset \mathbb{R}P^n_\infty$ and also contains the points $\mathcal{P}$ which are not in $\mathbb{R}P^n_\infty$.

• $w \in C \cap \mathcal{L}$ iff $w = \alpha z$ for some $z \in \mathcal{P}$ and $\alpha \geq 1$.

If w is not of this form then the segment $[0, w]$ is disjoint from $\mathcal{P}$. Since $\mathcal{P}$ is discrete there is a small neighborhood $U \subset \mathbb{R}^{n+1}$ of this segment that contains no point of $\mathcal{P}$. Hence there is a hyperplane that intersects $\mathcal{C}\overline{\Omega}$ in a small, convex, codimension-1 set in U and separates $[0, w]$ from $\mathcal{P}$, and hence from C. This means $w \notin C$.

For the converse, given $z \in \mathcal{P}$ the image $w \in \partial\Omega_\infty$ of z is in $\overline{C}$, hence $[z, w] \subset \overline{C}$. This contains all the points αz with $\alpha \geq 1$.

• Each ray $\lambda \subset \mathcal{C}\Omega$ that starts at 0 meets ∂C exactly once.

Since $\mathcal{P}$ is discrete in $\mathbb{R}^{n+1}$ it follows that $0 \notin \overline{C}$ so λ starts outside $\overline{C}$ and limits on $q \in \Omega_\infty \subset \overline{C}$. Thus λ contains points in the interior of $\overline{C}$. Since $\overline{C}$ is convex λ meets $\partial\overline{C}$ in a single point z. Since $\lambda \subset \mathcal{C}\Omega$ it follows that $z \in \mathcal{C}\Omega \cap \partial\overline{C} = \partial C$.

• If $W \subset \mathbb{R}^{n+1}$ is a supporting affine hyperplane for C at a point $z \in \partial C \cap \mathcal{C}\Omega$ then $W \cap \mathcal{C}(\overline{\Omega})$ is compact and convex.

The closure $\overline{W}$ of W in $\mathbb{R}P^{n+1}$ is a projective hyperplane that is a supporting hyperplane for $\overline{C}$ in $\mathbb{R}P^{n+1}$. Clearly $\overline{W}$ is disjoint from Ω_∞ and by the previous claim $0 \notin W$. The ray from 0 through z limits on Ω_∞ and crosses ∂C at z therefore $\overline{W}$ separates 0 from Ω_∞ in $\mathbb{A}^{n+1}$

Let V be the vector subspace parallel to W. Then $V = \ker\phi$ for some linear map ϕ. We claim that V is disjoint from $\mathcal{C}\overline{\Omega} = \mathcal{C}\Omega \sqcup \mathcal{L}$. Observe that $\overline{V}$ and $\overline{W}$ have the same intersection with $\mathbb{R}P^n_\infty$ and are disjoint from Ω_∞. Since V contains 0 it follows that V is disjoint from $\mathcal{C}\Omega$. It remains to show V is disjoint from $\mathcal{L}$.

Define an affine function ψ on $\mathbb{R}^{n+1}$ by $\psi(v) = \phi(v - z)$. Then $W = \psi^{-1}(0)$, and since W is a supporting hyperplane for C, this means that ψ has constant sign on C. By replacing ϕ by $-\phi$ if needed we may assume $\psi(v) \geq 0$ for all $v \in C$. Hence $\phi(v) \geq \phi(z)$ for all $v \in C$. Since V is disjoint from $\mathcal{C}\Omega$ it follows that ϕ has constant sign on $\mathcal{C}\Omega$. Since ψ takes arbitrarily large positive values on $\mathcal{C}\Omega$ it follows that $\phi \geq 0$ on $\mathcal{C}\Omega$ and hence $K = \phi(z) > 0$. This implies $\phi(v) \geq K$ for all $v \in C$. Since Γ preserves C it follows that for every $\gamma^* \in \Gamma^*$ that $\gamma^*\phi \geq K$ everywhere on C.

We claim that $\mathcal{L} \cap V = \emptyset$. For, suppose not and that $0 \neq x \in \mathcal{L} \cap V$.

Firstly, observe that $\mathbb{P}(x)$ cannot be a parabolic fixed point, otherwise points high on this ray are in C. However, V and W are parallel hyperplanes, so that these high points, which are all in V, must be below W, a contradiction.

However, consideration of stabilizers now implies that $\mathbb{P}(\phi) \in \partial\Omega^*$ is not a (dual) parabolic fixed point. Hence by 0.8, there is a sequence $\gamma_k^* \in \Gamma^*$ such that $\gamma_k^*\phi \to 0$. Thus for large k we have $\gamma_k^*\phi(z) < K$. This contradicts $\gamma_k^*\phi \geq K$ everywhere on C.

This proves the assertion that $\mathcal{L} \cap V = \emptyset$. Our main claim that $W \cap \mathcal{C}(\overline{\Omega})$ is compact and convex now follows: It is clear that this set is convex. Note that $\phi(W) = \phi(V + z) = \phi(z) = K$ is constant. However, since $\mathcal{L} \cap V = \emptyset$, for any ray $(\mathbb{R}_{>0}) \cdot v$ in $\mathcal{C}\overline{\Omega}$, $\phi(v) > 0$, so that very high points on that ray take values $> K$. It follows that W meets $\mathcal{C}\overline{\Omega}$ in a compact set.

• Every point in $\partial C \cap \mathcal{C}\Omega$ is contained in a supporting hyperplane that contains at least $(n+1)$ points in $\mathcal{P}$.

Given a supporting hyperplane H, rotate it around $H \cap C$ until it meets another point of $\mathcal{P}$. Since this set is discrete, there is a first rotation angle with this property. This process stops when $H \cap C$ contains an open subset of H. See [5] for more details.

• The set of codimension-1 faces is locally finite inside $\mathcal{C}\Omega$.

Let $K \subset \mathcal{C}\Omega$ be a compact set meeting faces $F_1, F_2, ...$ and suppose that these faces are defined by affine hyperplanes $A_1, A_2, ...$ Pick $x_i \in K \cap F_i$ and subconverge so that $x_i \to x$ and $A_i \to A$, an affine plane containing x. The A_i's are all support planes, whence so is A, thus it meets $\mathcal{C}\overline{\Omega}$ in a compact convex set. Move A upwards a small distance to obtain A^+. Then all but finitely many of $A_i \cap \mathcal{C}\overline{\Omega}$ lie below $A^+ \cap \mathcal{C}\overline{\Omega}$. Hence $\mathcal{P} \cap (\cup A_i)$ is finite and it follows that there were only a finite number of faces meeting K.

The locally finite cell structure on $\partial C \cap \mathcal{C}\Omega$ is Γ-equivariant and projects to a locally finite cell structure on $M = \Omega/\Gamma$. This completes the proof of 0.2. □

Proof of 0.3. In the case that M has only one cusp, the convex hull C is defined by the orbit of a single vector, which in turn is uniquely defined up to scaling. It follows that C is defined up to homothety and this is invisible when one projects into Ω/Γ. □

Proof of 0.4. If M is compact, this is well known (for example [6] and [1]), so we may assume M has cusps. The holonomy of a cusp has a unique fixed point. It follows that a sufficiently small deformation of the holonomy of a cusp has at most finitely many fixed points. The hypothesis then ensures that for a sufficiently small deformation each end has at least one isolated fixed point.

By 0.2 at $t = 0$ there is a compact convex polytope $P_0 \subset \mathbb{R}P^n$ with face pairings so that the quotient $X_0 = P_0/\sim$ is the compactification of the manifold M obtained by adding an ideal point for each cusp. One may regard X_0 as a projective manifold with a finitely many singular points. The vertices of P_0 are the parabolic fixed points $\mathcal{P}_0 = \{p_1(0), \cdots, p_k(0)\}$ of a finite set of (conjugates of) cusp subgroups $G_1, \cdots, G_k \subset \pi_1 M$.

Each face A_0 of P_0 is a convex polytope that is the convex hull of a subset of $\mathcal{P}_0$. In general this face need not be a simplex. Even if it is a simplex, it is possible that $|A_0 \cap \mathcal{P}_0| > 1 + \dim(A_0)$. Each face A_0 can be triangulated using 0-simplices $A_0 \cap \mathcal{P}_0$ in a way that respects the face pairings. This involves some arbitrary choices, for example if A_0 is a quadrilateral one chooses a diagonal.

From now on we regard P_0 as a triangulated convex polytope, with one vertex, x_0, in the interior of P_0 that is coned to the simplices in ∂P_0. Each face of P_0 (= simplex in ∂P_0) is the convex hull of a subset of $\mathcal{P}_0$. Adjacent codimension-1 faces might lie in the same hyperplane. Moreover the faces of P_0 are paired by projective maps, and the identification space is X_0.

By assumption $\rho_t(G_i)$ has at least one isolated fixed point $p_i(t)$ and by continuity it is close to $p_i(0)$. For each cusp $B \subset M$ choose a $\rho_t(\pi_1 M)$-orbit of isolated fixed point for $\rho_t(\pi_1 B)$. This is one choice per cusp of M; if M has one cusp then this is a single choice. Let $\mathcal{P}_t$ be the set of chosen fixed points for $\rho_t(G_i)$ for $1 \leq i \leq k$. There is a natural bijection $h_t : \mathcal{P}_0 \longrightarrow \mathcal{P}_t$.

For t sufficiently small this choice of fixed points and x_0 determines a (possibly non-convex) triangulated polytope $P_t \subset \mathbb{R}P^n$ close to P_0 and with the same combinatorics. A face A_0 of P_0 is the convex hull of a subset $\mathcal{A}_0 \subset \mathcal{P}_0$. Define A_t as the convex hull of the subset $\mathcal{A}_t = h_t(\mathcal{A}_0) \subset \mathcal{P}_t$. For t sufficiently small A_t is a simplex and the union of these simplices is the boundary of P_t. Moreover, ∂P_t is a simplicial complex and h_t extends to a simplicially isomorphism from P_0 to P_t, thus P_t is a cell.

We claim there are face pairings for P_t close to those of P_0. The reason is the following. Suppose A_0 and B_0 are two faces of P_0 and $\rho_0(g)[A_0] = B_0$ for some $g \in \pi_1 M$. For each vertex $p = p_i(0)$ of A_0 the vertex $(\rho_0 g)(p)$ of B_0 is in the same orbit as p. The vertex $p(t) = p_i(t)$ of A_t is sent by $\rho_t g$ to the vertex $(\rho_t g)(p(t))$ of B_t because our choices of isolated fixed points are preserved by the action of $\rho_t(\pi_1 M)$. It follows that $\rho_t(g)$ sends A_t to B_t.

The quotient gives a nearby singular structure on $P_t/\sim_t$, and by deleting the vertices of P_t a nearby projective structure on M. Moreover, it is clear from this description that the deformed manifold has radial ends. □

In fact, it is shown in [4], that if in addition ρ_t satisfies certain conditions in each cusp, then the nearby structure is properly (or even strictly) convex.

Remarks. The situation in the properly convex case is more involved; the authors hope to explore this in future work. There are other directions one might explore, for example [7] uses similar methods for hyperbolic 3-manifolds with totally geodesic boundary.

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Department of Mathematics, University of California Santa Barbara, CA 93106, USA
cooper@math.ucsb.edu
long@math.ucsb.edu