Grothendieck's problem for 3-manifold groups

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For Fritz Grunewald

1 Introduction

The profinite completion of a group Γ is the inverse limit of the directed system of finite quotients of Γ, and we shall denote this profinite group by $\hat{\Gamma}$. As is well-known, if Γ is residually finite then Γ injects into $\hat{\Gamma}$. In [19] the following problem was posed by Grothendieck (where it is pointed out that it is natural to assume the groups are residually finite):

Let $u : H \to G$ be a homomorphism of finitely presented residually finite groups for which the extension $\hat{u} : \hat{H} \to \hat{G}$ is an isomorphism. Is u an isomorphism?

This problem was solved in the negative by Bridson and Grunewald in [6] who produced many examples of groups G, and proper subgroups $u : H \hookrightarrow G$ for which $\hat{u}$ is an isomorphism, but u is not. The method of proof of [6] was a far reaching generalization of an example of Platonov and Tavgen [35] that produced *finitely generated* examples that answered Grothendieck's problem in the negative (see also [5]).

Notice that if $\hat{H} \to \hat{G}$ is an isomorphism, then the composite homomorphism $H \hookrightarrow \hat{H} \to \hat{G}$ is an injection. Hence, $H \to G$ must be an injection. Therefore, Grothendieck's Problem reduces to the case where H is a subgroup of G, and the homomorphism is inclusion. Henceforth, we will only consider the situation where $u : H \to G$ is the inclusion homomorphism. We will on occasion suppress u, however as we remark upon later, it is important for us that we do consider the case where the isomorphism $\hat{u}$ is induced by inclusion. All abstract groups are assumed infinite and finitely generated unless otherwise stated.

We introduce the following terminology. Let G be a group and $H < G$. We shall call (G, H) a *Grothendieck Pair* if $u : H \to G$ provide **negative answers** to Grothendieck's problem; i.e. $\hat{u}$ is an isomorphism and u is not. If for all finitely generated subgroups $H < G$, (G, H) is **never** a Grothendieck Pair then we will define G to be *Grothendieck Rigid*. Thus, if $H \to G$ is a counterexample to Grothendieck's original question, then G is **not** Grothendieck Rigid.

In [19], Grothendieck explored general conditions on groups H and G that are not Grothendieck Pairs. This theme was taken up in [29] and [35], and discussed more recently in Bridson's talk at Grunewald's 60th Birthday Conference. For example, in [35] it is shown that if G is a discrete subgroup of $\mathrm{SL}(2, \mathbf{Q}_p)$ or $\mathrm{SL}(2, \mathbf{R})$ then G is Grothendieck Rigid. The proof of this is a simple consequence of the fact that virtually free groups and Fuchsian groups are LERF (see §2.4 below for more on this). On the other hand, it still seems to be an open question in general as to

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whether arithmetic lattices in semi-simple Lie groups having the Congruence Subgroup Property are Grothendieck Rigid, even for $\mathrm{SL}(3, \mathbf{Z})$ (see [35]).

The aim of this paper is to explore Grothendieck's problem for (certain) 3-manifold groups. We will prove various results about Grothendieck Rigidity of such groups. As we discuss further below, Grothendieck's Problem seems naturally related to other problems concerning 3-manifold groups. Note that by Perelman's solution to the Geometrization Conjecture and a theorem of Hempel [22], the fundamental groups of all compact 3-manifolds are residually finite. Our first result generalizes part of Proposition 3 of [35].

Theorem 1.1 *Let M be a closed 3-manifold which admits a geometric structure. Then $\pi_1(M)$ is Grothendieck Rigid.*

We can clearly assume that $\pi_1(M)$ is infinite in our considerations, and the proof of Theorem 1.1 quickly reduces to the case of closed hyperbolic 3-manifolds; ie those that admit a complete finite volume metric of constant sectional curvature -1 (since manifolds with the other geometric structures are known to have LERF fundamental groups). It is worth remarking that it is still a major open problem (although there is some evidence that it is true) as to whether the fundamental groups of hyperbolic 3-manifolds are LERF. Therefore our methods do not appeal to LERF.

We can also extend our results to cusped hyperbolic 3-manifolds.

Theorem 1.2 *Let M be a finite volume, cusped hyperbolic 3-manifold. Then $\pi_1(M)$ is Grothendieck Rigid.*

A corollary of Theorems 1.1 and 1.2 that generalizes the Corollary to Proposition 3 of [35] is (in contrast with the discussion above regarding arithmetic lattices having the Congruence Subgroup Property):

Corollary 1.3 *Let M be an arithmetic hyperbolic 3-manifold. Then $\pi_1(M)$ is Grothendieck Rigid.*

The techniques in the proofs employ a combination of classical 3-manifold topology methods, as well as Thurston's hyperbolization theorem for atoroidal Haken manifolds (see [32] and [45]), applications of the character variety and more recent developments in the field. For example, we use the solutions to the Tameness Conjecture ([1], [9]), the Ending Lamination Conjecture ([7], [8]) and The Density Conjecture (see [11] for a discussion)

In addition to hyperbolic knots, we can also prove a Grothendieck Rigidity result for other knots in S^3 (see Theorem 5.1).

We finish this Introduction with a discussion of Grothendieck's Problem in the context of other problems concerning 3-manifold groups.

Our use of the term "Grothendieck Rigidity" is in part motivated by other rigidity phenomena about 3-manifolds. In our setting we are trying to distinguish a 3-manifold from its covering spaces with finitely generated fundamental group by its profinite completion. Classically, one of the basic problems about compact, irreducible 3-manifolds with infinite fundamental group (see §3 for terminology) was "Topological Rigidity"; namely, *given M_1 and M_2 as above, assume that M_1 and M_2 are homotopy equivalent, are M_1 and M_2 homeomorphic?* This was solved for Haken manifolds by Waldhausen [47], for Seifert fiber spaces in [40] and for hyperbolic manifolds this was solved in a series of papers ([14], [15] and [16]).

A more precise question of particular interest to us can be posed:

Suppose that M_1 and M_2 are geometric 3-manifolds with infinite fundamental group for which the profinite completions $\widehat{\pi_1(M_1)}$ and $\widehat{\pi_1(M_2)}$ are isomorphic. Are M_1 and M_2 homeomorphic?

The hypothesis of the above question can be weakened somewhat, however, there are examples like the fundamental groups of the square knot and granny knot which are isomorphic but the complements of the knots are non-homeomorphic.

Using a basic, but remarkable, result about profinite completions (see §2 for more on this) this question has a more concrete reformulation:

Suppose that M is a geometric 3-manifold with infinite fundamental group. Within the class of such 3-manifold groups, is M determined by the set $\{G : G \text{ is a finite quotient of } \pi_1(M)\}$?

This question has arisen, at least implicitly, in a totally different direction, in recent work of Calegari, Freedman and Walker, [10]. This paper addresses the properties of a "universal pairing" on a complex vector space that arises from looking at all compact oriented 3-manifolds that a fixed closed oriented surface bounds. To that end, in [10] a complexity function is defined on a compact oriented 3-manifold M, and part of of this complexity function involves listing all finite quotients of $\pi_1(M)$. The question as to whether the finite quotients of the fundamental group of a compact orientable irreducible 3-manifold M determine M arises naturally here (see Remark 3.7 of [10]).

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2 Profinite Preliminaries

Here we collect some basics about profinite groups, and discuss some connections of separability to Grothendieck's problem. For details about profinite groups see [37].

2.1

Suppose that Γ is a residually finite (abstract) group. We recall a basic but important fact relating the subgroup structures of Γ and $\hat{\Gamma}$ (see [37] Chapter 3.2).

Proposition 2.1 *With Γ as above, there is a one-to-one correspondence between the set $\mathcal{X}$ of subgroups of Γ that are open in the profinite topology on Γ, and the set $\mathcal{Y}$ of all open subgroups of $\hat{\Gamma}$. This is given by (where we assume Γ is embedded in $\hat{\Gamma}$):*

- *For $H \in \mathcal{X}$, $H \mapsto \overline{H}$, where $\overline{H}$ denotes the closure of H in $\hat{\Gamma}$.*
- *For $Y \in \mathcal{Y}$, $Y \mapsto Y \cap \Gamma$.*

Moreover, $[\Gamma : H] = [\hat{\Gamma} : \overline{H}]$.

Suppose now that Γ_1 and Γ_2 are finitely generated abstract groups such that $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ are isomorphic. Note that by "isomorphic" mean "isomorphic as groups", since every isomorphism between the profinite completions is continuous (we do not assume that there is a homomorphism $\Gamma_1 \to \Gamma_2$). It is easy to deduce from Proposition 2.1 that this implies that Γ_1 and Γ_2 have the same collection of finite quotient groups. In fact, the discussion on pp 88-89 of [37] gives a stronger statement that we record for convenience.

Theorem 2.2 *Suppose that* Γ_1 *and* Γ_2 *are finitely generated abstract groups. For* $i = 1, 2$, *let* $\mathcal{C}_i = \{G : G$ *is a finite quotient of* $\Gamma_i\}$. *Then* $\hat{\Gamma}_1$ *and* $\hat{\Gamma}_2$ *are isomorphic if and only if* $\mathcal{C}_1 = \mathcal{C}_2$.

An easy corollary of this that we will appeal to repeatedly is recorded below. We will use the notation:

$$b_1(\Gamma) = \mathrm{rank}(\Gamma/[\Gamma,\Gamma]) \otimes_{\mathbf{Z}} \mathbf{Q}, \quad \text{and if } X \text{ is a manifold}, \quad b_1(X) = b_1(\pi_1(X)).$$

Corollary 2.3 *Suppose that* Γ_1 *and* Γ_2 *are finitely generated abstract groups for which* $\hat{\Gamma}_1$ *and* $\hat{\Gamma}_2$ *are isomorphic. Then* $b_1(\Gamma_1) = b_2(\Gamma_2)$.

Proof: If not, then one easily constructs a finite abelian quotient of one that cannot be a finite quotient of the other, and this contradicts Theorem 2.2. □

Now suppose that $u : \Gamma_1 \to \Gamma_2$ is a homomorphism of finitely generated (residually finite abstract) groups. This determines a continuous homomorphism $\hat{u} : \hat{\Gamma}_1 \to \hat{\Gamma}_2$. Theorem 2.1 can be applied to show that (see [37] Lemma 3.2.6 for **2.**, and also [35]):

1. $\hat{u}$ *is surjective if and only if* $u(\Gamma_1)$ *is dense in the profinite topology on* Γ_2.

2. *Suppose that* u *is the inclusion homomorphism. Then* $\hat{u}$ *is injective if and only if the profinite topology of* Γ_2 *induces on* Γ_1 *(or* $u(\Gamma_1)$*) its profinite topology.*

With this discussion, we prove:

Lemma 2.4 *Let* Γ *be a finitely generated (abstract) group, let* $\Delta < \Gamma$ *be a subgroup of finite index, and* $H < \Gamma$. *If* (Γ, H) *is a Grothendieck Pair, then the inclusion map* $\Delta \cap H \hookrightarrow \Delta$ *induces an isomorphism of profinite completions.*

Proof: Let $\mathcal{T}$ denote the profinite topology on Γ. As above, we let $u : H \to \Gamma$ denote the inclusion homomorphism with $\hat{u} : \hat{H} \hookrightarrow \hat{\Gamma}$ an isomorphism. Let u' denote the restriction of u to $H' = H \cap \Delta$. Since Δ has finite index in Γ, Δ is an open subgroup in $\mathcal{T}$. By **1**. above, $u(H)$ is dense in $\mathcal{T}$. It now follows from elementary point set topology that $u'(H')$ is dense in the profinite topology on Δ, since this is the subspace topology induced from $\mathcal{T}$.

To prove injectivity, note first that by **2**. above, the profinite topology on H (or $u(H)$) coincides with that induced by $\mathcal{T}$. Since H' has finite index in H, and Δ has finite index in Γ it follows from the properties of the subspace topology that the profinite topology on $u'(H')$ conicides with that induced by Δ. Now **2**. proves that $\hat{u}'$ is injective. □

Remark: In our applications of Lemma 2.4 (see Corollary 4.5), the groups we are working with will have the property that $\Delta \cap H$ is not isomorphic to Δ, thereby showing that $(\Delta, \Delta \cap H)$ is a Grothendieck Pair.

2.2

Let G be a group and $H < G$. Then G is called H-separable if given any $g \in G \setminus H$, there is a finite index subgroup N of G such that $H < N$ but $g \notin N$. Equivalently, G is H-separable if H is closed in the profinite topology on G. The group G is called *LERF* or *subgroup separable* if G is H-separable for every finitely generated subgroup H of G. LERF has attracted considerable interest recently through its connections with problems in 3-manifold topology (see for example [2], [20] and [28]).

A weaker property than H-separability is *engulfing*, which is also related to problems in 3-manifold topology (see [26]). In the notation above, suppose that H is a proper subgroup of G. We say that H is engulfed if $H < K$ where K is a proper subgroup of G of finite index.

The following lemma illustrates the connection between the separability properties mentioned above, and Grothendieck's problem (see also [35] pp. 90-91). Again, we are implicitly assuming that if $H < \Gamma$, then $u : H \hookrightarrow \Gamma$ is the natural inclusion map.

Lemma 2.5 *Suppose Γ is a finitely generated abstract group and $H < \Gamma$. Assume that H is engulfed. Then (Γ, H) is not a Grothendieck Pair.*

Proof: We are assuming that $u : H \hookrightarrow \Gamma$ induces the isomorphism $\hat{u}$. In particular, $\hat{u}$ is onto, and so from **1.** of §2.1, $u(H)$ is dense in the profinite topology on Γ. On the other hand, if H is engulfed, H is contained in a proper subgroup of finite index, and so $u(H)$ cannot be dense. Hence we have a contradiction. $\square$

An immediate Corollary of Lemma 2.5 is the following statement that was made in the Introduction.

Corollary 2.6 *Suppose that Γ is LERF, then Γ is Grothendieck Rigid.*

2.3

We conclude this section with a discussion of how character varieties of Grothendieck Pairs are related. We will restrict attention to $\mathrm{SL}(2, \mathbf{C})$ and $\mathrm{PSL}(2, \mathbf{C})$ character varieties. For a finitely generated (abstract) group G, we will denote by $R(G)$ and $X(G)$ the $\mathrm{SL}(2, \mathbf{C})$ representation and character varieties of G. If V is an algebraic set, we will let $\dim(V)$ denote the maximal dimension of an irreducible component of V.

Suppose that H is a finitely generated subgroup of G. Then the inclusion mapping u induces maps:

$$\tilde{u} : R(G) \to R(H) \qquad u^* : X(G) \to X(H),$$

by

$$\tilde{u}(\rho) = \rho|H \ \text{ and } \ u^*(\chi_\rho) = \chi_{\rho|H}.$$

It is easy to check that these maps are algebraic maps.

The following result is a variation of Proposition 4 of [35] in terms of character varieties.

Proposition 2.7 *Let (G_1, G_2) be a Grothendieck Pair. Let $\mathcal{X}_i$ ($i = 1, 2$) denote the collection of components of $X(G_i)$ each of which contains the character of an irreducible representation. Then u^* determines a bijection from $\mathcal{X}_1$ to $\mathcal{X}_2$. Furthermore, u^* is a birational isomorphism on each $X \in \mathcal{X}_1$.*

Proof: Firstly, we note that u^* maps $\mathcal{X}_1$ to $\mathcal{X}_2$. The reason is this. Pick some irreducible representation ρ of G_1, and suppose its restriction to G_2 were reducible. In our setting, this means it is conjugate to a group of upper triangular matrices and therefore soluble. Such a subgroup is verbal for the word $[[a, b], [c, d]]$ and therefore separable in $\rho(G_1)$ (see [25] for example). In particular, it can be engulfed in the image, so that G_2 is engulfed in G_1, a contradiction.

We next claim that u^* is onto a Zariski dense subset of $\mathcal{X}_2$: Let $X \in \mathcal{X}_2$, and let χ_ρ be the character of an irreducible representation $\rho : G_2 \to \mathrm{SL}(2, \mathbf{C})$. Since (G_1, G_2) is Grothendieck Pair, it follows from [19] (see also [35]), that there exists a representation $\rho_1 : G_1 \to \mathrm{SL}(2, \mathbf{C})$ such that $\rho = \rho_1 \circ u$; i.e. ρ_1 restricted to G_2 is ρ. Since X contains the character of an irreducible representation, the generic character of X is the character of an irreducible representation (see [13]) and the claim follows.

Finally, we show that u^* is injective. Suppose that $\chi_\rho, \chi_\phi \in \mathcal{X}_1$ are characters of irreducible representations ρ and ϕ (as above the generic character is the character of an irreducible representation) with $u^*(\chi_\rho) = u^*(\chi_\phi)$. Let $\rho(G_1) = G$ and $\phi(G_1) = G'$.

Since $u^*(\chi_\rho) = u^*(\chi_\phi)$, and the restrictions to G_2 determine the same *irreducible* character, we can conjugate G and G' so that they agree on the image, H say, of G_2 (see [13] Proposition 1.5.2). Following the idea in [25], we can construct a homomorphism:

$$\Phi : G_1 \to G \times G'$$

by setting

$$\Phi(g) = (\rho(g), \phi(g)).$$

By construction $\Phi(G_2)$ maps to the diagonal subgroup $H \times H$, whereas $\Phi(G_1)$ does not map diagonally, since the representations ρ and ϕ are distinct. Since G and G' are finitely generated linear groups, they admit many non-trivial finite quotients, and so it follows that G_2 can be engulfed. This contradicts Lemma 2.5. $\square$

Remark: Proposition 2.7 can be formulated in exactly the same way for the $\mathrm{PSL}(2, \mathbf{C})$ character variety (see [24] for more on this character variety). We will use the notation $Y(G)$ for the $\mathrm{PSL}(2, \mathbf{C})$ character variety.

3 3-Manifold Preliminaries

We include some background from the topology of 3-manifolds that will be useful in the sequel.

3.1

Recall that a compact orientable 3-manifold M is called *irreducible* if every embedded 2-sphere bounds a 3-ball. A properly embedded orientable surface $S \neq S^2, D^2$ in a compact orientable 3-manifold M is called *incompressible* if $\ker(\pi_1(S) \to \pi_1(M)) = 1$. The surface S is *essential* if it is in addition non-peripheral; that is to say $\pi_1(S)$ is not conjugate into the fundamental group of a boundary component of M. M is called *atoroidal* if every π_1-injective map of a torus into M is homotopic into ∂M. A compact, orientable, irreducible 3-manifold M is called *Haken* if it contains an incompressible surface.

Remark: We are assuming that our incompressible surfaces are embedded, which is somewhat non-standard these days.

3.2

The following theorem of Jaco-Shalen and Johannson provides a canonical decomposition for compact orientable irreducible 3-manifolds.

Theorem 3.1 *Let M be a compact, orientable, irreducible 3-manifold. There exists a finite collection $\mathcal{T}$ of disjoint incompressible tori such that each component of $M \setminus \mathcal{T}$ is either a Seifert fibered space or is atoroidal. A minimal such collection T is unique up to isotopy.*

We will be interested only in the case when ∂M is a single incompressible torus and assume this in the following discussion. In this case, this minimal decomposition of M is called the JSJ decomposition of M. The collection $\mathcal{T}$ in this case will be called the JSJ tori. It is easy to see that naturally attached to the JSJ decomposition is a dual graph with edges the JSJ tori and vertices the connected components of $M \setminus \mathcal{T}$. We will denote the submanifold determined by the vertex v by M_v. This in turn leads to a graph of group decomposition of $\pi_1(M)$ that we also refer to as the JSJ decomposition for $\pi_1(M)$.

By Thurston's hyperbolization theorem ([32] and [45]) the atoroidal pieces are typically hyperbolic (i.e. admit a complete finite volume hyperbolic structure), the only exceptions being $S^1 \times S^1 \times [0,1]$ and the twisted I-bundle over the Klein bottle (note that $S^1 \times D^2$ does not arise in this situation). We will refer to a $S^1 \times S^1 \times [0,1]$ piece as a collar.

We now recall some of the terminology and results of [18] that are important for us. We will do this only for manifolds with a single torus boundary component and have no atoroidal piece that is a twisted I-bundle over the Klein bottle.

Notation: *Let M be a compact orientable irreducible 3-manifolds with ∂M a single incompressible torus. Let $\mathcal{T}$ be the collection of JSJ tori.*

Σ denotes the characteristic submanifold of M. For convenience we spell out the construction.

Suppose that $T \in \mathcal{T}$. If T lies in no Seifert piece add to $\mathcal{T}$ a parallel copy of T. If T lies between two Seifert fibered pieces, remove the interior of a regular neighbourhood of T from M. Split M along the new $\mathcal{T}$, and let Σ' denote the Seifert fibered pieces obtained. If the boundary of M does not lie in a Seifert fibered piece add to Σ' the closure of a collared neighbourhood of ∂M. The collection of Seifert fibered pieces so obtained is Σ.

Let βM denote the piece of Σ that contains ∂M.

Let γM denote the geometric piece adjacent to ∂M, together with a collar on ∂M.

Remark: Since M has one boundary component, if the unique piece of the JSJ decomposition that contains the boundary is hyperbolic then βM is a collar. In this case γM can be identified with M_v where M_v is the vertex manifold containing the boundary of M.

Let M_1 and M_2 be compact, orientable, irreducible 3-manifolds with ∂M_i $(i = 1, 2)$ being a single incompressible torus. An essential map $g : M_1 \to M_2$ (i.e. g_* is injective on fundamental groups) is *loose* if it is homotopic to a map $f : M_1 \to M_2$ for which $f(M_1) \cap \gamma M_2 = \emptyset$. Otherwise g is *tight.*

A subgroup H of $\pi_1(M)$ is *loose* if for some component C of $M \setminus \gamma M$, there is a conjugate of H in $i_*(\pi_1(C))$, where $i : C \to M$ is the inclusion map. Otherwise H is called *tight.*

The two notions of tight just described are consistent, since it is shown in [18] Proposition 6.5 that if M_1 and M_2 are as above and $f : M_1 \to M_2$ is an essential map, then f is tight if and only $f_*(\pi_1(M_1))$ is a tight subgroup in $\pi_1(M_2)$.

We record the following proposition (which is a corollary of results in [18]).

Proposition 3.2 *Let M_1 and M_2 be Haken manifolds whose boundaries both consist of a single incompressible torus, and assume that βM_2 is a collar. Assume that $f : M_1 \to M_2$ is a tight essential map. Then βM_1 is a collar and $f|_{\beta M_1} : \beta M_1 \to \beta M_2$ is a covering map.*

Proof: Since f is a tight essential map, we can homotope f so that it satisfies the conclusions of Theorem 6.1 of [18]. Consider $f(\beta M_1)$. By properties of tight essential maps, this is a component Q of the JSJ decomposition of M_2 (see the Deformation Theorem 1.2 and Theorem 6.1 of [18]). Indeed, tightness shows that Q must coincide with that piece of the JSJ decomposition of M_2 containing ∂M_2, and this is a collar by assumption. It now follows that βM is a collar, and the map f is a covering map. $\square$

3.3

We will be particularly interested in the discussion of the previous section when the manifold M_2 is the exterior of a knot $K \subset S^3$; that is the closure of the complement of a small open tubular neighbourhood of K. We will denote the exterior of a knot K by $E(K)$. Recall that a knot K is prime if it is not the connect sum of non-trivial knots.

The JSJ graph associated to $E(K)$ is a rooted tree, where the root vertex v_0 corresponds to the unique vertex manifold containing $\partial E(K)$. In the notation §3.2, this is simply the manifold γM.

The Seifert fibered pieces that can arise in the JSJ decomposition of $E(K)$ are also well-understood (see [23] Lemma VI 3.4). These are the following:

torus knot exterior;

a cable space: ie a manifold obtained from $D^2 \times S^1$ by removing an open regular neighbourhood in $\mathrm{Int}(D^2) \times S^1$ of an essential simple closed curve c which lies in a torus $J \times S^1$, where J is a simple closed curve in $\mathrm{Int}(D^2) \times S^1$ and c is non-contractible in $J \times S^1$.

an (n-fold) composing space: i.e. a compact 3-manifold homeomorphic to $W \times S^1$, where W is an n-times punctured disc.

In addition, since a Klein bottle does not embed in S^3, the only exceptional atoroidal piece is $S^1 \times S^1 \times [0, 1]$. If γM is a cable space, then K is a cable knot, and when γM is a composing space, K is a composite knot; that is a non-prime knot (see [23]).

A corollary of this discussion and that contained in §3.1 that will be useful to record is the following.

Corollary 3.3 *Suppose that K is a prime satellite knot that is not a cable knot. Then $\beta E(K)$ is a collar and $\gamma E(K)$ is the piece of the JSJ graph containing the root vertex.*

Proof: Only the statement about $\beta E(K)$ needs any comment. Since K is a satellite knot it is not a torus knot. In addition, since K is a prime knot that is not a cable knot, it follows from above that $\gamma E(K)$ is hyperbolic. The statement about $\beta E(K)$ now follows. □

3.4

A standard property about compact 3-manifolds with non-empty boundary is the so-called "half-lives, half-dies" which is a consequence of Poincare-Lefschetz duality.

Theorem 3.4 *Let M be a compact orientable 3-manifold with non-empty boundary. Then the rank of*

$$\ker(H_1(\partial M; \mathbf{Z}) \to H_1(M; \mathbf{Z}))$$

is $b_1(\partial M)/2$.

A corollary of this that will be useful for us is the following.

Corollary 3.5 *Let M be a compact orientable irreducible 3-manifold with a single torus boundary component, $H < \pi_1(M)$ a finitely generated non-abelian subgroup and let X_H denote the cover of M corresponding to H. Assume that $H_1(X_H; \mathbf{Q}) \cong \mathbf{Q}$. Then X_H is homotopy equivalent to $\Sigma \setminus K$, a knot complement in a closed orientable 3-manifold Σ.*

Proof: Since H is finitely generated, [38] guarantees the existence of a compact core C_H for X_H; that is a compact co-dimension zero submanifold C_H of X_H such that the inclusion mapping $C_H \hookrightarrow X_H$ induces a homotopy equivalence. In particular, $\pi_1(C_H) \cong H$.

In addition since M is irreducible, C_H is irreducible and so we may assume that there are no 2-sphere boundary components in ∂C_H. We are assuming that $b_1(X_H) = 1$, and so C_H is a compact manifold with non-empty boundary and $b_1(C_H) = 1$. It follows from Theorem 3.4 that ∂C_H is therefore a torus.

Since H is non-abelian, C_H is not a solid torus, and by irreducibility ∂C_H must be incompressible. Thus C_H is a compact, orientable, 3-manifold with incompressible torus boundary. This proves the corollary. $\square$

4 Proofs

Before commencing with the proofs of Theorems 1.1 and 1.2, we make some preliminary comments and prove a proposition and a lemma which require some additional notation.

4.1

Let W be a compact orientable 3-manifold that is hyperbolizable—so that the interior of W admits at least one complete hyperbolic structure. If this is of finite volume, then this is the unique hyperbolic structure.

We will let $Y(W)$ denote the character variety $Y(\pi_1(W))$. If ∂W is empty or consists of a disjoint union of n_T incompressible tori, then Thurston proved [44] (see also [33]) that $Y(W)$ contains a so-called *canonical component* (denoted by X_0) that contains the character of the faithful discrete representation. In the former case, a well-known consequence of Mostow-Weil Rigidity is that X_0 is a single point. In the latter case, Thurston [44] proved that X_0 has complex dimension n_T.

In the following discussion, we will assume that ∂W contains a non-toroidal component. In which case the interior of W admits many hyperbolic structures. Denote by $AH(W)$ the subset of $Y(W)$ consisting of all the characters of discrete faithful representations. In addition, $AH_T(W)$ will denote the subset of characters of *minimally parabolic representations*, ie those $\chi_\rho \in AH_T(W)$ satisfying the condition that for $g \in \pi_1(W)$, $\rho(g)$ is parabolic if and only if g lies in a rank 2 abelian subgroup. We will also call the group $\rho(\pi_1(W))$ minimally parabolic.

It will be convenient for our purposes to record some facts about $AH(W)$. A convenient reference for the first result stated below is [24] Theorem 8.44. Recall that a Kleinian group Γ (or the quotient $\mathbf{H}^3/\Gamma$) is called *geometrically finite* if Γ admits a finite sided convex fundamental polyhedron. Otherwise, Γ (or $\mathbf{H}^3/\Gamma$) is called geometrically infinite.

Theorem 4.1 *Let W be as above and χ_ρ a minimally parabolic geometrically finite representation. Then χ_ρ is a smooth point of $Y(W)$ and the complex dimension of a component of $Y(W)$ containing χ_ρ is $-3\chi(\partial W)/2 + n_T = -3\chi(W) + n_T$.*

Understanding the detailed structure of $AH(W)$ (resp. $AH_T(W)$) has been one of the main goals in the deformation theory of Kleinian groups. For example, combining work of Marden [31] and Sullivan [42] shows that the subset $GAH_T \subset AH_T(W)$ consisting of characters of geometrically finite representations is the interior of $AH(W)$. The Density Conjecture asserts that $AH(W)$ is the closure of $GAH_T(W)$. This has been recently established as a consequence of the solutions to the Tameness and Ending Lamination Conjectures ([1], [9], [7] and [8]), as well as many other develpoments (see the survey papers [11] and [12] for more on this).

Theorem 4.2 *(Density Theorem) With W as above, then $AH(W)$ is the closure of its interior.*

The topology of $AH_T(W)$ has also been investigated in some detail. It is known that $AH_T(W)$ can be highly disconnected ([3], [4]) and it remains an open problem to determine if these components can be contained in more than one irreducible component of $Y(W)$ (see [11] §10). However, for our purposes all we require is the following.

Proposition 4.3 *Let W be as above. Let V be an irreducible component of $Y(W)$ such that $V \cap AH(W) \neq \emptyset$. Then* $\dim(V) = -3\chi(\partial W)/2 + n_T = -3\chi(W) + n_T$.

Proof: By Theorem 4.2 (The Density Theorem), and the discussion preceding it, every component of $AH(W)$ contains a component of $GAH_T(W)$. Thus if V is a component of $Y(W)$ such that $V \cap AH(W) \neq \emptyset$, we can assume that V contains a component $U \subset GAH_T(W)$. By Theorem 4.1, $GAH_T(W)$ is contained in the set of smooth points of $Y(W)$. Thus the dimension of V is now given by Theorem 4.1. □

Two special cases that are worth pointing out are the cases when $W = S \times I$ where S is either a closed orientable surface of genus ≥ 2 or an orientable punctured surface $\neq$ disc or annulus. In the former case Goldman [17] established that $Y(W)$ has two irreducible components of dimension $6g-6$, and the latter case is that of a free group, and in this case, $Y(W)$ is affine space of dimension $3r - 3$ where r is the rank of the free group.

Before commencing with the proofs, we prove a general lemma that will be useful in the setting of hyperbolic 3-manifolds.

Lemma 4.4 *Let $M = \mathbf{H}^3/\Gamma$ be a finite volume orientable hyperbolic 3-manifold, and $H < \Gamma$ a finitely generated subgroup such that (Γ, H) is a Grothendieck Pair. Then H must be geometrically finite.*

Proof: Let χ_0 denote the character of the faithful discrete representaion of Γ. From the discussion above, if M is closed, the canonical component $X_0 \subset Y(\Gamma)$ consist only of $\{\chi_0\}$, and if M has m cusps then X_0 has dimension m. Proposition 2.7 implies that $u^*(X_0)$ is a component of $Y(H)$. Furthermore, this component contains the character $u^*(\chi_0)$ (the character of identity representation of H).

Suppose that H were geometrically infinite. It follows from the solution to the Tameness Conjecture ([1], [9]) and work of Canary (see [12]) that H is isomorphic to the fundamental group of either a closed surface or a punctured surface. Indeed, in either case, the surface in question is the fibre in a fibration over the circle (or a bundle over a mirrored interval in the non-orientable case) of some finite cover of M. However, in both cases the surface group is separable and so we can apply Lemma 2.5.

This completes the proof that H is geometrically finite. □

4.2 Proof of Theorem 1.1

Since $\pi_1(M)$ can be assumed to be infinite, M admits a geometric structured modelled on $\mathbf{H}^3$, SOLV, $\mathbf{E}^3$, $S^2 \times \mathbf{R}$, $\mathbf{H}^2 \times \mathbf{R}$, $\widetilde{\mathrm{PSL}}_2$, or NIL (see [46]). The last five of these are Seifert fibered geometries and so [39] implies that Γ is LERF. Hence Corollary 2.6 applies. Manifolds in SOLV have virtually solvable fundamental groups, and these are also known to be LERF; since for example the fundamental group of any torus bundle with SOLV geometry is a subgroup of $\mathrm{SL}(3, \mathbf{Z})$ and Theorem 2 of [48] applies. Thus it remains to deal with the case of hyperbolic 3-manifolds.

It will be convenient to state the following corollary of Lemma 2.4 in the hyperbolic setting that we will make use of on several occasions below. Recall that a group is called *cohopfian* if it does not inject as a proper subgroup of itself.

Corollary 4.5 *Let $M = \mathbf{H}^3/\Gamma$ be a finite volume hyperbolic 3-manifold, $\Delta < \Gamma$ a subgroup of finite index and $H < \Gamma$. If (Γ, H) is a Grothendieck Pair, then $(\Delta, \Delta \cap H)$ is a Grothendieck Pair.*

Proof: From Lemma 2.4 and the Remark that follows it, we simply need to note that it is known that the fundamental group of a finite volume hyperbolic 3-manifold is cohopfian (see for example [36]). □

Returning to the proof of Theorem 1.1, let $M = \mathbf{H}^3/\Gamma$ and $u : H \hookrightarrow \Gamma$ determine a Grothendieck Pair. Note that by Corollary 4.5, it suffices to deal with the case that M is orientable, and we assume this henceforth.

Let X_H (resp. C_H) denote the cover of M corresponding to H (resp. denote a compact core C_H for X_H). Thus C_H is a compact manifold with non-empty boundary (H must have infinite index in Γ). This boundary may or may not be incompressible.

By Lemma 4.4, H is geometrically finite, and since M is closed, Γ has no parabolic elements. Hence H is minimally parabolic. Proposition 2.7 implies that $u^*(X_0)$ is a component of $Y(H)$. Furthermore, this component contains the character $u^*(\chi_0)$ which by definition is the character of a minimally parabolic representation onto H. Since the core corresponding to H has non-empty boundary, Proposition 4.3 applies to give a contradiction. □

When M is not geometric, one can still prove some results.

Theorem 4.6 *Suppose that M is a closed 3-manifold that is an irreducible rational homology 3-sphere. Then $\pi_1(M)$ is Grothendieck Rigid.*

Proof: Since M is a rational homology 3-sphere it is orientable. The theorem easily follows from Theorem 3.4 and the observation that any finitely generated infinite index subgroup H of $\pi_1(M)$ has infinite abelianization since after capping off any 2-spheres, a compact core C_H of H must have non-empty boundary. □

4.3 Proof of Theorem 1.2:

As above, we will assume that M is orientable, and also as above, we let $\Gamma = \pi_1(M)$, with $H < \Gamma$ for which (Γ, H) is a Grothendieck Pair.

To illustrate some of the ideas in the proof, we first give the proof in a special case, that of a 1-cusped manifold.

M is 1-cusped.

The case of $b_1(M) = 1$ can be handled without resort to the character variety and we give this argument first. As in the proof of Theorem 1.1, we consider the compact core C_H. Note that by Corollary 2.3, the assumption that $\hat{H} \cong \hat{\Gamma}$, implies that $b_1(C_H) = 1$. Hence we deduce from Corollary 3.5 that C_H is homotopy equivalent to a compact 3-manifold with non-empty incompressible torus boundary. Furthermore C_H is irreducible since X_H is a quotient of $\mathbf{H}^3$. Hence C_H is Haken.

Since $H < \Gamma$, it follows that C_H satisfies the criteria required for Thurston's hyperbolization theorem for Haken manifolds (see [32] and [45]). That is to say, $C_H \subset \mathbf{H}^3/H$ with $\mathbf{H}^3/H$ having finite volume. This is a contradiction, since X_H is then a finite cover of M.

We now deal with the general 1-cusped case. Note that in this case X_0 has dimension 1. With notation as above, some component of ∂C_H is not an incompressible torus, for otherwise, as in the previous paragraph, $\mathbf{H}^3/H$ has finite volume which is a contradiction.

Thus, there is some component S of ∂C_H of genus at least two. As in the closed case, Proposition 2.7 implies that $u^*(X_0)$ is a component of $Y(H)$. Furthermore, this component contains the character $u^*(\chi_0)$ which by definition is the character of the discrete faithful representation of H. Now by Lemma 4.4, H is geometrically finite, and if it is also minimally parabolic then we can argue as in the closed case and apply Proposition 4.3. If H is not minimally parabolic, we simply note that nearby to H in the component containing $u^*(\chi_0)$, there are geometrically finite minimally parabolic representations and the dimension argument can still therefore be made. $\square$

M **has** $m \geq 2$ **cusps.**

As above, we let $H < \Gamma$ with (Γ, H) a Grothendieck Pair. Again, as before, the compact core C_H is a compact manifold with non-empty boundary. As in the 1-cusped case, H is geometrically finite by Lemma 4.4.

Notice that any torus in ∂C_H must be incompressible, for if it is not, then the Loop Theorem gives a compression and hence an embedded 2-sphere. By irreducibility, this sphere bounds a ball in the manifold $\mathbf{H}^3/H$ and this ball cannot contain the non-compact end which abuts the torus in question. It follows that ∂C_H is a solid torus, which is impossible.

It follows that if ∂C_H consists of m tori, then these are all incompressible and we can then argue exactly as in the case of one cusp.

Thus we can assume that some component of ∂C_H has genus at least two, and so $\chi(C_H) < 0$. We will assume that C_H has $n_T < m$ (possibly zero) torus cusps. As in the argument for the Theorem 1.1 and the case of one cusp given above, the character χ_0 of the faithful discrete representation of $\pi_1(M)$ determines a character $u^*(\chi_0)$ of H, and furthermore $u^*(X_0) = V$, a component of $Y(C_H)$ with $\dim(X_0) = m = \dim(V)$. Note that, as in the argument for the 1-cusped case if H is not minimally parabolic, nearby to $u^*(\chi_0)$ in V, there are geometrically finite minimally parabolic representations and the dimension argument can still therefore be made. We therefore deduce from Proposition 4.3 that $\dim(V) = -3\chi(C_H) + n_T$, and so $m = -3\chi(C_H) + n_T$.

The idea now is to exhibit a finite index subgroup $\Delta < \Gamma$ such that the dimension of the canonical component of the manifold $\mathbf{H}^3/\Delta$ is strictly less than the dimension of the component containing the character associated to the faithul discrete representation given by $H \cap \Delta$. In particular, $(\Delta, H \cap \Delta)$ is not a Grothendieck Pair which contradicts Corollary 4.5. This will complete the proof.

To that end, [27] provides an infinite collection of rational primes $p > 3$ so that reduction homomorphisms surject both Γ and H onto the finite simple groups $\mathrm{PSL}(2, p)$. Let N_p denote the kernel of the reduction epimorphism $\Gamma \to \mathrm{PSL}(2, p)$, and M_p the finite cover of M determined by N_p. We let C_p denote the cover of the compact core C_H determined by $H \cap N_P$.

Now we can, by discarding a further finite number of p, assume that for each of the primes p, the image under these reduction homomorphisms of any of the m peripheral subgroups of Γ, and any of the n_T toroidal peripheral subgroups of H is non-trivial. In particular, using the structure of subgroups of $\mathrm{PSL}(2, p)$ generated by unipotent elements (see [43] Chapter 3, §6 for example), it follows that any of the aforementioned peripheral subgroups is mapped to a cyclic group of order p. The order of $\mathrm{PSL}(2, p)$ is $p(p^2 - 1)/2$, and so we deduce that M_p has $m(p^2 - 1)/2$ cusps.

In addition, H surjects onto $\mathrm{PSL}(2, p)$, and so by multiplicativity of Euler characteristic, we deduce that the Euler characteristic of C_p is $\chi(C_H)p(p^2 - 1)/2$. Furthermore, we also arranged that C_p has $n_T(p^2-1)/2$ torus cusps. We now compare the dimensions of the canonical component of M_p with the dimension of the component V_p of $X(C_p)$ containing the character associated to $H \cap N_p$.

From above, M_p has $m(p^2-1)/2$ cusps, and therefore the dimension of $X_0(M_p)$ is $m(p^2-1)/2$. By Proposition 4.3 the dimension of V_p is $-3\chi(C_p) + n_T(p^2 - 1)/2$. Now, using the earlier computation for $\dim(V)$ we have:

$$-3\chi(C_p) = -3\chi(C_H)p(p^2 - 1)/2 = p(p^2 - 1)(m - n_T)/2.$$

Since $(N_p, H \cap N_p)$ is a Grothedieck Pair, Proposition 2.7 allows us to equate the dimensions of

$X_0(M_p)$ and V_p. This gives:

$$m(p^2-1)/2 = p(p^2-1)(m-n_T)/2 + n_T(p^2-1)/2.$$

This is easily rearranged to show that $m = n_T$ which is a contradiction. □

Remarks:(1) We are very grateful to a referee who suggested the argument above in the proof of the case when M has at least 2 cusps.

(2) We are grateful to a second referee who made the following observation in the context of trying to prove Theorems 1.1 and 1.2 using only pro-p completions.

If two groups have isomorphic profinite completions, then they have isomorphic pro-p completions for any prime p. However, having isomorphic pro-p completions is not enough to prove the main results of the paper.

For example, let $L \subset S^3$ be a boundary link (ie the components of L bound disjoint Seifert surfaces) and let $G = \pi_1(S^3 \setminus L)$. There is a map from $S^3 \setminus L$ to a wedge of circles which induces a surjective homomorphism from G to a non-abelian free group F. This has a section $F \to G$. By Stallings' theorem (see [41]), any homomorphism between groups G and K that induces an isomorphism on $H_1(-;\mathbf{F}_p)$ and a surjection on $H_2(-;\mathbf{F}_p)$ induces an isomorphism of pro-p completions. Applying this to G and F we see that the homomorphism $G \to F$ induces an isomorphism of pro-p completions. Hence, the inclusion $F \hookrightarrow G$ induces an isomorphism of pro-p completions.

5 Knots in S^3

In this section we prove the following result. As in §3.3, $E(K)$ will denote the exterior of K.

Theorem 5.1 *Let $K \subset S^3$ be a knot in S^3 which is either a torus knot or for which $\beta E(K)$ is a collar. Then $\pi_1(S^3 \setminus K)$ is Grothendieck Rigid.*

It is a consequence of Thurston's hyperbolization theorem for Haken manifolds (see [32] and [45]), that a knot K is either hyperbolic (so that $S^3 \setminus K$ is hyperbolic), is a torus knot (in which case $S^3 \setminus K$ is a Seifert fibered space) or is a satellite knot (in which case $S^3 \setminus K$ contains an embedded incompressible torus that is not boundary parallel). It is known by [39] that the fundamental groups of Seifert fibered spaces are LERF, which proves Grothendieck Rigidity for torus knot groups. Thus, given Theorem 1.2, the main part of the proof of Theorem 5.1 deals with the case that $\beta E(K)$ is a collar.

Proof: Following the discussion above, we will assume that $\beta E(K)$ is a collar. In particular, the discussion in §3.3 (see Corollary 3.3) shows that $\gamma E(K)$ is hyperbolic.

Let H be a finitely generated subgroup of Γ such that the inclusion of H into Γ induces an isomorphism of profinite completions. Let X_H (resp. C_H) denote the cover of $E(K)$ corresponding to H (resp. denote a compact core C_H for X_H). Corollary 3.5 shows that C_H is homotopy equivalent to a compact 3-manifold with non-empty incompressible torus boundary.

Note that the assumption that $\hat{H} \cong \hat{\Gamma}$ implies that H has infinite index in Γ by Lemma 2.5. In addition, H is not infinite cyclic since Γ is a non-abelian group. Also note that $\pi_1(\partial C_H)$ is not peripheral; ie it is not conjugate into $\pi_1(\partial E(K))$. For if this were the case, C_H would be a finite sheeted covering of $E(K)$ and so H would be a finite index subgroup of Γ which is a contradiction.

The proof will be completed by the following two lemmas and Lemma 2.5 (recall the terminology from §3.2).

Lemma 5.2 *In the notation above, H is a loose subgroup of Γ.*

Proof: Suppose that H is not loose, and so by definition H is a tight subgroup of Γ. From §3.2, we can suppose that there is a tight essential map $f : C_H \to E(K)$ realizing $H = f_*(\pi_1(C_H))$. Now Proposition 3.2 shows that βC_H is a collar, and f can be deformed to a covering map of collars. In particular, this covering map has finite degree. Hence H has finite index in Γ which is a contradiction. $\square$

Lemma 5.3 *A loose subgroup of* Γ *can be engulfed in* Γ.

Proof: Suppose that H is a loose subgroup of $\Gamma = \pi_1(S^3 \setminus K)$, so we can assume that $H < \pi_1(C)$ where C is a component of $E(K) \setminus \gamma E(K)$.

By definition, C does not contain $\partial E(K)$. This affords the following decomposition of the JSJ graph of $E(K)$; namely it has the form of a rooted tree $T = T_C \cup \{v_0\} \cup T_D$ where T_C and T_D are subtrees of the JSJ graph of $E(K)$, and meet precisely in $\{v_0\}$. This yields the free product with amalgamation decomposition $\Gamma = G_C *_{\pi_1(P)} G_D$ where P is some JSJ torus lying in the boundary of γM, and G_C and G_D the fundamental groups of the connected submanifolds associated to T_C and T_D, with $H < G_C$. Note that $G_D \neq \pi_1(P)$, so that this is a non-trivial amalgamated product decomposition.

Now $\pi_1(P)$ is Abelian, so it is separable in $\pi_1(E(K))$ (see [21] and [25]). A standard consequence of the separability of $\pi_1(P)$ is that we can arrange finite groups A, B and C and an epimorphism $\phi : G_C *_{\pi_1(P)} G_D \to A *_Q B$ with $\phi(G_C) = A$, $\phi(G_D) = B$ and $\phi(\pi_1(P)) = Q$ (see for example [30]). In particular, $A *_Q B$ is a non-trivial free product with amlagamation of finite groups, and as such, is itself LERF ([39]). Thus if $g \in B \setminus Q$, we can find a finite index subgroup of $A *_Q B$ that contains A but not g. Since A contains $\phi(H)$, it follows that $\phi(H)$ is contained in a proper subgroup of finite index in $A *_Q B$. That is to say, $\phi(H)$ is engulfed, and so H is engulfed. $\square$

Remarks: (1) Even when K is a cable knot or composite knot, Lemma 5.3 still holds. Thus to prove Theorem 5.1 for cable knots and composite knots, one needs to rule out the case of a Grothendieck Pair $(\pi_1(S^3 \setminus K), H)$ where H is a tight subgroup of $\pi_1(S^3 \setminus K)$.

(2) If $(\pi_1(S^3 \setminus K), H)$ is a Grothendieck Pair, then as in the proof of Theorem 5.1, X_H is homotopy equivalent to a knot complement in some closed manifold. Furthermore, the Grothendieck Pair assumption implies that $H_1(X_H; \mathbf{Z}) \cong \mathbf{Z}$. It follows that X_H is homotopy equivalent to a knot complement in integral homology 3-sphere (see Lemma 3.0 of [18] for example). At present we do not know that X_H is homotopy equivalent to a knot complement in S^3.

(3) There are satellite knot groups that are known not to be LERF [34], for example connect sums of torus knots.

(4) Lemma 5.3 is in the spirit of a result of Wilton and Zalesskii [49], where it is proved that fundamental groups of the vertex manifolds in the JSJ decomposition are separable.

(5) Note that if a group is not cohopfian, it will contain a proper subgroup with isomorphic profinite completion. The point here is that the isomorphism is not induced by the canonical inclusion map. Now much is known about which 3-manifold groups are cohopfian (see [18] and [36] for example), and indeed it is shown in [18] that a non-trivial knot group is cohopfian if and only K is not a torus knot, a cable knot or a non-trivial connect sum. In this latter case, the knot group injects in itself as a subgroup of infinite index. The proof of Theorem 5.1 relied on the techniques of [18], and the hypothesis of Theorem 5.1 in part indicates the fact that these are the knot groups that are not cohopfian. In particular, this discussion also illustrates why using $\hat{u}$, with u the inclusion homomorphism is crucial in our discussions.

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