MOST HITCHIN REPRESENTATIONS ARE STRONGLY DENSE

D. D. LONG, A. W. REID, AND M. WOLFF

Abstract. We prove that generic Hitchin representations are strongly dense: every pair of non commuting elements in their image generate a Zariski-dense subgroup of $\mathrm{SL}_n(\mathbb{R})$. The proof uses a theorem of Rapinchuk, Benyash-Krivetz and Chernousov, to show that the set of Hitchin representations is Zariski-dense in the variety of representations of a surface group in $\mathrm{SL}_n(\mathbb{R})$.

1. Introduction

Following Breuillard, Green, Guralnick and Tao [2], we say that a subgroup $\Gamma \subset \mathrm{SL}_n(\mathbb{R})$ is *strongly dense* if any pair of non-commuting elements of Γ generate a Zariski-dense subgroup of $\mathrm{SL}_n(\mathbb{R})$. They proved that, among many other semisimple algebraic groups, the group $\mathrm{SL}_n(\mathbb{R})$ contains a strongly dense non abelian free subgroup [2, Theorem 4.5]. In this note, we extend the Breuillard, Green, Guralnick and Tao result to certain (discrete and) faithful representations of surface groups of genus at least two into $\mathrm{SL}_n(\mathbb{R})$.

To describe this more carefully, we introduce some background and terminology. For fixed $g \geqslant 2$, and base field k, the set of representations of the surface group $\pi_1(\Sigma_g)$ to $\mathrm{SL}_n(k)$ is denoted by $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(k))$ and is naturally an affine subvariety of k^{2gn^2} known as the *representation variety*. In the case of $k = \mathbb{R}$, those representations of interest to us, the Hitchin representations, are of particular geometric importance and can be defined as follows.

The *Teichmüller representations* in $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{R}))$ are those obtained by composing any faithful and discrete representation $\pi_1(\Sigma_g) \to \mathrm{SL}_n(\mathbb{R})$ with an irreducible representation $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_n(\mathbb{R})$. The Hitchin representations are those that lie in the same connected component (for the usual, Euclidean topology) of $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{R}))$ as a Teichmüller representation. Note that, depending on the parity of n, there may be more than one such component, but we simply choose one and denote it by HIT_n. [1]

We say that a representation is strongly dense if its image is a strongly dense subgroup of $\mathrm{SL}_n(\mathbb{R})$, and we say that a subset of $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{R}))$ is *generic* if its complement consists of a countable union of proper subvarieties. The main result of this note is:

Theorem 1.1. *Let $n \geqslant 3$. Then the set of strongly dense representations of $\pi_1\Sigma_g$ is generic in HIT_n.*

It is known that all the representations in HIT_n are faithful and discrete (see [8, Theorem 1.5]), so this provides the representations promised in the first paragraph. In fact, it is also known that generic Hitchin representations are Zariski-dense (see [7, 12]). We note that the result of Theorem 1.1 was obtained recently in [9] in the

[1] We note that a *Hitchin component* more usually refers to a connected component of the *character variety* $X(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{R})))$ and the notation Hit_n is frequently used, but in this note it will be technically simpler to work at the level of representations.

case of $n = 3$ by direct geometric methods.

To prove Theorem 1.1 we prove the following result, which seems independently interesting, and uses a result of Rapinchuk, Benyash-Krivetz and Chernousov [11], that $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{C}))$ is an irreducible subvariety of $\mathbb{C}^{2gn^2}$; in fact, it is connected for the Zariski topology and for the classical (Euclidean) topology.

Theorem 1.2. *For all $n \geqslant 2$, the set HIT_n is Zariski-dense in the affine algebraic set $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{C}))$.*

The case $n = 2$ was already essentially observed in [5, Chapter 3].

As we describe below, Theorem 1.1 follows from Theorem 1.2 together with [2] and the fact that surface groups are residually free [1]. The idea of combining the irreducibility of representation spaces with residual properties of surface groups was already used, for example in [3, 4].

Acknowledgements. The authors wish to thank Bill Goldman, Eran Iton, Fanny Kassel, Julien Marché, Andrés Sambarino, and Nicolas Tholozan for encouragement and helpful conversations. The second author gratefully acknowledges the financial support of the N.S.F. and the Max-Planck-Institut für Mathematik, Bonn, for its financial support and hospitality during the preparation of this work.

2. Proofs.

Proof of Theorem 1.2. As noted in §1, $R(\mathbb{C}) = \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{C}))$ is an affine subvariety of $\mathbb{C}^{2g \cdot n^2}$, and it was proved in [11, Theorem 3] to be irreducible of dimension $(2g-1)(n^2-1)$.

The set HIT_n is, by definition, a (topological) connected component of $R(\mathbb{R})$, which is a real algebraic variety, and hence HIT_n is open. We claim that it contains smooth points of $R(\mathbb{R})$, or equivalently, of $R(\mathbb{C})$: in fact, we will show that all its points are regular.

Indeed, by a result of Goldman [6, Propositon 1.2], at each point ρ of $R(\mathbb{R})$, the dimension of the Zariski tangent space at ρ equals $(2g-1)(n^2-1)+\dim(\zeta(\rho(\pi_1\Sigma_g)))$, where $\zeta(\rho(\pi_1\Sigma_g))$ is the centralizer of the image group $\rho(\pi_1\Sigma_g)$ in $\mathrm{SL}_n(\mathbb{R})$.

We will make use of the following facts proved by Labourie (see [8, Theorem 1.5 and Paragraph 10]). First, if $\rho \in \mathrm{HIT}_n$, then ρ is irreducible, and second, for all nonidentity elements $\gamma \in \pi_1(\Sigma_g)$, the matrix $\rho(\gamma)$ is diagonalizable with pairwise distinct real eigenvalues.

Fix such a γ_0; by conjugating the image of ρ in $\mathrm{SL}_n(\mathbb{R})$, we may suppose that $\rho(\gamma_0)$ is diagonal. Let ξ be an element of $\zeta(\rho(\pi_1(\Sigma_g)))$. Since ξ commutes with $\rho(\gamma_0)$, it is also diagonal, and if λ is an eigenvalue of ξ, the matrix $\xi - \lambda I$ also commutes with $\rho(\pi_1(\Sigma_g))$. Hence $\ker(\xi - \lambda I)$ is invariant by $\rho(\pi_1(\Sigma_g))$. However, ρ is irreducible, and so this implies that ξ is a scalar matrix, that is to say, $\xi = \pm I$.

Thus, the Zariski tangent space at any representation $\rho \in \mathrm{HIT}_n$ has minimal dimension, $(2g-1)(n^2-1)$, in other words, these are regular points of the varieties $R(\mathbb{R})$ and $R(\mathbb{C})$.

Now, the result follows from the following general fact from real algebraic geometry: suppose V is an irreducible complex affine variety defined by real polynomials, and suppose H is a connected component of $V(\mathbb{R})$ which has a smooth real point. Then H is Zariski-dense in V. This is a slight variation of the statement [10, Theorem 2.2.9] (with the same proof). $\square$

Proof of Theorem 1.1. For every pair of non commuting elements $a, b \in \pi_1(\Sigma_g)$, let $\mathrm{Bad}(a,b)$ denote the subset of $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{R}))$ consisting of representations ρ such that $\rho(a)$ and $\rho(b)$ do not generate a Zariski-dense subgroup of $\mathrm{SL}_n(\mathbb{R})$, and let $\mathrm{Good}(a,b)$ denote its complement.

The proof will be complete once we know that for every pair of non commuting elements $a, b \in \pi_1(\Sigma_g)$, the set $\mathrm{Bad}(a,b) \cap \mathrm{HIT}_n$ is Zariski-closed, and that it is a proper subset of HIT_n.

The fact that the sets $\mathrm{Bad}(a,b)$ are Zariski-closed follows from [2, Theorem 4.1].

Now let us check that $\mathrm{Bad}(a,b) \cap \mathrm{HIT}_n$ is a proper subset of HIT_n, or equivalently, that $\mathrm{Good}(a,b) \cap \mathrm{HIT}_n$ is nonempty. Since $\mathrm{Good}(a,b)$ is Zariski-open, and since HIT_n is Zariski-dense in $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{R}))$ by Theorem 1.2, it suffices to check that $\mathrm{Good}(a,b)$ is nonempty.

By [2, Theorem 4.5], there exists a strongly dense representation $\rho_0 \colon F_2 \to \mathrm{SL}_n(\mathbb{R})$. Let $a, b \in \pi_1(\Sigma_g)$ be a pair of non commuting elements. Since $\pi_1(\Sigma_g)$ is residually free (see Baumslag [1]) and $[a,b] \neq 1$, there exists a surjective morphism ψ from $\pi_1\Sigma_g$ onto a free group F, such that $\phi([a,b]) \neq 1$. By composing ψ with an injective morphism $F \to F_2$, this yields a morphism $\varphi \colon \pi_1(\Sigma_g) \to F_2$ such that $\varphi([a,b]) \neq 1$. Thus, $\varphi(a)$ and $\varphi(b)$ do not commute, hence $\rho_0(\varphi(a))$ and $\rho_0(\varphi(b))$ generate a Zariski dense subgroup of $\mathrm{SL}_n(\mathbb{R})$. In other words, $\rho_0 \circ \varphi$ lies in $\mathrm{Good}(a,b)$, so this set is non empty. □

References

[1] G. Baumslag. On generalised free products. *Math. Z.*, 78:423–438, 1962.

[2] E. Breuillard, B. Green, R. Guralnick, and T. Tao. Strongly dense free subgroups of semisimple algebraic groups. *Israel J. Math.*, 192(1):347–379, 2012.

[3] Emmanuel Breuillard, Tsachik Gelander, Juan Souto, and Peter Storm. Dense embeddings of surface groups. *Geom. Topol.*, 10:1373–1389, 2006.

[4] Jason Deblois and Richard P. Kent, IV. Surface groups are frequently faithful. *Duke Math. J.*, 131(2):351–362, 2006.

[5] W. M. Goldman. *Disontinuous groups and the Euler class.* PhD thesis, University of California, Berkeley, 1980.

[6] W. M. Goldman. The symplectic nature of fundamental groups of surfaces. *Adv. in Math.*, 54(2):200–225, 1984.

[7] O. Guichard. Zariski closure of positive and maximal representations, 2022. In preparation.

[8] F. Labourie. Anosov flows, surface groups and curves in projective space. *Invent. Math.*, 165(1):51–114, 2006.

[9] D. D. Long and A. W. Reid. Strongly dense representations of surface groups, 2021. Preprint.

[10] Frédéric Mangolte. *Real algebraic varieties.* Springer Monographs in Mathematics. Springer, Cham, [2020] ©2020. Translated from the 2017 French original [3727103] by Catriona Maclean.

[11] A. S. Rapinchuk, V. V. Benyash-Krivetz, and V. I. Chernousov. Representation varieties of the fundamental groups of compact orientable surfaces. *Israel J. Math.*, 93:29–71, 1996.

[12] Andrés Sambarino. Infinitesimal zariski closures of positive representations, 2020.

Department of Mathematics, University of California, Santa Barbara, CA 93106, USA.

Email address: long@math.ucsb.edu

Department of Mathematics, Rice University, Houston, TX 77005, USA.

Max-Planck-Insititut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany
Email address: alan.reid@rice.edu, areid@mpim-bonn.mpg.de

Sorbonne Universités, UPMC Univ. Paris 06, Institut de Mathématiques de Jussieu-Paris Rive Gauche, UMR 7586, CNRS, Univ. Paris Diderot, Sorbonne Paris Cité, 75005 Paris, France
Email address: maxime.wolff@imj-prg.fr