Fundamental groups, geometry and some papers of Scott

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Abstract

Throughout the history of 3-manifolds, the fundamental group has played a central role. There is a list of reasons for that, and exactly what that role is has evolved over time, but it has always been a player. The papers under consideration here[1], all written by G. Peter Scott (1944 – 2023) in a period 1972 – 1978, highlight and reflect the beginnings of a transitional period for the subject viewed through the prism of fundamental groups.

1 Subgroup separability.

We begin by considering what is historically the last paper of the three, [12] *Subgroups of surface groups are almost geometric* which appeared in 1978. It was hugely influential both in terms of the result and the proof, and reflected the burgeoning new emphasis on geometric methods pioneered around this time by Thurston.

We begin by recalling a few definitions; most of these can be couched purely in terms of group theory or purely geometrically. The fact that this interplay exists is one of the powerful features of this direction.

A group G is said to be *residually finite*, if given any nontrivial $g \in G$, there is a subgroup $G_F \leq G$, where the index $[G : G_F] < \infty$ and with the property that $g \notin G_F$. It's an easy exercise that this is equivalent to the property that given any nontrivial g, there homomorphism $\theta : G \longrightarrow F$ with the property that F is a finite group and $\theta(g)$ is nontrivial in F.

This can be recast geometrically as follows. Suppose X is a connected space with pleasant local properties (X a manifold is more than enough) and $\pi_1(X, p) = G$ for some choice of basepoint p. Denoting the universal covering of X by $\widetilde{X}$, it is a basic fact that G acts freely and it follows p and $g \cdot p$ are distinct points in $\widetilde{X}$. Now with these hypotheses, every covering of X has the form $\widetilde{X}/H$ for some $H \leq G$ so the

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statement that G is residually finite now translates to the fact that there should be some finite sheeted covering X_F of X in which p and $g \cdot p$ continue to be distinct. Guided by this observation, one can usefully go further in this direction: The two point set is the smallest compact subset of $\widetilde{X}$ which might or might not embed in some other covering and it is a somewhat less easy exercise (it uses the stronger fact that G acts properly discontinuously on its universal covering) that G is residually finite if and only if the following condition holds: For every compact $C \subset \widetilde{X}$, there is a finite sheeted covering $\widetilde{X}/G_F$ so that the obvious composition $C \subset \widetilde{X} \longrightarrow \widetilde{X}/G_F$ embeds C.

A generalization was proposed by Hall [6], replacing the trivial group by any finitely generated subgroup of G: Precisely, one says that a subgroup $H \leq G$ is *separable* if given any element $g \notin H$, there is a subgroup $G_F \leq G$ where the index $[G : G_F] < \infty$ with the properties that $H \leq G_F \leq G$ and $g \notin G_F$. The geometric incarnation is: Given a compact subset $C \subset \widetilde{X}/H$, there is a subgroup G_F for which $H \leq G_F \leq G$, $[G : G_F] < \infty$, and so that the obvious composition $C \subset \widetilde{X}/H \longrightarrow \widetilde{X}/G_F$ embeds C. One says that G is *subgroup separable* if all the finitely generated subgroups of G are separable in this sense.

Residual finiteness is a relatively soft condition to impose upon a group and there are fairly general conditions that guarantee it, for example it is a classical theorem of Mal'cev that a finitely generated linear group is residually finite. In contrast to this, subgroup separability is extremely delicate – nothing of the broad generality of Mal'cev's theorem is known or seems likely. Hall proved that finitely generated free groups have this property, but, for example, even as apparently innocuous a group as $Free(2) \times Free(2)$ does not. In this setting, the result of [12] came as something of a surprise: The fundamental group of a closed hyperbolic 2-manifold is subgroup separable. Even more interesting, and what makes this paper something of a watershed moment is the method, which uses the geometry of the hyperbolic plane in an essential way. This was a reflection of the times: Thurston and his geometric viewpoint were emerging as a huge influence on the subject.

It's possible to sketch some of the ideas of how one proves what appears to be a purely algebraic result from the geometry of the hyperbolic plane $\mathbf{H}^2$. We reference [4] for some of the basic facts about the hyperbolic plane upon which we draw.

We begin by observing that it is not difficult to see that if $A \leq B$ and B is subgroup separable, then so is A, and if A is subgroup separable and $[B : A] < \infty$, then so is B. This means that showing some carefully chosen group is subgroup separable will be sufficient to prove the result for all hyperbolic surface groups simultaneously. Our carefully chosen group here is geometrically constructed: It's not difficult to show that $\mathbf{H}^2$ contains a regular pentagon all of whose interior angles are $\pi/2$. Reflections in the sides of this pentagonal tile generate a discrete group of isometries Γ and a tiling of $\mathbf{H}^2$ by regular right-angled pentagons. Very general considerations show that Γ contains a torsion free subgroup of finite index which is therefore a closed surface group of genus at least two. It follows that to prove the result for all closed hyperbolic surfaces, it suffices to show that Γ is subgroup separable.

Suppose then, that we've fixed a finitely generated subgroup $H < \Gamma$; without much loss of generality H is torsion-free. It's now useful to recall a rather general construction in hyperbolic geometry, namely the notion of *convex hull* of a group of hyperbolic isometries. Each of the nontrivial elements of H have associated to them a hyperbolic axis along which the element is a hyperbolic translation. The convex hull of H is the smallest hyperbolically convex set $\mathcal{H}(H)$ which contains all these axes. This can be usefully thought of as the intersection of all the hyperbolic half-spaces which contain all the axes of H. By construction this is invariant for the action of H, so $\mathcal{H}(H)/H$ is a canonical convex subset of $\mathbf{H}^2/H$ which contains the axes of all the elements of H. (We note in passing that in higher dimensions this set can be somewhat exotic, but in dimension two it is rather well behaved.) Now, with a view to showing we can satisfy the geometric version of subgroup separability, we suppose that we are given a compact subset C of $\mathbf{H}^2/H$. Now geometric considerations show that by taking a sufficiently large R, we can arrange that the R-neighbourhood of $\mathcal{H}(H)/H$ is a compact convex subset of $\mathbf{H}^2/H$ which contains not only all the axes of H but also the compact subset C. Denoting this set by $\mathcal{H}_R(H)/H$, we can lift it to a connected convex H-invariant subset $\mathcal{H}_R(H)$ of $\mathbf{H}^2$.

Up this this point, the ambient group Γ has played no role. However, it enters now in the following fashion. Associated to Γ is a countable collection of hyperbolic half-spaces, namely the half-spaces cut out by the reflection axes of the order two elements of Γ. We can use only these axes to make a tiling hull $\mathcal{T}$ of the convex set $\mathcal{H}_R(H)$, i.e. take the intersection of all these Γ defined half-spaces which contain $\mathcal{H}_R(H)$. The convex set $\mathcal{T}$ is visibly H invariant and is obviously a union of pentagons, however there is a danger that it is far too big. (The reader might want to contemplate the following example: Tile the Euclidean plane by unit squares in the obvious way, and consider the tiling hull for the line $x = y$ using these squares.) However an argument using the geometry of the hyperbolic plane shows that very distant pentagons cannot be involved in $\mathcal{T}$, so that $\mathcal{H}_R(H)/H \subset \mathcal{T}/H$ is a compact convex union of tiles, and reflections in its sides give the required subgroup of finite index.

While the methods of [12] were not adequate to address them, the result incentivized questions concerning subgroup separability in the context of hyperbolic 3-manifolds, which were a magnet for much research in the area in the years that followed. Briefly, the history is this: The work of Waldhausen [15] showed that one could answer (almost) any question about a closed irreducible[2] 3-manifold M if it was *sufficiently large*, that is to say, there is a π_1-injective embedding of a closed orientable surface of genus at least one; Waldhausen called such manifolds *Haken.* For a while, an optimism prevailed that if M was irreducible and $\pi_1(M)$ was infinite

[2]i.e. every embedded 2-sphere bounds a 3-ball

then it was Haken, a hope that was dashed with the publication of Thurston's notes. There it was shown that with finitely many exceptions, surgeries on the figure eight knot complement admitted a hyperbolic structure, which in turn implies they are irreducible and have infinite fundamental group. However, relatively simple topological considerations imply that these manifolds do not possess closed embedded incompressible surfaces.

As understanding evolved, however, it became clear that although it was too much to ask for an embedding, it was in many cases (in particular, in the hyperbolic case) possible to hope that manifolds always contained a π_1-injective *immersion* of a closed orientable surface of genus at least one. The question then arose: If this is true, how could that information be used? This story has several threads, but most relevant for this article is the line of argument that asked if, given a π_1-injective immersion of a surface, could it be lifted to an embedding in some finite sheeted covering of M? Such manifolds are said to be *virtually Haken* and enough methods of a purely topological nature were available that this class was regarded as largely understood. Subgroup separability in its geometric incarnation then becomes relevant, since given a π_1-injective immersion $i : F \longrightarrow M$, the surface group $i_*\pi_1(F)$ defines an infinite sheeted covering of M and classical 3-manifold arguments show that the Scott-Shalen core (see the subsequent section of this article) of this covering must be topologically $F \times I$, so that it contains an embedding of the closed surface F. Taking this as the compact subset C in the geometric description above, one sees that if one knew that $\pi_1(M)$ were subgroup separable, then there would be a finite sheeted covering M_F in which F embeds and therefore M_F would be Haken, i.e. M would be virtually Haken.

Thus there was a huge incentive to prove that the fundamental group of any closed hyperbolic 3-manifold was subgroup separable, or at least that their surface subgroups were separable. This generated a vast amount of mathematics, too much to delve into here (for a wide-ranging overview of this and contiguous topics, we refer the untiring reader to [2]), but we can sketch some of the initial reductions. These involve three-dimensional hyperbolic geometry in an essential way.

For example, we can make closed 3-manifolds as the mapping torus $M(\theta)$ of a homeomorphism $\theta : F \longrightarrow F$ of a closed orientable surface F. Such manifolds fibre over the circle $F \longrightarrow M(\theta) \longrightarrow S^1$ and the long exact sequence of a fibration shows that such surfaces are incompressible. Early work of Thurston specifies exactly when they are hyperbolic in terms of properties of θ. Notice that such a surface group has an interesting element in its normalizer and it follows from the paper of Bonahon alluded to below that if one could identify an immersion of a surface which contained such a normalizing element inside the closed hyperbolic manifold X, there is a finite sheeted covering of X which fibres over the circle and so the manifold is in particular virtually Haken.

This leads one to reduce to the case that any surface groups one can locate do not have such normalizing elements; locating such surface groups proved to be enormously challenging, but there was a breakthrough made by Kahn & Markovic [8] that showed they were plentiful. A good deal more work remained to be done but the final chapter of the story was provided by Agol [1]; with his work, the surfaces provided by Kahn-Markovic could be embedded in a finite sheeted covering so that hyperbolic manifolds are virtually Haken.

2 Cores & finite generation.

In this section we turn our attention to the consideration of the two papers *Finitely generated 3-manifold groups are finitely presented*, [10] and *Compact submanifolds of 3-manifolds*, [11]; it makes both good mathematical and historical sense to consider these as somewhat intertwined. To explain why this should be so, we first discuss the result of [10]. It does exactly what it says on the tin: Suppose G is a finitely generated group which in addition is the fundamental group of a 3-manifold, then in fact G is finitely presented. For historical context, I note that I am assured by topologists from the era that this result was "in the air" in the sense that it was inspired by, and built upon, other results that were available at the time, in particular due to Jaco [7] and Swarup [14]. It's also historically important to note that while [10] has become the standard reference, this theorem was simultaneously proved by P. B. Shalen, as indeed [10] is at pains to point out.

Of course finitely generated, non-finitely presentable 4-manifold groups are well known, and indeed, there are finitely presented groups with finitely generated, non--finitely presentable subgroups [13], so this result highlights the fact that the fundamental groups of 3-manifolds have special properties, a theme which has continued to develop and over time has become central.

A very reasonable question to ask at this stage is why in an area that focusses largely upon compact manifolds one should be led to thinking about such a theorem. Here is a very natural way the question arises: Suppose that M is a closed (i.e. compact with empty boundary) connected 3-manifold; it's not difficult to see from the van Kampen theorem that with these hypotheses, $\pi_1(M)$ is finitely presented. However, elementary covering space theory shows that given any subgroup H of $\pi_1(M)$, there is a covering space $p : \widetilde{M} \longrightarrow M$ with the property that $H = p_*(\pi_1(\widetilde{M}))$, so that in particular, H is the fundamental group of a 3-manifold. Moreover, when the index $[\pi_1(M) : p_*(\pi_1(\widetilde{M}))]$ is infinite, the manifold $\widetilde{M}$ is noncompact. In the case of most interest here, namely when H is finitely generated, we have exhibited a noncompact 3-manifold with finitely generated fundamental group. Absent any theorem, that is all one would know, but the magic here is that it follows from the Scott-Shalen theorem that H is in fact finitely presented.

In this way then, we are led naturally to the considerations addressed in [11]:

One is given M, a 3-manifold whose fundamental group is finitely generated and one asks the question: Is there a *compact* submanifold N of M so that the inclusion map $i : N \longrightarrow M$ induces an isomorphism $i_* : \pi_1(N) \longrightarrow \pi_1(M)$. Such a submanifold is often called a *core* (perhaps even a Scott-Shalen core) for M and as we have already observed, $\pi_1(N)$ must be finitely presented, so the isomorphism shows that the fundamental group of the covering space M is finitely presented. The result of [11] (also proved independently by Shalen) is that such cores always exist.

The proofs of both these theorems involve an interplay of group theory and classical topological methods which are not easily summarized, although it is worth saying that as in so much 3-manifold theory of the time, a crucial role is played by the loop theorem of Papakyriakopoulos [9].

As evidenced, for example, by the number of citations, both of these theorems have been enormously influential and while they are largely topological in spirit, this influence has carried on into the geometric era of topology. To briefly describe one such geometric example which makes essential use of the core, we mention a wonderful paper of Bonahon [3], which solves or partially solves some important results in the theory of hyperbolic 3-manifolds. It is too technical to say anything with real content about this paper, but here is the spirit of it and how the core theorem is relevant.

In his celebrated Princeton notes, Thurston considered geometric properties of the ends of a noncompact hyperbolic manifold N and in particular introduced the notion of what it means for an end of N to be *geometrically tame*. Rather than formally define this, it is somewhat easier to speaking roughly about *topologically tame* manifolds: a manifold is topologically tame if it is homeomorphic to the interior of a compact manifold with a closed subset of the boundary removed. Marden had conjectured in the seventies that every hyperbolic 3-manifold M with finitely generated fundamental group was homeomorphic to the interior of a compact manifold and so was tame in this sense. Thurston had shown in his notes that a certain conjecture about geometric tameness implied that Marden's conjecture was true, at least in the case that $\pi_1(M)$ was indecomposable as a free product. A key case to understand is when the fundamental group of the noncompact hyperbolic manifold N is that of a closed surface, and Thurston conjectured that in this case the ends are geometrically tame, in particular that topologically the noncompact manifold is $F \times \mathbf{R}$ for a compact surface F.

Of course, the hypothesis of finite generation is where the core theorem is relevant, so that one has a compact core M_C, for which the inclusion $M_C \longrightarrow M$ is a homotopy equivalence. The key is that the algebraic condition that $\pi_1(M)$ is freely indecomposable has topological implications for the way any core sits inside M. For example, it implies that if S is a component of ∂M_C, then the inclusion of S into the relevant component of $\overline{M \setminus M_C}$ is a homotopy equivalence; indeed the inclusion of $\pi_1(S)$ is an injection into $\pi_1(M)$. In this way one sees that one can reduce Marden's theorem in the case that the fundamental group is freely indecomposable to the case of the noncompact hyperbolic manifold with fundamental group a closed surface group. This topological picture is the starting point for Bonahon's proof and his intricate geometric arguments take over, but at the outset an absolutely essential role is played by the core theorem.

We mention that subsequent work of Gabai-Calegari [5] (and independently Agol) proved that all hyperbolic 3-manifolds with finitely generated fundamental group are topologically tame, but while those arguments came from a totally different direction, the impetus provided by Bonahon's result which in turn rested upon the Scott-Shalen core theorem cannot be understated.

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