Zariski dense surface groups in $SL(2k+1, \mathbf{Z})$

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Abstract

[1] We show that for all k, $SL(2k+1, \mathbf{Z})$ contains surface groups which are Zariski dense in $SL(2k+1, \mathbf{R})$.

1 Introduction.

Let G be a semi-simple Lie group, and $\Gamma < G$ a lattice. Following Sarnak (see [15]), a subgroup Δ of Γ is called *thin* if Δ has infinite index in Γ and is Zariski dense. There has been an enormous amount of interest in the nature of thin subgroups of lattices, motivated in part by work on expanders, and in particular the so-called "affine sieve" of Bourgain, Gamburd and Sarnak [3].

Since it is quite standard to exhibit Zariski dense subgroups of lattices that are free products, the case of most interest is when the (finitely generated) thin group Δ does not decompose as a free product. Despite their importance and interest, non-free thin subgroups in higher rank are extremely difficult to exhibit since the Zariski dense condition makes any given subgroup hard to distinguish from a lattice and freely indecomposable isomorphism types in the higher rank situation are poorly understood. Our main theorem is the following:

Theorem 1.1. *For every $k \geq 1$, the group* $\mathrm{SL}(2\mathrm{k}+1, \mathbb{Z})$ *contains a faithful representation of a surface group which is Zariski dense in* $\mathrm{SL}(2\mathrm{k}+1, \mathbb{R})$.

To the authors' knowledge, this is the first result of this type concerning a freely indecomposable isomorphism type with Zariski closure $\mathrm{SL}(\mathrm{n}, \mathbb{R})$ for infinitely many n.

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In fact our argument shows that there are infinitely many non-conjugate such representations. We note that previous work of the authors (see [8] and [9]) using a totally different approach proved this to be true for $2k+1 = 3, 5$ (and also in fact for $k = 3/2$). However, that method seems to have no hope of generalizing for infinitely many n.

Here is an outline of our argument, with careful definitions deferred to the sections below. The starting point is a certain discrete faithful representation of the triangle group $\phi_n : \Delta(3, 4, 4) \to \mathrm{PSL(n, \mathbb{R})}$ obtained by composing a discrete faithful representation coming from the hyperbolic structure with the irreducible representation

$$\tau_n : \mathrm{PSL}(2, \mathbb{R}) \longrightarrow \mathrm{PSL(n, \mathbb{R})}$$

obtained by the action on homogeneous polynomials of degree $n = 2k + 1$ in two variables. Such representations lie on the so-called *Hitchin component* (see [6] and its generalization to orbifolds coming from [1]), with the key fact being that all representations on the Hitchin component are discrete and faithful. (See Theorem 1.1 of [1]).

Since n is odd, we can show (Theorem 2.1) that this representation can be conjugated to be integral. Of course, this representation cannot be Zariski dense since it lies inside an algebraic group isomorphic to $\mathrm{PSL}(2, \mathbb{R})$. Indeed, standard theory shows that it has image inside $\mathrm{SO}(J, \mathbb{R})$, where J is a certain quadratic form of signature $(k + 1, k)$.

It is this representation we seek to ameliorate using the well known bending construction, however the triangle group does not lend itself to that, so we pass to a subgroup of index four which is the fundamental group of the orbifold $S^2(3, 3, 3, 3)$. The bending construction is described in detail in §3, but briefly: One takes an integral element $\delta \in \mathrm{PSL(n, \mathbb{R})}$ which centralizes the image of an essential simple closed curve γ; this curve splits the surface into two subsurfaces L and R (in the initial case, each is a disc with two orbifold points) and defines a new representation by conjugating ϕ_n by δ on the R surface. This new representation is obviously still inside $\mathrm{SL(2k + 1, \mathbb{Z})}$ and it continues to be faithful since we take care to arrange that it lies on the Hitchin component.

If the bent group is not Zariski dense, we conclude that it lies inside $\mathrm{SO}(J)$, and we appeal to a result of Guichard (see Theorem 3.1 & [5]) to prove that it must have Zariski closure all of $\mathrm{SO}(J)$.

The strategy now is to perform a suitable second bend. This is much more subtle, for example one needs at least to be sure that one can find an element for which all the integral centralising elements do not lie in $\mathrm{SO}(J)$. However, we can use the extra information that the Zariski closure of the bent group is all of $\mathrm{SO}(J)$ to appeal to the Weisfeiler-Nori version of the Strong Approximation theorem (see the discussion of §3.2). From this follows that for all but finitely many primes p, if one reduces the bent group modulo p it surjects $\Omega(J, p)$, the commutator subgroup of $\mathrm{SO}(J; \mathbb{Z}/p)$.

(It is classical that $[\mathrm{SO}(J;\mathbb{Z}/p) : \Omega(J,p)] = 2$ for odd p.) Since we can show that group $\Omega(J,p)$ contains elements whose characteristic polynomials are of the form $(Q-1)f(Q)$ where $f(Q)$ is irreducible modulo p, it follows that the original group contains an element η mapping onto such an element, in particular its characteristic polynomial has the form $(Q-1)F(Q)$ where $F(Q)$ is $\mathbb{Z}$ irreducible.

Of course the element η may not be a simple loop, but we show in §4.1 one can ascend a carefully constructed tower of regular coverings so that η lifts to each step of this tower and ultimately becomes a (power of an) essential simple loop on some orbifold surface $S^2(3,3,.....,3)$ which must still surject $\Omega(J,p)$.

One can then show using rank considerations that the characteristic polynomial condition implies that there is an integral element in the centralizer of η inside $\mathrm{SL}(2\mathrm{k}+1,\mathbb{Z})$ which does not power into $\mathrm{SO}(J)$ and from this, it is not hard to see (appealing again to the classification result of Guichard) that bending the group using that element makes the resulting orbifold group Zariski dense. The resulting representation is faithful since it continues to be on the Hitchin component.

We note that the methods of this paper can be used to show that in any dimension, if the Hitchin component contains an integral representation, then it contains an integral Zariski dense representation. This is because although the finite group situation for n even is slightly more involved, there is no real difficulty extending the observations of 3.2 to the symplectic case. However, finding such an integral representation in even dimensions seems to pose significant difficulties.

For the rest of this article, we restrict to the case $n \geq 9$. This is not for any truly essential reason as the argument present here works if $n > 3$. However, as mentioned above, the cases $n = 3, 5$ are already in the literature and while the case $n = 5$ can be dealt with by the method described here, the case $n = 7$ involves a technical detour which is hardly worth the number of words it would take. This case is resolved explicitly in [10].

2 Integrality of certain representations of the 344–triangle group.

This section analyses certain representations of the triangle group

$$\Delta(3,4,4) = \langle a\,,\, b\,,\, c \;\mid\; a^3 = b^4 = c^4 = abc = 1\rangle$$

which is the group of orientation-preserving symmetries of the tiling of the hyperbolic plane by triangles with angles $\{\frac{\pi}{3}\,,\,\frac{\pi}{4}\,,\,\frac{\pi}{4}\}$.

Let $\phi_2 : \Delta(3,4,4) \to \mathrm{PSL}(2,\mathbb{R})$ be the faithful representation of $\Delta(3,4,4)$ into $\mathrm{Isom}^+(\mathbb{H}^2)$. Hyperbolic triangle groups are rigid and so ϕ_2 is uniquely determined up to conjugacy.

Let $\tau_n : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL(n,\mathbb{R})}$ $(n \geq 3)$ denote the irreducible representation obtained from the standard action on homogeneous polynomials in two variables. We denote the composite representation $\tau_n \circ \phi_2 : \Delta(3,4,4) \to \mathrm{PSL(n,\mathbb{R})}$ by ϕ_n.

Remark. For odd n (the situation which primarily concerns us here) the representation ϕ_n lifts to a representation of $\Delta(3,4,4)$ into $\mathrm{SL(n,\mathbb{R})}$, but for even n we have to be content with the situation we already see for $n = 2$, namely that we have a representation of a certain pullback group U_{344} into $\mathrm{SL(n,\mathbb{R})}$, the pullback being that determined by the representation ϕ_2 together with the natural projection $\mathrm{SL(n,\mathbb{R})} \to \mathrm{PSL(n,\mathbb{R})}$.

For ease of notation we denote elements of $\mathrm{PSL(n,\mathbb{R})}$ by representative matrices in $\mathrm{SL(n,\mathbb{R})}$; also, given a subring A of $\mathbb{C}$ we say that a representation into $\mathrm{PSL(n,\mathbb{R})}$ *can be written over* A if the corresponding lifted representation (of $\Delta(3,4,4)$ or U_{344}) can be so written.

Since $\Delta(3,4,4)$ is the fundamental group of a compact orbifold, specifically S^2 with cone points of orders $3,3,4$, it follows that $\phi_2(\Delta(3,4,4))$ contains no parabolic. Therefore all matrices in $\phi_2(\Delta(3,4,4))$ are diagonalizable, and application of τ_n to a diagonal matrix shows that if $A \in \phi_2(\Delta(3,4,4))$ has eigenvalues $\lambda\,,\ \frac{1}{\lambda}$, then the eigenvalues of $\tau_n(A)$ are

$$\lambda^{n-1}\,,\ \lambda^{n-3}\,,\ \ldots\,,\ \lambda^{-(n-3)}\,,\ \lambda^{-(n-1)}$$

In this section, we prove the following.

Theorem 2.1. *For odd n, the representation ϕ_n be written over $\mathbb{Z}$.*
For even n, the representation ϕ_n can be written over $\mathbb{Z}[\sqrt{2}\,]$ but not over $\mathbb{Z}$,

We choose the following faithful representation ϕ_2 of $\Delta(3,4,4)$ into the group $\mathrm{PSL}(2,\mathbb{R})$ of orientation-preserving isometries of $\mathbb{H}^2$, obtained by placing the isosceles $(\frac{\pi}{3}\,,\ \frac{\pi}{4}\,,\ \frac{\pi}{4})$–triangle symmetrically about the y–axis in the upper-half plane and then putting the matrix for the generator a in rational canonical form:

$$\phi_2(a) = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \quad , \quad \phi_2(b) = \begin{bmatrix} 0 & -1-\sqrt{2} \\ -1+\sqrt{2} & \sqrt{2} \end{bmatrix} \quad , \quad \phi_2(c) = \begin{bmatrix} 1-\sqrt{2} & -\sqrt{2} \\ -1+\sqrt{2} & -1 \end{bmatrix}$$

Thus the representation ϕ_2 of $\Delta(3,4,4)$ can be written over the ring $\mathbb{Z}[\sqrt{2}\,]$, and since application of τ_n to a 2×2 matrix A produces an $n \times n$ matrix whose entries are integer polynomial expressions of those of A, we see that ϕ_n $(n \geq 3)$ can also be written over $\mathbb{Z}[\sqrt{2}\,]$. We shall show that for odd n, ϕ_n can actually be written over $\mathbb{Z}$ and for even n this is not possible.

Our next basic ingredient is the fact that the representation ϕ_3 can be written over the integers. Here is an example of an integral representation ϕ_3' of $\Delta(3,4,4)$, conjugate to ϕ_3:

$$\phi_3'(a) = \begin{bmatrix} 1 & 1 & -2 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix} \qquad \phi_3'(b) = \begin{bmatrix} 1 & 0 & -1 \\ 4 & 1 & -1 \\ 2 & 0 & -1 \end{bmatrix} \qquad \phi_3'(c) = \begin{bmatrix} 1 & 1 & 0 \\ -2 & -1 & 0 \\ -4 & -1 & 1 \end{bmatrix}$$

Lemma 2.2. *The representation ϕ_n of $\Delta(3,4,4)$ has integral character if and only if n is odd. For even n the character of ϕ_n takes values in $\mathbb{Z}[\sqrt{2}\,]$.*

Proof. Let A be any matrix in $\phi_2(\Delta(3,4,4))$, and let the eigenvalues of A be $\lambda\,,\,\frac{1}{\lambda}$. Since ϕ_3 can be written over the integers, we see that $\lambda^2+1+\lambda^{-2}\in\mathbb{Z}$, hence also $\lambda^2+\lambda^{-2}\in\mathbb{Z}$. For odd n, $\lambda^{n-1}+\lambda^{-(n-1)} = f(\lambda^2+\lambda^{-2})$ for a polynomial $f\in\mathbb{Z}[x]$, so we deduce inductively that the trace of $\tau_n(A)$ is an integer. On the other hand, if n is even, then

$$\lambda^{n-1}+\lambda^{n-3}+\cdots+\lambda^{-(n-3)}+\lambda^{-(n-1)} \;=\; (\lambda+\lambda^{-1})(\lambda^{n-2}+\lambda^{n-6}+\cdots+\lambda^{-(n-6)}+\lambda^{-(n-2)}) \;\;.$$

A similar argument shows that the second factor is an integer, whereas the first factor might not be, *e.g.* the trace of $\phi_2(a^{-1}b)$ is $2\sqrt{2}$. We deduce that for even n, the trace of $\phi_n(a^{-1}b)$ is not an integer; however, as ϕ_n can be written over $\mathbb{Z}[\sqrt{2}\,]$, its character takes values in that ring. □

Lemma 2.3. *For odd n the representation ϕ_n of $\Delta(3,4,4)$ can be written over the rational numbers.*

Proof. We have established that for odd n, ϕ_n has integral character and can be written over $\mathbb{Z}[\sqrt{2}\,]$. Thus ϕ_n is realizable over a field of degree 2 over $\mathbb{Q}$. Suppose that ϕ_n is not realizable over a field of smaller degree over $\mathbb{Q}$; then the Schur index of the irreducible representation ϕ_n is $[\mathbb{Q}(\sqrt{2})\,:\,\mathbb{Q}]=2$. However the Schur index divides the degree n of the representation [2], contradicting the fact that n is odd. □

The conclusion of Proposition 2.1 has already been established for even n; for odd n it follows directly from Lemmas 2.1, 2.3 together with the proof of Proposition 2.1 of [8]. □

3 The bending construction.

Our construction is reliant upon an orbifold which has somewhat more geometric flexibility than the triangle group $\Delta(3,4,4)$. To this end we note that there is a homomorphism $\Delta(3,4,4) \longrightarrow \mathbf{Z}/4$ which sends the two elements of order 4 to 1 and -1. This defines an orbifold covering $S^2(3,3,3,3) \longrightarrow \Delta(3,4,4)$. We may restrict the representation ϕ_n to the subgroup defined by the covering to yield a discrete and faithful representation corresponding to a hyperbolic structure

$$\phi_n : \pi_1(S^2(3,3,3,3)) \longrightarrow \mathrm{PSL(n,\mathbb{Z})}$$

The orbifold $S^2(3,3,3,3)$ has an obvious flexibility (often called *bending* in the setting of $\mathrm{SO}(n,1)$ representations) coming from the following construction: Let $d_1.d_2 = \gamma$ be the simple closed curve on the two sphere which separates two of the orbifold points from the other two; denote the two sides by L and R. Each contains two cone points of order three. Let δ be any element of $\mathrm{PSL(n,\mathbb{R})}$ which centralizes $\phi_n(\gamma)$. Then we may form a bent representation of $S^2(3,3,3,3)$ by matching $\phi_n(\pi_1(L))$ with $\delta.\phi_n(\pi_1(R)).\delta^{-1}$; these representations agree on $\phi_n(\gamma)$ by the choice of δ. We denote this bent representation by ρ^δ (even though this is a bit of an abuse).

With a view to questions concerning faithfulness, we will invariably use bending elements which are in the image of the exponential map (for example they will be diagonalizable and with positive real eigenvalues) that if, for example, $\delta = exp(\mathbf{v})$, then all the elements $exp(t\mathbf{v})$ centralize $\phi_n(\gamma)$. Then there is a path of bendings from ρ to ρ^δ given by $\rho^{exp(t\mathbf{v})}$. It follows that if ρ lies on the Hitchin component, then so does ρ^δ and in particular, this implies that the latter is a faithful representation.

An important ingredient of our approach which gives the requisite control hangs upon the following theorem of Guichard:

Theorem 3.1. *(Guichard, [5], see also [1].) Suppose that* $\rho : \pi_1(S) \longrightarrow \mathrm{SL(m,\mathbb{R})}$ *is a representation on the Hitchin component and* G *is the Zariski closure of* $\rho(\pi_1(S))$ *then*

- *If* $m = 2n$ *is even then* G *is conjugate to one of* $\tau_m(\mathrm{SL(2,\mathbb{R})})$, $\mathrm{Sp}(2m,\mathbf{R})$ *or* $\mathrm{SL(m,\mathbb{R})}$.
- *If* $m = 2n+1 \neq 7$ *is odd, then* G *is conjugate to one of* $\tau_m(\mathrm{SL(2,\mathbb{R})})$, $\mathrm{SO}(n+1,n)$ *or* $\mathrm{SL(m,\mathbb{R})}$.
- *If* $m = 7$*, then* G *is conjugate to one of* $\tau_m(\mathrm{SL(2,\mathbb{R})})$, $\mathrm{SO}(4,3)$, G_2 *or* $\mathrm{SL(7,\mathbb{R})}$.

The relevance of this theorem is that if one can show that a given representation of a surface group leaves no form invariant, then the image is Zariski dense[2].

[2]Note that $G_2 < \mathrm{SO}(4,3)$

3.1 The first bend.

The first step is to bend $\phi_n : \pi_1(S^2(3,3,3,3)) \longrightarrow \mathrm{PSL(n,\mathbb{Z})}$. There is a good deal of flexibility in this part of our construction.

To this end, fix any simple closed curve γ which separates two of the orbifold points from the other two and we claim (recall that we are assuming that $n \geq 5$) that there is an element δ in the $\mathrm{PSL(n,\mathbb{Z})}$-centralizer of γ which does not lie in the image $\phi_n(\mathrm{PSL}(2,\mathbb{R}))$. We argue this as follows. As in Lemma 2.2, all the hyperbolic elements in the image $\phi_n(\pi_1(S^2(3,3,3,3))$ have integral character and with eigenvalues 1 and $(n-1)/2$ pairs of the form μ^{2j}, μ^{-2j}. Since $\phi_n(A)$ is integral, it follows that can be conjugated *over the rationals* into the block form consisting of a single 1 and $(n-1)/2$ block matrices $exp(j\mathbf{w}) = \begin{pmatrix} 0 & -1 \\ 1 & K \end{pmatrix}^j$ for $1 \leq j \leq (n-1)/2$. This latter matrix has $\mathbb{Z}$-centralizer isomorphic to $(\pm 1) \oplus \mathbb{Z}^{(n-1)/2}$.

The following is well known:

Proposition 3.2. *Let $M \in M_n(\mathbb{Q})$ such that $det(M) = \pm 1$ and the characteristic polynomial of M has integer coefficients. Then some power of M is integral.*

Since the conjugation matrix is rational, it follows from this proposition that the centralizer of $\phi_n(A)$ in $\mathrm{PSL(n,\mathbb{Z})}$ is $\mathbb{Z}^{(n-1)/2}$. Since $n \geq 5$ and the centralizer in $\phi_n(\mathrm{PSL}(2,\mathbb{R}))$ is $\mathbb{Z}$ our claim follows. In fact, with a view to our argument it is important to note a little more is true, namely that this argument shows that we may choose these centralising elements to be in the image of the exponential map, since we may choose hyperbolic elements with distinct positive eigenvalues.

Accordingly, we may fix some element δ lying in in the $\mathrm{PSL(n,\mathbb{Z})}$-centralizer of γ which does not lie in the image $\phi_n(\mathrm{PSL}(2,\mathbb{R}))$, in the interests of being specific, we choose δ as the relevant conjugate of a power of $1 \oplus Id_2 \oplus Id_2 .... \oplus exp(\mathbf{w})$. Notice that if we write $\delta = exp(\mathbf{v})$, then the entire path $exp(t\mathbf{v})$ centralises γ.

Theorem 3.3. *The bent representation ϕ_n^δ (which henceforth we denote by ρ) is a representation of $\pi_1(S^2(3,3,3,3))$ into $\mathrm{PSL(n,\mathbb{Z})}$ lying on the Hitchin component. In particular, ρ is discrete and faithful.*

Moreover, ρ is Zariski dense in $\mathrm{SO}(J)$ for the form J of signature $(k+1,k)$ left invariant by $\phi_n(\mathrm{PSL(n,\mathbb{R})})$

Proof. If $\delta = exp(\mathbf{v})$, then ρ is the endpoint of the path of representations $\phi_n^{exp(t\mathbf{v})}$ which has one endpoint on the Hitchin component. Since this is a path-component of $Hom(\pi_1(S^2(3,3,3,3)), \mathrm{PSL}(2,\mathbb{R}))$, it follows that ρ is on the Hitchin component and is therefore discrete (which was obvious anyway) and faithful.

The second claim is our first application of Theorem 3.1. The argument is the following. Notice (in the notation of §3) that $\pi_1(L)$ (and of course $\pi_1(R)$) is Zariski dense in $\mathrm{PSL}(2, \mathbb{R})$ since this Lie group has no interesting algebraic subgroups. It follows that the algebraic group $\phi_n(\mathrm{PSL}(2, \mathbb{R}))$ is determined by the image $\phi_n(\pi_1(L))$. Referring to the list of Theorem 3.1, we see that the image of the bent representation ρ must be larger than $\phi_n(\mathrm{PSL}(2, \mathbb{R}))$ unless δ normalises $\phi_n(\pi(R))$ and hence $\phi_n(\mathrm{PSL}(2, \mathbb{R}))$.

We claim that this is impossible. For if conjugacy by δ preserved $\phi_n(\mathrm{PSL}(2, \mathbb{R}))$ it would act as an automorphism which commuted with the action by conjugacy of the hyperbolic element γ. However it is well known that $Aut(\mathrm{PSL}(2, \mathbb{R}))) \cong \mathrm{GL}(2, \mathbb{R})$ and we deduce that there would be some nontrivial word $\delta^a \gamma^b$ which centralized the absolutely irreducible representation $\phi_n(\mathrm{PSL}(2, \mathbb{R}))$ and would therefore be trivial, which is impossible by the choice of δ.

Thus we have proved that the Zariski closure of $\rho(\pi_1(S^2(3,3,3,3))$ is strictly larger than $\phi_n(\mathrm{PSL}(2, \mathbb{R}))$.

One now consults the list provided by Theorem 3.1 and we see that (since in particular $n > 7$) either we are done or the Zariski closure of $\rho(S^2(3,3,3,3))$ is SO of some form of signature $(k+1, k)$. We claim that this form is necessarily $\mathrm{SO}(J)$, where J is the form mentioned in the introduction. The reason is that on $\pi_1(L)$, the representation ϕ_n and ρ agree and are absolutely irreducible. We now appeal to

Lemma 3.4. *Suppose that* $\rho : G \longrightarrow \mathrm{SO}(J) \leq \mathrm{SL}(\mathrm{n}, \mathbb{R})$ *is an absolutely irreducible representation.*

Then J *is unique up to a real scaling.*

Proof. If J_1 and J_2 are two such forms then the equations $A^T.J_i.A = J_i$ for $i = 1, 2$ and any A in the image of ρ imply that $J_2^{-1}.J_1$ centralises the image of ρ, whence by Schur's lemma is a scalar matrix. □

Returning to the proof of Theorem 3.3, since the group $\phi_n(\pi_1(L)) < \mathrm{SO}(J)$, the Lemma implies that the Zariski closure of $\rho(S^2(3,3,3,3))$ must be $\mathrm{SO}(J)$ as required. □

Remark. In fact, it's useful to note that if ρ is a Hitchin representation, one can weaken the hypothesis in Lemma 3.4 to ask only that ρ be *real* irreducible. The point is that for a Hitchin representation, the infinite order elements have the property that they have distinct real eigenvalues. Fix such an element and suppose that we have diagonalised it over the reals. Then in the proof above $J_2^{-1}.J_1$ is real and commutes with a diagonal matrix with distinct real eigenvalues and is therefore diagonal with real eigenvalues. The usual Schur argument using real irreducibility now implies that $J_2^{-1}.J_1 = r.Id$ for some real r.

Remark. Questions concerning *integral* centralizers can be quite delicate because of the rational conjugacy; the need to appeal to Proposition 3.2 results in the loss of any real control.

3.2 The second bend: Finding η.

To perform the second bend requires a more careful choice of bending curve and this necessitates a discussion of some notation and results from the theory of finite simple groups, in particular from the theory of (special) finite orthogonal groups. This has two steps. We will appeal to a theorem that follows from work of Weisfeiler [18] or Nori [12] to construct an essential curve η on $S^2(3,3,3,3)$ for which we can use purely algebraic considerations to show that $\rho(\eta)$ has a large centralizer in $\mathrm{PSL}(\mathrm{n},\mathbb{Z})$. We cannot use η directly to bend, since it may not be simple on $S^2(3,3,3,3)$, however in §4.1 we show how to improve this situation.

Here is a summary of the algebraic facts that we require. Let J be an m-dimensional quadratic form over the finite field $GF(p^n)$ of cardinality $q = p^n$. It simplifies the discussion (and this is no loss of generality for us) to assume p is odd. We are interested only in the case that m is also odd; we assume this in the following without further comment. (The situation is slightly more complicated for m even.)

In this case, there is a unique orthogonal group up to isomorphism $\mathrm{O}(J,q) = \mathrm{O}(2k+1,q)$ which is independent of J. (see [17] p 377 Theorem 5.8). Let $\mathrm{SO}(m,q)$ denote the special orthogonal group and set $\Omega(m,q) = [\mathrm{O}(m,q),\mathrm{O}(m,q)]$ where $[G,G]$ denotes the commutator subgroup of a group G. We summarize the important fact for us in the following theorem (see [17] pp 383 - 384 for a discussion):

Theorem 3.5. *When m is odd, $\Omega(J,q)$ is a simple subgroup of $\mathrm{O}(J,q)$ of index 4.*

Recall the first bending provided a representation $\rho = \phi_n^\delta : \pi_1(S^2(3,3,3,3)) \longrightarrow \mathrm{SL}(\mathrm{n},\mathbb{Z})$ lying on the Hitchin component and whose image is Zariski dense in $\mathrm{SO}(J,\mathbb{R})$. Given a rational prime p, we may compose with the obvious reduction map modulo p, $\mathrm{SL}(\mathrm{n},\mathbb{Z}) \longrightarrow \mathrm{SL}(\mathrm{n},\mathbb{Z}/\mathrm{p})$. The following comes from the Strong Approximation Theorem (cf. [12] and [18]):

Theorem 3.6. *In the notation above, when m is odd, for all but a finite number of rational primes p, we have $\Omega(J,p) = \pi_p(\rho(S^2(3,3,3,3))$.*

Proof. It follows from Strong Approximation (cf. [12] and [18]) that except for finitely many primes we have

$$\Omega(J,p) \leq \pi_p(\rho(S^2(3,3,3,3)) \leq \mathrm{SO}(J;p)$$

However, Theorem 3.5, together with the fact that $\pi_1(S^2(3,3,3,3))$ has no subgroup of index 2 implies the result. □

To apply this, we also require the following fact:

Theorem 3.7. *There is an element $\eta \in S^2(3,3,3,3)$ with the property that the integer matrix $\rho(\eta)$ has characteristic polynomial of the form $(Q-1)F(Q)$ where $F(Q)$ is irreducible over $\mathbb{Z}$.*

The main ingredient in the proof is the following:

Proposition 3.8. *For every prime p, there is a matrix in $\Omega(J,p)$ with the property that its characteristic polynomial has the form $(Q-1)f(Q)$, where $f(Q)$ is irreducible modulo p.*

This Proposition now implies Theorem 3.7. For, by Strong Approximation, we may choose a prime p so that $\pi_p\rho$ maps $S^2(3,3,3,3)$ onto $\Omega(J,p)$. Pick an element $A(p) \in \Omega(J,p)$ whose characteristic polynomial has the form $(Q-1)f(Q)$, where $f(Q)$ is irreducible modulo p, and choose η so that $\pi_p\rho(\eta) = A(p)$. Then the characteristic polynomial of $\rho(\eta)$ has the form $(Q-1)F(Q)$, and $F(Q)$ is necessarily $\mathbb{Z}$ irreducible, since it is irreducible modulo p.

Proof of Theorem 3.7. We have a fixed $n = 2k+1$ and a prime p. We claim that there is an element τ in the algebraic closure of $\mathbb{Z}/p$ which has degree $n-1 = 2k$ over $\mathbb{Z}/p$ with the property that its minimal polynomial over $\mathbb{Z}/p$, which we denote by $a(Q)$, is symmetric. Further, τ^2 also has degree $n-1$ with (therefore irreducible) symmetric minimal polynomial which we denote $a_2(Q)$.

Deferring this claim temporarily, we proceed as follows. Let g be any $\mathbb{Z}/p$ matrix whose characteristic polynomial is $(Q-1)a(Q)$ (for example use the rational canonical form). Let K be a splitting field for $a(Q)$ over $\mathbb{Z}/p$, so that there is a K-matrix m for which $m.g.m^{-1}$ is diagonal, with the eigenvalues arranged $1, \lambda_1, \lambda_1^{-1}, ..., \lambda_n, \lambda_n^{-1}$. This matrix is an isometry of the form Σ whose $(1,1)$ entry is any element of K and then has $(n-1)/2$ blocks which are K multiples of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This gives an $(n+1)/2$ dimensional family of solutions and using the obvious ordered basis, the determinant of such a form Σ is $\pm a_0 \cdot a_1 \cdot ..... \cdot a_{(n-1)/2}$ is therefore nondegenerate as long as no a_j is zero. Therefore the original matrix g has a K-family of nondegenerate solutions, namely $m^T.\Sigma.m$. Now, when we regard the entries of a symmetric matrix σ as indeterminates, the question of whether a form σ satisfies $g^T.\sigma.g = \sigma$ is a family of homogeneous linear equations with $\mathbb{Z}/p$ coefficients and we have just shown that there are nondegenerate solutions over K. It follows that there are nondegenerate $\mathbb{Z}/p$ solutions and such solution σ gives $g \in \mathrm{SO}(\sigma,p)$ with characteristic polynomial $(Q-1)a(Q)$. As noted above, since n is odd there is only one such orthogonal group up to change of basis, so this characteristic polynomial also occurs in $\mathrm{SO}(J,p)$. After squaring if necessary, one finds the promised element in $\Omega(J,p)$.

It remains to find the polynomials promised in the first paragraph; we sketch an argument. Denote the finite field of degree r over $\mathbb{Z}/p$ by $GF(p^r)$, it's known there is exactly one such field up to isomorphism for every r. Take an element $x \in GF(p^k)$ with the property that the polynomial $T^2 - x \cdot T + 1$ is irreducible over $GF(p^k)$. Let τ be a root of this polynomial so that $GF(p^k)(\tau) = GF(p^{2k})$. By construction, the polynomial for τ over $\mathbb{Z}/p$ has degree $2k = n - 1$ and is symmetric & irreducible. Moreover, considering the equation $\tau^2 - x \cdot \tau + 1 = 0$, we see that τ^2 cannot lie in $GF(p^k)$. Therefore the polynomial for τ^2 over $\mathbb{Z}/p$ also has the required properties. □

4 Improving η.

At this stage we have a representation $\rho : S^2(3,3,3,3) \longrightarrow \mathrm{PSL}(2k+1, \mathbb{Z})$ so that with finitely many exceptions, the reduction modulo p yields a surjection

$$\pi_p : \rho(S^2(3,3,3,3)) \longrightarrow \Omega(J, p)$$

This was used to find an element η with the property that the characteristic polynomial of $\rho(\eta)$ is of the form $(Q-1)F(Q)$, where it follows from the construction that $F(Q)$ is irreducible over $\mathbb{Z}$. This section is devoted to a proof of the following:

Theorem 4.1. *Denote the orbifold surface which is a S^2 with k-cone points all of order 3 by $F(k)$. Then given η as above, there is a tower of 3-fold regular coverings*

$$F(4) \longleftarrow F(u_1) \longleftarrow F(u_2) \longleftarrow .... \longleftarrow F(u_k).$$

with the property that η lifts to each covering and in the covering $F(u_k)$, η is (a power of) a simple loop which encloses at least two cone points on each side.

Proof. The construction here is based upon the following simple observation. Fix some surface $F(k)$ and fix two of the cone points, c_1 and c_2, enclose them with a simple closed curve C. Then C splits the surface into two pieces, one of which is a disc with two cone points of order 3. There is a 3-fold covering of $F(k)$ given by the homomorphism $c_1 \to 1$, $c_2 \to -1$ and all other cone points mapping to zero. It's easy to see that the resulting covering is planar: The disc with two cone points becomes a S^2 with three discs removed, each of which corresponds to a lift of C. The other side of C lifts to three copies each a trivial covering which is attached to one of these C-lifts. Notice that the number of cone points in the covering strictly increases since it has $3(k-2)$ cone points and this is strictly larger than k for $k \geq 4$.

The coverings being used are all abelian, and we indicate homology class by $[*]$. We begin by observing that it is easy to make coverings where η lifts: For $F(4)$, for example, if $[\eta]$ is not zero, it represents a generator of $H_1(F(4)) \cong (\mathbb{Z}/3)^3$, we can choose two other cone points x and y so that $H_1(F(4)) =< \eta, x, y >$ and we form a covering as above with $c_1 = x$ and $c_2 = y$. Clearly η lifts to this covering. If $[\eta]$ is zero in $H_1(F(4))$, we may choose any pair of the cone points. Since the number of cone points only goes up, this procedure can be iterated. (Increasing the number of supplementary generators if need be.)

The proof of Theorem 4.1 is accomplished by showing one can find a tower of coverings where η lifts and has a strictly decreasing number of self-intersections up to the point that we obtain the conclusion of the theorem. For future reference we note that the shape of the characteristic polynomial implies that η has infinite order in the fundamental group of the orbifold so that it cannot encircle just one cone point on either side. In particular, it is a hyperbolic element on the hyperbolic orbifolds in question, so we may make η geodesic.

We construct a planar subsurface X of $F(k)$ as follows. Take a thin regular neighbourhood of η and attach discs to all the boundary components which bound discs in the complement of η. By construction the boundary of the subsurface X consists of simple closed curves all of which are essential and therefore bound discs which have on them at least one cone point.

Suppose that X has at least three boundary components. Then at least two of these boundary components $\partial_1 X$ and $\partial_2 X$, say, contain cone points c_1 and c_2 (respectively) with the property that $< c_1, c_2, \eta > \cong (\mathbb{Z}/3)^3 < H_1(F(k))$. Define a covering of $F(k)$ as above, mapping $c_1 \to 1$, $c_2 \to -1$, $\eta \to 0$ and extend to $H_1(F(k))$. This arranges $\partial_1 X \to 1$, $\partial_2 X \to -1$ and $[\eta] \to 0$. The preimage of X in this covering is connected so that the three preimages $\tilde{\eta}_i$ $1 \leq i \leq 3$ of η form a connected graph. A point where $\tilde{\eta}_1$ meets $\tilde{\eta}_2$ corresponds to a self-intersection of η in $F(k)$ which has now disappeared from $\tilde{\eta}_1$. Thus the number of self-intersections of $\tilde{\eta}_1$ is strictly less than the number of self-intersections of η.

We can repeat this process as long as the planar neighbourhood X is not an annulus. However in this case, η must be a power of a simple loop, namely the core of the annulus. $\square$

Remark. Notice that in the latter case, the annulus cannot contain just one cone point on either side, since in this case the loop η would have order dividing 3.

5 Proof of Theorem 1.1.

We may now complete the argument of Theorem 1.1.

At the top of the tower provided by Theorem 4.1, either we have a simple loop corresponding to η on a planar surface for which the characteristic polynomial of $\rho(\eta)$ is $(Q-1)F(Q)$ where $F(Q)$ is $\mathbb{Z}$-irreducible, or we have that $\eta = (\eta')^r$ where η' is a simple loop. In this latter case, the characteristic polynomial of $\rho(\eta')$ is also of the form $(Q-1)G(Q)$ where $G(Q)$ is $\mathbb{Z}$-irreducible, since the $r-th$ power of the roots of $G(Q)$ give the roots of $F(Q)$ and these all have maximal degree over the rationals.

We economize on notation by replacing η' by η and $G(Q)$ by $F(Q)$.

Notice that since ρ was on the Hitchin component for $S^2(3,3,3,3)$, $F(Q)$ is totally real. (See [6] Theorem 1.5.) The simple curve η cannot encircle just one cone point, else it would have order 3 so η splits the surface into two pieces each of which must have at least two cone points in it. By applying the construction of 4.1 if necessary, we may assume that we have constructed a planar orbifold surface (denote it Σ) in which η is simple and each of the two sides of η contains a large number of cone points.

We have already observed that the initial representation ρ lies on the Hitchin component for $S^2(3,3,3,3)$ and it follows that ρ restricted to $\pi_1(\Sigma)$ lies on the Hitchin component for Σ (for example, one can use the bending used to construct ρ restricted to the corresponding subgroup of finite index). Moreover, each side of η is a hyperbolic orbifold with totally geodesic boundary, so that the restriction of the given representation to either side gives an element of the Hitchin component of that side in the sense of Theorem 2.28 of [1]. In particular, each side is represented irreducibly into $\mathrm{SO}(J;\mathbb{R}) < \mathrm{SL}(\mathrm{n},\mathbb{R})$.

The main claim now is that there is a path of elements $\delta_t = exp(t\mathbf{v})$ all centralising η and with $\delta = \delta_1$ in the centralizer of η in $\mathrm{SL}(\mathrm{n},\mathbb{Z})$. Moreover, the element δ does not preserve J (even up to scaling).

Once this is established, Theorem 1.1 is proved with mild variations of the arguments we have already used in the first bending: By choice, we have a 1-parameter family of bendings $\rho^{exp(t\mathbf{v})}$ connecting ρ^δ to ρ. The latter representation lies on the Hitchin component and therefore ρ^δ lies in the Hitchin component. It is therefore faithful.

Moreover, after the δ-bending, one side of the orbifold surface Σ represents into $\mathrm{SO}(J;\mathbb{R})$ and the other in $\mathrm{SO}(\delta^T J\delta;\mathbb{R})$; by absolute irreducibility and the property of δ we claimed above, it follows that there is no form left invariant by the whole orbifold surface group.

The proof of Theorem 1.1 is now completed by the following theorem applied to $\xi = \rho^\delta$.

Theorem 5.1. *Suppose that ξ is any representation on the Hitchin component of a hyperbolic orbifold group Γ which leaves no form invariant.*

Then ξ restricted to any surface subgroup of finite index in Γ is Zariski dense in $\mathrm{SL}(\mathrm{n},\mathbb{R})$.

Proof. It is shown in [6] Theorem 1.5, that for any representation on the Hitchin component, the nonidentity elements are *loxodromic*, that is to say that their eigenvalues are distinct real numbers and moreover, since n is odd these eigenvalues are all positive. In particular, it follows from Theorem 2 of [4] that all the infinite order elements of $\xi(\Gamma)$ are in a unique one parameter subgroup of the exponential map $exp : \mathfrak{sl}(n,\mathbf{R}) \to \mathrm{SL}(\mathrm{n},\mathbb{R})$.

Suppose then that some surface subgroup H of finite index in Γ, lies inside a proper algebraic subgroup of $\mathrm{SL(n,\mathbb{R})}$; by Guichard's result Theorem 3.1, it must be contained in $SO(J)$ for some form J of signature $(k+1,k)$. Take any loxodromic element $exp(\mathbf{v}) = \gamma \in \xi(\Gamma)$ and choose r so that $exp(r\mathbf{v}) = \gamma^r \in H$. The condition that $\gamma^r \in SO(J)$ is equivalent to $J\mathbf{v}J^{-1} = -\mathbf{v}^{tr}$ so the entire one parameter subgroup $exp(t\mathbf{v})$ lies in $SO(J)$ and in particular therefore, $\gamma \in SO(J)$. However, it is clear that Γ is generated by its loxodromic elements and we would deduce that $\Gamma < SO(J)$, a contradiction. It follows that H must have Zariski closure $\mathrm{SL(n,\mathbb{R})}$. □

In particular, since any subgroup of finite index in $\xi(\Gamma)$ contains a surface group, it shows that the Zariski closure of any subgroup of finite index (in particular, index $= 1$) is all of $\mathrm{SL(n,\mathbb{R})}$.

5.1 The existence of δ.

Recall that the characteristic polynomial of the integer matrix $\rho(\eta)$ is $(Q-1)F(Q)$ where where $F(Q)$ is symmetric $\mathbb{Z}$-irreducible and with (distinct) real roots, since ρ is on the Hitchin component. One can see (for example by diagonalising the element $\rho(\eta)$ and considering the possible forms it could leave invariant) that the centralizer of $\rho(\eta)$ in $\mathrm{SO}(J;\mathbb{R})$ has rank $(n-1)/2$. On the other hand, the totally real number field K defined by a root of $f(Q) = 0$ has degree $n-1$ so that the unit group of its ring of integers has rank $n-2$ which is $> (n-1)/2$ for $n \geq 5$.

Make a rational change of basis so that $M^{-1}\rho(\eta).M = (1) \oplus A$, where A is an integer matrix in rational canonical form. The ring $\mathbb{Z}[A]$ is a matrix representation of the ring of integers $\mathcal{O}_K$ which therefore contains a multiplicative subring of units of rank $n-2$, i.e. matrices which have determinant ± 1. Since all elements of the form $M.((1) \oplus \Sigma r_j A^j).M^{-1}$ clearly commute with $\rho(\eta)$, it follows from the rank considerations described above that we may find a rational matrix with determinant $= 1$ and integer characteristic polynomial in the $\mathrm{SL(n,\mathbb{R})}$-centralizer of $\rho(\eta)$ which does not power into in $\mathrm{SO}(J;\mathbb{R})$. By Lemma 3.2 there is some power of this matrix which is integral, this is a choice for δ with the required properties. As observed above, δ commutes with an element which has distinct positive real eigenvalues, so that it is diagonalizable and by squaring if need be, we arrange that δ has positive eigenvalues. Therefore, it is in the image of the exponential map, so that $exp(\mathbf{v}) = \delta$. From this it follows that the entire path $exp(t\mathbf{v})$ centralizes $\rho(\eta)$, so that the bent representation lies on the Hitchin component.

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