Geometriae Dedicata
https://doi.org/10.1007/s10711-018-0376-8

ORIGINAL PAPER

The dimension of the Hitchin component for triangle groups

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Received: 21 July 2017 / Accepted: 20 July 2018

Abstract
Let p, q, r be positive integers satisfying $1/p + 1/q + 1/r < 1$, and let $\Delta(p, q, r)$ be a geodesic triangle in the hyperbolic plane with angles π/p, π/q, π/r. Then there exists a tiling of the hyperbolic plane by triangles congruent to $\Delta(p, q, r)$, and we define the *triangle group* $T(p, q, r)$ to be the group of orientation preserving isometries of this tiling. Representation varieties of closed surface groups into $\mathrm{SL}(n, \mathbb{R})$ have been studied extensively by Hitchin and Labourie, and the dimension of a certain distinguished component of the variety was obtained by Hitchin using Higgs bundles. Here we determine the corresponding dimension for representations of triangle groups into $\mathrm{SL}(n, \mathbb{R})$, generalising some earlier work of Choi and Goldman in the case $n = 3$.

Keywords Discrete subgroups of Lie groups · Fuchsian groups and their generalizations · Representation varieties

Mathematics Subject Classification 22E40 · 20H10

1 Introduction

It is well known that the angle sum of a triangle is greater than π in the round 2-sphere, equal to π in the Euclidean plane and less than π in the hyperbolic plane. Let p, q, r be integers greater than or equal to 2, and let $\Delta(p, q, r)$ be a triangle in the appropriate geometry with angles π/p, π/q, π/r at vertices P, Q, R respectively. We assume that the ordering (P, Q, R) of vertices is consistent with a chosen orientation of the ambient space. The *triangle group* associated with $\Delta(p, q, r)$ is the group of isometries $T(p, q, r)$ generated by rotations r_P, r_Q, r_R about P, Q, R through angles $+2\pi/p$, $+2\pi/q$, $+2\pi/r$ respectively. It is easily checked that the orbit of $\Delta(p, q, r)$ under the action of $T(p, q, r)$ constitutes a tiling of the homogeneous space (S^2, $\mathbb{R}^2$ or $\mathbb{H}^2$) by triangles congruent to $\Delta(p, q, r)$, and that the product $r_P \circ r_Q \circ r_R$ (composition being as usual from right to left) is the identity. The quotient under

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Published online: 24 July 2018

the action of $T(p,q,r)$ is a 2-sphere with three cone points of orders p, q, r, whose (orbifold) fundamental group is easily shown to be $\langle a\ ,\ b\ ,\ c \mid a^p = b^q = c^r = abc = 1\rangle$, the generators corresponding to the rotations $r_P\ ,\ r_Q\ ,\ r_R$. From elementary covering space theory in the orbifold setting it follows that the group $T(p,q,r)$ also admits this presentation, i.e.

$$T(p,q,r) \ = \ \langle a\ ,\ b\ ,\ c \mid a^p = b^q = c^r = abc = 1\rangle.$$

For triangles $\Delta(p,q,r)$ in the 2-sphere, the group $T(p,q,r)$ is finite; for each of the cases $(p,q,r) = (2,2,n)$ it is a dihedral group, and for each of the remaining cases $(p,q,r) = (2,3,3)\ ,\ (2,3,4)\ ,\ (2,3,5)$ it is the group of rotational symmetries of a platonic solid. The three Euclidean triangle groups $T(2,3,6)\ ,\ T(2,4,4)\ ,\ T(3,3,3)$ all generate familiar tilings of the Euclidean plane. In this paper we consider exclusively *hyperbolic* triangle groups, namely those for which $1/p + 1/q + 1/r < 1$.

Let us fix a triangle $\Delta(p,q,r)$ in the hyperbolic plane. The holonomy representation ϕ embeds the resulting triangle group $T(p,q,r)$ as a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^2) \approx \mathrm{PSL}(2,\mathbb{R})$. Since a triangle in the hyperbolic plane is determined up to congruence by its angles, ϕ is *rigid*, in the sense that its only deformations are those afforded by conjugation. However, if we compose ϕ with the (unique) irreducible representation ρ_n of $\mathrm{PSL}(2,\mathbb{R})$ into $\mathrm{PSL}(\mathrm{n},\mathbb{R})$, for all sufficiently large n we obtain a representation of degree n of $T(p,q,r)$ that admits essential deformations, i.e. deformations that do not arise from post-composition with an inner automorphism of $\mathrm{PSL}(\mathrm{n},\mathbb{R})$. The purpose of this article is to establish a general formula for the dimension of the deformation space of such a representation.

The interest in representation varieties of hyperbolic triangle groups lies partly in the fact that these groups contain surface groups as subgroups of finite index, allowing one to draw on deep work of Hitchin [2] and Labourie [3]. Borrowing terminology of [3], we call the component of the representation variety containing $\rho_n \circ \phi$ the *Hitchin component*. The Hitchin component is homeomorphic to $\mathbb{R}^d$ for some d, and all representations therein are discrete and faithful [3]. In the case where there are no essential deformations, we have $d = n^2 - 1$.

The statement of the main theorem involves a certain arithmetic function of two variables $\sigma(n,k)$ $(n,k \geq 2)$. Let $Q\ ,\ R$ be the quotient and remainder on dividing n by k, i.e.

$$n \ = \ Qk + R \qquad (0 \leq R \leq k-1).$$

Then

$$\sigma(n,k) \ = \ (n+R)Q + R.$$

For example, for $n = 7\ ,\ k = 4$, we have $Q = 1\ ,\ R = 3$; thus $\sigma(7,4) = 13$.

It will be seen later that the function $\sigma(n,k)$ is related to the dimension of the centralizer in $\mathrm{SL}(\mathrm{n},\mathbb{R})$ of a certain diagonalizable matrix whose eigenvalues are roots of unity.

Theorem 1.1 *Let $\mathcal{H}$ be the Hitchin component for the representation $\rho_n \circ \phi$ of the triangle group $T(p,q,r)$. Then*

$$\dim \mathcal{H} \ = \ (2n^2+1) - (\sigma(n,p) + \sigma(n,q) + \sigma(n,r)).$$

Remark (i) For $k = n$, we have $Q = 1$ and $R = 0$, and for $k > n$, we have $Q = 0$ and $R = n$. It follows that for $k \geq n$, $\sigma(n,k) = n$; therefore if none of p, q, r is less than n, $\dim \mathcal{H} = 2n^2 + 1 - 3n = (2n-1)(n-1)$.

(ii) If one wishes to "factor out" conjugation and consider the moduli space $\mathcal{H}^*$ of essential deformations, one simply subtracts $\dim \mathrm{PSL}(\mathrm{n},\mathbb{R}) = n^2 - 1$ from the expression of Theorem 1.1, obtaining

$$\dim \mathcal{H}^* \ = \ (n^2+2) - (\sigma(n,p) + \sigma(n,q) + \sigma(n,r)).$$

From the definition of $\sigma(n,k)$ a quick computation shows that $\sigma(n,k) \equiv n \pmod 2$, from which we see that $\dim \mathcal{H}^*$ is always even.
(iii) The case $n = 3$ is covered in S. Choi's and W.M. Goldman's paper [1]. The special case of their result that pertains to hyperbolic triangle groups can be summarized as follows: if one of p, q, r is 2, $T(p,q,r)$ is rigid, and otherwise $\dim \mathcal{H}^* = 2$. From Theorem 1.1 one finds that triangle groups of type $(2, 3, r)$ $(r \geq 7)$ cease to be rigid at $n = 6$, and those of type $(2, q, r)$ $(4 \leq q\ ,\ 4 < r)$ cease to be rigid at $n = 4$ (see [4], §4 for examples).
(iv) Using Higgs bundle techniques, Hitchin [2] proved that for a compact surface S, the Hitchin component of the representation variety of $\pi_1(S)$ into PSL(n, $\mathbb{R}$) has dimension $\chi(S)(1 - n^2)$.

In outline, the proof of Theorem 1.1 proceeds as follows: in §3 the deformation spaces of the finite cyclic groups $\langle a \rangle$, $\langle b \rangle$, $\langle c \rangle$ are studied, and then in §4 the role of the relation $abc = 1$ is determined by means of a transversality argument in conjunction with Schur's Lemma.

2 From PSL(n, $\mathbb{R}$) to SL(n, $\mathbb{R}$)

Our primary aim is to examine deformations of $\rho_n \circ \phi : T(p,q,r) \to \mathrm{PSL}(n, \mathbb{R})$, where $\phi : T(p,q,r) \to \mathrm{PSL}(2, \mathbb{R})$ is the holonomy representation and $\rho_n : \mathrm{PSL}(2, \mathbb{R}) \to \mathrm{PSL}(n, \mathbb{R})$ is the irreducible representation, namely the projective version of the familiar representation $\sigma_n : \mathrm{SL}(2, \mathbb{R}) \to \mathrm{SL}(n, \mathbb{R})$ induced by the action of SL(2, $\mathbb{R}$) on two-variable homogeneous polynomials of degree $n - 1$.

Naturally, in discussions we prefer to work with matrix representatives of elements of projective linear groups, being mindful that each matrix is deemed equivalent to its negative. Therefore we would like to translate the task of proving Theorem 1.1 to a problem involving matrix groups. This involves a certain technical issue for the case where n is even and at least one of p, q, r is even.

Let $U(p,q,r)$ be the pullback of the pair consisting of the natural projection $\varpi_2 : \mathrm{SL}(2, \mathbb{R}) \to \mathrm{PSL}(2, \mathbb{R})$ and the injection $\phi : T(p,q,r) \to \mathrm{PSL}(2, \mathbb{R})$. Thus $U(p,q,r)$ comes with an injection $\psi : U(p,q,r) \to \mathrm{SL}(2, \mathbb{R})$ and a two-to-one epimorphism $\varpi_0 : U(p,q,r) \to T(p,q,r)$, as part of the following commutative diagram:

$$\begin{array}{ccccc} U(p,q,r) & \xrightarrow{\quad\psi\quad} & \mathrm{SL}(2,\mathbb{R}) & \xrightarrow{\quad\sigma_n\quad} & \mathrm{SL}(n,\mathbb{R}) \\ \downarrow{\scriptstyle \varpi_0} & & \downarrow{\scriptstyle \varpi_2} & & \downarrow{\scriptstyle \varpi_n} \\ T(p,q,r) & \xrightarrow{\quad\phi\quad} & \mathrm{PSL}(2,\mathbb{R}) & \xrightarrow{\quad\rho_n\quad} & \mathrm{PSL}(n,\mathbb{R}) \end{array}$$

We must address the mildly inconvenient fact that if at least one of p, q, r is even, ϕ does not admit a lift to SL(2, $\mathbb{R}$). To see this, note that a rotation through $2\pi/k$ about a point of $\mathbb{H}^2$ is represented by a matrix conjugate to

$$\tau_k = \begin{bmatrix} \cos\frac{\pi}{k} & -\sin\frac{\pi}{k} \\ \sin\frac{\pi}{k} & \cos\frac{\pi}{k} \end{bmatrix},$$

of order $2k$ rather than k; moreover, if k is even, this order doubling feature cannot be corrected by negating the matrix τ_k. The issue disappears in $\mathrm{SL(n, \mathbb{R})}$ for odd n, as then $\mathrm{SL(n, \mathbb{R})} = \mathrm{PSL(n, \mathbb{R})}$; however the issue does persist in $\mathrm{SL(n, \mathbb{R})}$ with n even.

Under the circumstances, a reasonable option is simply to work with the group $U(p, q, r)$. This group has generators α , β , γ with respective orders $2p, 2q, 2r$; their images under ψ are conjugates of τ_p , τ_q , τ_r, and their images under $\varpi_2 \circ \psi$ are $\phi(a)$, $\phi(b)$, $\phi(c)$, respectively. The central element $z = \alpha^p = \beta^q = \gamma^r$ is mapped by ψ to $-I$, and $\alpha\beta\gamma \in \{1\ , \ z\}$. We may identify $T(p, q, r)$ with the quotient group $U(p, q, r)/\langle z\rangle$. The homomorphism $\sigma_n \circ \psi$ is an absolutely irreducible representation of $U(p, q, r)$ into $\mathrm{SL(n, \mathbb{R})}$.

Although it is not needed for the sequel, the apparent ambiguity regarding the product $\alpha\beta\gamma$ can be resolved as follows.

Proposition 2.1 *Let α , β , γ , $z \in U(p, q, r)$ be as above. If none of p, q, r is 2, then $\alpha\beta\gamma = z$, and if $p = 2$, then α can be chosen so that $\alpha\beta\gamma = z$. It follows that for appropriately chosen generators α , β , γ, $U(p, q, r)$ admits the presentation*

$$U(p, q, r) \ = \ \langle \alpha\ , \ \beta\ , \ \gamma \mid \alpha^p = \beta^q = \gamma^r = \alpha\beta\gamma = z\ , \ z^2 = 1\rangle.$$

Proof The reason why the case $p = 2$ is considered separately is that it is the only case where τ_p is conjugate to its negative. If $p > 2$, both eigenvalues of τ_p have strictly positive real part, and the condition that α is conjugate to τ_p excludes ambiguity in the choice between α and its negative.

If Δ is an arbitrary triangle in the hyperbolic plane, or indeed in the Euclidean plane, with angles θ_1 , θ_2 , θ_3, then it is an easy geometric fact that the product of rotations through $2\theta_1$, $2\theta_2$, $2\theta_3$ about the respective vertices, taken in the correct order, is the identity. The reason is simply that a rotation through $2\theta_i$ about a vertex with angle θ_i is the product of reflections in the two sides incident to the vertex. Therefore to prove the Proposition it is sufficient to check the conclusion for a single case, the general case following by continuity.

Here are matrices for α , β , γ in the case $p = q = r = 4$, obtained from a specific triangle in the hyperbolic plane; it is easily checked that each of α , β , γ is conjugate to τ_4 and that $\alpha\beta\gamma = -I$.

$$\alpha = \frac{1}{2}\begin{bmatrix} \sqrt{2} & 1 \\ -2 & \sqrt{2} \end{bmatrix}$$

$$\beta = \frac{1}{2}\begin{bmatrix} \sqrt{2} & 1+\sqrt{2}-\sqrt{2(1+\sqrt{2})} \\ -2\left(1+\sqrt{2}+\sqrt{2(1+\sqrt{2})}\right) & \sqrt{2} \end{bmatrix}$$

$$\gamma = \frac{1}{2}\begin{bmatrix} \sqrt{2}\left(1-\sqrt{1+\sqrt{2}}\right) & 1+\sqrt{2}-\sqrt{1+\sqrt{2}} \\ -2\left(1+\sqrt{2}+\sqrt{1+\sqrt{2}}\right) & \sqrt{2}\left(1+\sqrt{1+\sqrt{2}}\right) \end{bmatrix}$$

□

Let us define $\psi_n = \sigma_n \circ \psi$, $\phi_n = \rho_n \circ \phi$, i.e. ψ_n , ϕ_n are the representations given by the top and bottom rows of the above commutative diagram.

Proposition 2.2 *Let $\mathcal{H}$, $\mathcal{K}$ be the deformation spaces of ϕ_n , ψ_n respectively. Then $\mathcal{H}$, $\mathcal{K}$ are diffeomorphic.*

Proof We proceed to define a smooth map $F : \mathcal{H} \to \mathcal{K}$ and a smooth inverse to F, $G : \mathcal{K} \to \mathcal{H}$.

Let $\phi'_n \in \mathcal{H}$, and let Φ_t $(0 \le t \le 1)$ be a path in $\mathcal{H}$ from ϕ_n to ϕ'_n. For each $g \in T(p, q, r)$, $\Phi_t(g)$ can be regarded as a pair of paths $\{m_t(g)\,,\ -m_t(g)\}$ in SL(n, $\mathbb{R}$), this pair being well defined by continuity. We define a path Ψ_t in $\mathcal{K}$ by

$$\Psi_t(g) = \begin{cases} m_t(g) & \text{if } \psi_n(g) = m_0(g) \\ -m_t(g) & \text{if } \psi_n(g) = -m_0(g) \end{cases}$$

and define $F(\phi'_n)$ by $F(\phi'_n)(g) = \Psi_1(g)$.

Conversely, let $\psi'_n \in \mathcal{K}$, and let Ψ_t $(0 \le t \le 1)$ be a path in $\mathcal{K}$ from ψ_n to ψ'_n. Since $\psi_n(z) = \pm I_n$ is central in SL(n, $\mathbb{R}$), $\Psi_t(z)$ is kept constant at $\pm I_n$. It follows that Ψ_t induces a path Φ_t in $\mathcal{H}$ with $\varpi_n \circ \Psi_t = \Phi_t \circ \varpi_0$ for each $t \in [0, 1]$, and we define $G(\psi'_n) = \Phi_1$. □

3 Deformations of finite cyclic groups

Any deformation of ψ_n restricts to deformations of the cyclic subgroups $\langle\alpha\rangle\,,\ \langle\beta\rangle\,,\ \langle\gamma\rangle$ of orders $2p, 2q, 2r$ respectively.

Let g be one of the generators $\alpha\,,\beta\,,\gamma$, and let us consider the subgroup $\langle g\rangle$. By the *deformation space of* $\langle g\rangle$ we mean the component of the representation variety of $\langle g\rangle$ in SL(n, $\mathbb{R}$) containing the restriction of $\psi_n : U(p, q, r) \to$ SL(n, $\mathbb{R}$) to the subgroup $\langle g\rangle$. Let us denote this component $\mathcal{D}_n(\langle g\rangle)$. As representations ψ'_n of $\langle g\rangle$ travel through $\mathcal{D}_n(\langle g\rangle)$, $\psi'_n(g)$ must remain within the conjugacy class of $\psi_n(g)$, as the order of the element $\psi'_n(g)$ remains constant; also, by connectedness of SL(n, $\mathbb{R}$) any conjugate of $\psi_n(g)$ is so attainable. Since any homomorphism of $\langle g\rangle$ is determined by its effect on g, the correspondence $\psi'_n \mapsto \psi'_n(g)$ defines a diffeomorphism from $\mathcal{D}_n(\langle g\rangle)$ to the conjugacy class of $\psi_n(g)$ in SL(n, $\mathbb{R}$), or equivalently the space of right cosets of the centralizer of $\psi_n(g)$ in SL(n, $\mathbb{R}$). Denoting this centralizer $C(\psi_n(g))$, we have

$$\dim \mathcal{D}_n(\langle g\rangle) = \dim \mathrm{SL(n, \mathbb{R})} - \dim C(\psi_n(g)).$$

We now put this into a more practical form. Let $g \in U(p, q, r)$ be as in the previous paragraph, and suppose that g has order k. Recall that $\psi(g) \in$ SL(2, $\mathbb{R}$) is a matrix conjugate to

$$\tau_k \ = \begin{bmatrix} \cos\frac{\pi}{k} & -\sin\frac{\pi}{k} \\ \sin\frac{\pi}{k} & \cos\frac{\pi}{k} \end{bmatrix}$$

with eigenvalues $\zeta\,,\ 1/\zeta$, where ζ is the primitive $(2k)$th root of unity $e^{\pi i/k}$. It is easily determined that the image under σ_n of a diagonal matrix $\begin{bmatrix} x & 0 \\ 0 & 1/x \end{bmatrix}$ is an $n \times n$ diagonal matrix, with diagonal entries

$$x^{-n+1}\,,\ x^{-n+3}\,,\ \ldots\,,\ x^{n-3}\,,\ x^{n-1}.$$

Therefore the eigenvalues of $\psi_n(g) = \sigma_n(\tau_k)$ are

$$\zeta^{-n+1}\,,\ \zeta^{-n+3}\,,\ \ldots\,,\ \zeta^{n-3}\,,\ \zeta^{n-1}.$$

The origin of the slightly arcane function $\sigma(n, k)$ of Theorem 1.1 should now be apparent: taking the multiset

$$\mathcal{E} = \{\zeta^{-n+1}\,,\ \zeta^{-n+3}\,,\ \ldots\,,\ \zeta^{n-3}\,,\ \zeta^{n-1}\},$$

and writing $n = Qk + R$ $(0 \le R \le p-1)$, we see that there are R eigenvalues each occurring with multiplicity $Q+1$, and that each of the remaining eigenvalues in $\mathcal{E}$ occurs with multiplicity Q. The dimension of the centralizer $C(g)$ is one less than the sum of the squares of the multiplicities, i.e.

$$\dim C(g) \;=\; R(Q+1)^2 + \frac{n - R(Q+1)}{Q}\, Q^2 - 1 \;=\; (n+R)Q + R - 1,$$

the reduction by 1 corresponding to the constraint that matrices have determinant 1.

Proposition 3.1 *Let g be any one of the three generators α, β, γ of $U(p,q,r)$, and let k be the order of g. Then*

$$\dim \mathcal{D}_n(\langle g\rangle) \;=\; (n^2-1) - (\sigma(n\,,k) - 1) \;=\; n^2 - \sigma(n\,,k).$$

We are grateful to E. Weir for providing the simplified expression for $\sigma(n,k)$.

4 Proof of Theorem 1.1

For $g \in \mathrm{SL(n,\mathbb{R})}$, we adopt the notation

$$[g] = \{xgx^{-1} : x \in \mathrm{SL(n,\mathbb{R})}\}.$$

Also we define α_n, β_n, γ_n, z_n to be the images of α, β, γ, $z \in U(p,q,r)$ under the representation $\psi_n : U(p,q,r) \to \mathrm{SL(n,\mathbb{R})}$.

There is a natural map

$$\Pi :\; [\alpha_n] \times [\beta_n] \times [\gamma_n] \to \mathrm{SL(n,\mathbb{R})}\;,\;\; \Pi\,(\alpha_n'\,,\,\beta_n'\,,\,\gamma_n') \;=\; \alpha_n'\,\beta_n'\,\gamma_n'$$

which Proposition 4.1 below shows to be a submersion near to the point corresponding to the canonical representation. Since our deformation space $\mathcal{K}$ is naturally diffeomorphic to $\Pi^{-1}(z_n)$, we deduce that

$$\dim \mathcal{K} \;=\; \dim\,[\alpha_n] + \dim\,[\beta_n] + \dim\,[\gamma_n] - \dim \mathrm{SL(n,\mathbb{R})}.$$

In light of Proposition 3.1 and the fact that $\mathcal{H}$, $\mathcal{K}$ have the same dimension, this completes the proof of Theorem 1.1.

Thus it remains only to prove the claim that Π is a submersion close to the canonical representation. The following proposition shows that this objective is achieved even if we hold γ_n fixed, allowing only α_n', β_n' to vary within their conjugacy classes:

Proposition 4.1 *Let us define*

$$\mathcal{S} = \{\alpha_n'\,\beta_n' \;:\; \alpha_n' \in [\alpha_n]\,,\;\beta_n' \in [\beta_n]\}.$$

Then the tangent space of $\mathcal{S}$ at $\alpha_n\beta_n$ is equal to the tangent space of $\mathrm{SL(n,\mathbb{R})}$ *at $\alpha_n\beta_n$ (thus $\mathcal{S}$ contains a neighbourhood in* $\mathrm{SL(n,\mathbb{R})}$ *of $\alpha_n\beta_n$).*

Since our discussion takes place entirely within $\mathrm{SL(n,\mathbb{R})}$, we may lighten notation by suppressing the subscripts of α_n, β_n without fear of confusion.

The proof of Proposition 4.1 makes use of the well-known non-degenerate, indefinite bilinear form on the Lie algebra $\mathfrak{sl}(\mathrm{n},\mathbb{R})$:

$$\langle x\,,\,y\rangle \;:=\; \mathrm{tr}(x\,y).$$

We note immediately that $\langle\ ,\ \rangle$ is respected by Ad_g for any $g \in \mathrm{SL}(n, \mathbb{C})$, since

$$\langle \mathrm{Ad}_g(x)\,,\ \mathrm{Ad}_g(y)\rangle \ = \ \mathrm{tr}((gxg^{-1})(gyg^{-1})) \ = \ \mathrm{tr}(g(xy)g^{-1}) \ = \ \mathrm{tr}(xy) \ = \ \langle x\,,\ y\rangle.$$

Lemma 4.1.1 *Let $g \in \mathrm{SL}(n, \mathbb{R})$ be any element of the image of $U(p, q, r)$ under the representation ψ_n. Then $\mathfrak{sl}(n, \mathbb{R})$ admits a direct sum decomposition*

$$\mathfrak{sl}(n, \mathbb{R}) \ = \ \mathrm{Ker}(\mathrm{Ad}_g - 1) \oplus \mathrm{Im}(\mathrm{Ad}_g - 1),$$

orthogonal with respect to $\langle\ ,\ \rangle$.

Proof Let $g \in \psi_n(U(p, q, r))$. Since a triangle group has no parabolic elements, all elements of $U(p, q, r)$ (considered as a subgroup of $\mathrm{SL}(2, \mathbb{R})$) are diagonalizable (over $\mathbb{C}$). This property is preserved by the representation ψ_n, and it is a simple matter to check that it holds also for the endomorphism Ad_g. Therefore the endomorphism $\mathrm{Ad}_g - 1$ of $\mathfrak{sl}(n, \mathbb{R})$ has a full eigenspace for the eigenvalue 0. This eigenspace constitutes precisely the kernel of $\mathrm{Ad}_g - 1$, and the direct sum decomposition follows.

On the matter of orthogonality, let $\xi \in \mathrm{Ker}(\mathrm{Ad}_g - 1)$, $\eta \in \mathrm{Im}(\mathrm{Ad}_g - 1)$. Let us write $\eta = (\mathrm{Ad}_g - 1)(\zeta)$. Then

$$\begin{aligned}
\langle \xi\,,\ \eta\rangle \ &= \ \langle \xi\,,\ (\mathrm{Ad}_g - 1)(\zeta)\\
&= \ \langle \xi\,,\ \mathrm{Ad}_g(\zeta)\rangle - \langle \xi\,,\ \zeta\rangle\\
&= \ \langle \mathrm{Ad}_g(\xi)\,,\ \mathrm{Ad}_g(\zeta)\rangle - \langle \xi\,,\ \zeta\rangle\\
&= \ \langle \xi\,,\ \zeta\rangle - \langle \xi\,,\ \zeta\rangle\\
&= \ 0.
\end{aligned}$$

□

Lemma 4.1.2 *Let W be a linear subspace of $\mathfrak{sl}(n, \mathbb{R})$, and let $W^{\perp}$ be the orthogonal complement of W with respect to $\langle\ ,\ \rangle$. Then*

$$\dim W + \dim W^{\perp} \ = \ \dim \mathfrak{sl}(n, \mathbb{R}).$$

Proof The non-degenerate bilinear form $\langle\ ,\ \rangle$ induces an isomorphism from $W^{\perp}$ to the annihilator of W in the dual space $\mathfrak{sl}(n, \mathbb{R})^*$. □

We recall the following facts from basic Lie theory:

(i) There exists a neighbourhood of $0 \in \mathfrak{sl}(n, \mathbb{R})$ on which the exponential map is a diffeomorphism to a neighbourhood of $I_n \in \mathrm{SL}(n, \mathbb{R})$;
(ii) For $\xi \in \mathfrak{sl}(n, \mathbb{R})$ and $g \in \mathrm{SL}(n, \mathbb{R})$, $\exp(\mathrm{Ad}_g\,\xi) = g\exp(\xi)\,g^{-1}$, from which it follows that $\mathrm{Ad}_g(\xi) - \xi = 0 \iff \exp(\xi) \in C(g)$.

Proof of Proposition 4.1 We define subsets $\mathcal{S}_1$, $\mathcal{S}_2$ of $\mathcal{S}$, each containing $\alpha\beta$, as follows:

$$\mathcal{S}_1 = \{\alpha'\beta \ : \ \alpha' \in [\alpha]\}\,,\ \mathcal{S}_2 = \{\alpha\,\beta' \ : \ \beta' \in [\beta]\}.$$

It is sufficient to show that the tangent spaces of $\mathcal{S}_1$, $\mathcal{S}_2$ at $\alpha\beta$ generate the tangent space of $\mathrm{SL}(n, \mathbb{R})$ at $\alpha\beta$.

Let $g = \exp(\xi)$, $h = \exp(\eta)$ be elements of $\mathrm{SL}(n, \mathbb{R})$ close to the identity. Then the typical element $g\alpha g^{-1}\beta$ of $\mathcal{S}_1$ can be written as $\exp(\xi)(\exp(\mathrm{Ad}_\alpha\,\xi))^{-1}(\alpha\beta)$, and to first order this is equal to $\exp((1 - \mathrm{Ad}_\alpha)\,\xi)\,(\alpha\beta)$. Similarly, $\alpha h\beta h^{-1} \in \mathcal{S}_2$ has the first order approximation $\exp((\mathrm{Ad}_\alpha - \mathrm{Ad}_{\alpha\beta})\,\eta)\,(\alpha\beta)$.

Our task is therefore reduced to checking that the images of $1-\mathrm{Ad}_\alpha$, $\mathrm{Ad}_\alpha - \mathrm{Ad}_{\alpha\beta}$ satisfy

$$\mathrm{Im}(1-\mathrm{Ad}_\alpha) + \mathrm{Im}(\mathrm{Ad}_\alpha - \mathrm{Ad}_{\alpha\beta}) = \mathfrak{sl}(n, \mathbb{R}).$$

Suppose that the sum of these images is a proper subspace of $\mathfrak{sl}(n, \mathbb{R})$. Then from Lemma 4.1.2 there exists a non-zero element ζ in the orthogonal complement of $\mathrm{Im}(1-\mathrm{Ad}_\alpha) + \mathrm{Im}(\mathrm{Ad}_\alpha - \mathrm{Ad}_{\alpha\beta})$. In particular, ζ is orthogonal to $\mathrm{Im}(1-\mathrm{Ad}_\alpha)$, hence from Lemma 4.1.1 is in $\mathrm{Ker}(1-\mathrm{Ad}_\alpha)$. Note that we have just established that $\zeta = \mathrm{Ad}_\alpha(\zeta)$. Similarly, ζ is in the orthogonal complement of $\mathrm{Im}(\mathrm{Ad}_\alpha - \mathrm{Ad}_{\alpha\beta}) = \mathrm{Ad}_\alpha(\mathrm{Im}(1-\mathrm{Ad}_\beta))$. Since orthogonality is preserved by Ad_α, this last orthogonal complement is precisely $\mathrm{Ad}_\alpha(\mathrm{Ker}(1-\mathrm{Ad}_\beta))$. Therefore, using $\zeta = \mathrm{Ad}_\alpha(\zeta)$, we have $\mathrm{Ad}_\alpha(\zeta) \in \mathrm{Ad}_\alpha(\mathrm{Ker}(1-\mathrm{Ad}_\beta))$, implying $\zeta \in \mathrm{Ker}(1-\mathrm{Ad}_\beta)$. Now Schur's Lemma applied to the absolutely irreducible representation ψ_n gives us

$$\mathrm{Ker}(1-\mathrm{Ad}_\alpha) \cap \mathrm{Ker}(1-\mathrm{Ad}_\beta) = \{0\},$$

from which we deduce $\zeta = 0$, a contradiction. □

Acknowledgements The first author is partially supported by the National Science Foundation (Grant No. 1045292), and the second author is partially supported by the Simons Foundation (Grant No. 208565).

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