Zariski dense surface subgroups in $\mathrm{SL}(3, \mathbb{Z})$

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1 Introduction

The nature of finitely generated infinite index subgroups of $\mathrm{SL}(3, \mathbb{Z})$ remains extremely mysterious. It follows from the famous theorem of Tits [12] that free groups abound and, moreover, Zariski dense free groups abound. Less trivially, classical arithmetic considerations (see for example §6.1 of [9]) can be used to construct surface subgroups of $\mathrm{SL}(3, \mathbb{Z})$ of every genus $\geq 2$. However these are constructed using the theory of quadratic forms and their Zariski closures in $\mathrm{SL}(3, \mathbb{R})$ are $\mathrm{SO}(f, \mathbb{R})$ for some appropriate ternary quadratic form $f$; in particular these surface groups are not Zariski dense in $\mathrm{SL}(3, \mathbb{R})$.

The purpose of this note is two-fold. First, we aim to shed new light on the structure of Zariski dense faithful representations of surface groups into $\mathrm{SL}(3, \mathbb{Z})$, and second we will use this to compare and contrast the surface subgroup structure of $\mathrm{SL}(3, \mathbb{Z})$ with many other examples of groups arising naturally in geometry and topology; for example Kleinian groups, word hyperbolic groups, Mapping Class groups and $\mathrm{SL}(n, \mathbb{Z})$ with $n > 3$.

The main result is the following.

Theorem 1.1. The family of representations of the triangle group

$$\Delta(3, 3, 4) = \langle a, b \mid a^3 = b^3 = (a.b)^4 = 1 \rangle$$

given by

$$a \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$b \mapsto \begin{pmatrix} 1 & 2 - t + t^2 & 3 + t^2 \\ 0 & -2 + 2t - t^2 & -1 + t - t^2 \\ 0 & 3 - 3t + t^2 & (-1 + t)^2 \end{pmatrix}$$

are discrete and faithful for every $t \in \mathbb{R}$.

Moreover, for all integral values of $t$ the image groups are non-conjugate subgroups of $\mathrm{SL}(3, \mathbb{Z})$ which are Zariski dense in $\mathrm{SL}(3, \mathbb{R})$.

Since for any $t \in \mathbb{R}$, these subgroups are all isomorphic to $\Delta(3, 3, 4)$, by taking any $\mathbb{Z}$ specialization of $t$ and passing to a subgroup of finite index, we obtain a family of Zariski dense surface groups (for every fixed genus $\geq 2$) inside $\mathrm{SL}(3, \mathbb{Z})$.

To the authors’ knowledge, this is the first family of non-conjugate, infinite index, Zariski-dense freely indecomposable subgroups of $\mathrm{SL}(3, \mathbb{Z})$ that has been constructed. In fact, as far as

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we know there was only one such subgroup known previously, an unpublished example due to W. Goldman produced by a completely different method (see [9] for a discussion of this example). It is our understanding that his method is restricted to producing only a finite number of such representations. Goldman’s example appears in our family as the case \( t = 1 \) (after a suitable conjugation).

Moreover, since the representations given by Theorem 1.1 lie on the so-called Hitchin component (see [2] and \( \S \)), any element of infinite order necessarily has three distinct real eigenvalues (see [7] and [2]). For convenience, we will refer to these as purely semisimple groups. Recall from [10] that the group \( \Delta(3,3,4) \) has Property FA. Hence we have the following corollary.

**Corollary 1.2.** There exists an infinite family of non-conjugate Zariski dense, infinite index, freely indecomposable, purely semisimple subgroups of \( \text{SL}(3,\mathbb{Z}) \) isomorphic to a fixed group \( G \) which is word hyperbolic and has Property FA.

In general, the existence of a family of subgroups with the properties as given by Corollary 1.2 seem rare: by way of contrast, there are many results having the flavour that the number of conjugacy classes of images of a fixed group \( G \) into target groups \( \Gamma \) that are constrained to have various geometric properties is finite. Expanding on this theme, the first result of this kind goes back to Thurston ([11] Corollary 8.8.6) who uses compactness of pleated surfaces to show that if \( G \) is a surface group and \( \Gamma \) the fundamental group of a finite volume hyperbolic 3-manifold, then there are only finitely many conjugacy classes of faithful images of \( G \) that have no accidental parabolic (i.e. the analogue of purely semisimple). There are now broad ranging generalizations of this idea. For example, an analogue of this has also been established in the setting of word hyperbolic target groups (see [8] Theorem 5.3C and [6]), and relatively hyperbolic groups with conditions that the image of \( G \) does not have accidental parabolics [4]. Also striking is the result of Bowditch, which shows that if \( \Gamma \) is a Mapping Class group, then these have only finitely many conjugacy classes of purely pseudo-Anosov surface subgroups [1]. An extension of this by Dahmani and Fujiwara [5] extended this to \( G \) an arbitrary 1-ended group.

On the other hand, our results should also be compared with [13], which produces families of faithful discrete representations of surface groups into \( \text{SL}(n,\mathbb{R}) \) for \( n \geq 3 \). However, as is pointed out in [13], the representations constructed in [13] do not lie in the Hitchin component. Furthermore, his methods do not produce representations into \( \text{SL}(3,\mathbb{Z}) \). His methods do produce representations into \( \text{SL}(n,\mathbb{Z}) \) for \( n \geq 5 \), however, it is not easily checked whether these give purely semisimple groups.

Our method is to exploit convex real projective structures, coupled with an explicit understanding of these structures derived from [3]. It is known [2], that if \( \Delta(p,q,r) \) is a hyperbolic triangle group with \( p,q,r \) all greater than 2, then the space \( \text{Hom}(\Delta(p,q,r),\text{PGL}(3,\mathbb{R}))/\text{PGL}(3,\mathbb{R}) \) contains a two dimensional component (the Hitchin component, which we will denote by \( X(p,q,r)^{\text{Hit}} \)), which is characterised by the property that the corresponding representations are precisely those that determine convex real projective structures on the quotient 2-orbifold. In particular every such representation is discrete and faithful. It is this powerful fact that we are able to exploit using the method of [3], which computes the representation variety and identifies this component explicitly. This, taken together with some elementary Diophantine analysis yields the curve of representations defined above.

For character reasons, the values of \( p,q,r \) for which a hyperbolic triangle group could have a faithful representation into \( \text{SL}(3,\mathbb{Z}) \) must be drawn from \( \{2,3,4,6,\infty\} \). We also include a family for the triangle group \( \Delta(3,3,\infty) \).

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2 The construction.

Using the methods of [3], we constructed a two dimensional family of \( SL(3, \mathbb{R}) \) representations of \( \Delta(3,3,4) \). The actual matrices are a little cumbersome and they are relegated to the Appendix, but the structural aspects that we need can easily be described. As remarked above, it follows from [2] that there is a certain two (real) dimensional set \( X(3,3,4)^{Hit} \) where all the corresponding representations define convex real projective structures. In particular, these all are discrete and faithful.

This picture is reflected in our representation family as follows. There is a discriminant

\[
D = -4u^2(5 + u) + 4u(8 + u)v + (-20 + u(4 + u))v^2 - 4v^3
\]

which needs to be positive in order that the matrix entries be real. When one plots the locus \( D = 0 \) in the \( uv \)-plane it appears as in the figure below.

![Figure 1](image_url)

Each value in the region \( D > 0 \) corresponds to two representations, depending on which sign one takes for \( \sqrt{D} \). In this way one sees that each of the two components is a topological disc. (See [2])

The triangle group \( \Delta(3,3,4) \) admits essentially one discrete faithful representation into \( SO(2, 1; \mathbb{R}) \). Of course, this can then be embedded in \( SL(3, \mathbb{R}) \) in many ways upon changing the quadratic form of signature \( (2, 1) \). In our family, the point \( (2\sqrt{2}, 2\sqrt{2}) \) can be shown to define such an embedding, and this representation appears (as it must) on the locus \( D = 0 \). It follows that, given the structure we have set forth above, that all the representations in the quadrant \( u > 0, v > 0 \) and which lie in the region \( D \geq 0 \) correspond to discrete faithful representations of \( \Delta(3,3,4) \) (since they correspond to points in \( X(3,3,4)^{Hit} \)). This completes the first part of the proof of Theorem 1.1.

We now focus on which of these representations can be conjugated to be integral. To this end one computes that the characteristic polynomial of the element \( a.b^{-1} \) is

\[
1 - Q^3 - Q(2 + u) + Q^2(2 + v)
\]

and for \( [a, b] \) it is

\[
1 - Q^3 - Q(2 + 2u + 2v + uv - \sqrt{D})/2 + Q^2(2 + 2u + 2v + uv - \sqrt{D})/2
\]

from which it follows that we need to restrict to those rational integers \( u, v \) making \( D \) a square.
A complete Diophantine analysis of this situation seems rather complicated as it involves a study of the integral points on a complex surface, however a very simple-minded analysis using an interpolation method gives that if one takes $u = 4 - 3k + 2k^2$ and $v = 6 - 5k + 2k^2$, then $D = (-3 + 2k)^2(-1 + 2k)^2(4 - 2k + k^2)^2$. This substitution yields a representation of $\Delta(3, 3, 4)$ of integral character, which it turns out can be conjugated into $\text{SL}(3, \mathbb{Z})$ by using the rational canonical form for $a$ applied to the vector $(0, 1, 0)$.

One finds in this way the family of (necessarily) discrete faithful representations of $\Delta(3, 3, 4)$ into $\text{SL}(3, \mathbb{Z})$ described in the introduction. As mentioned in §1, the representation found by Goldman is a conjugate of that obtained by setting $t = 1$.

We now turn to the proof of the Zariski density claim of 1.1. To that end, we recall the following result that was proved in [9]:

**Theorem 2.1.** Let $G$ be a finitely generated nonsoluble subgroup of $\text{SL}(3, \mathbb{Z})$. Suppose that there is an element $g \in G$ whose characteristic polynomial is $\mathbb{Z}$-irreducible and non-cyclotomic.

Then $G$ is Zariski dense in $\text{SL}(3, \mathbb{Z})$.

We will apply this in the following way. The characteristic polynomial of $[a, b]$ is 

$$p(Q) = 1 - Q^3 + Q^2(17 - 8t + 4t^2) + Q(-29 + 46t - 39t^2 + 16t^3 - 4t^4).$$

It is clear that the polynomial $p(Q)$ cannot have a root $\pm1$ for $t > 0$, so that by the Gauss Lemma, $p(Q)$ is irreducible for large $t$. Moreover, it cannot be cyclotomic when $17 - 8t + 4t^2 > 3$, so that even this soft computation shows that the representations are Zariski dense for all but finitely many $t$. In fact an easy computer computation shows that $p(Q)$ satisfies the conditions of the theorem for all integral $t$. Thus the representations are Zariski dense for all integral $t$. This completes the proof of Theorem 1.1. □

Notice the only elements of order three in $\Delta(3, 3, 4)$ are conjugate to the elements $a^{\pm1}$ and $b^{\pm1}$, so that $p(Q)$ is an invariant of conjugacy and automorphisms of the triangle group. It follows that our family is indeed a family of distinct representations.

### 2.1 Comments

1. The family of integral representations constructed above is by no means exhaustive. For, by construction the representations of §1 all lie on the parameter curve $6 + u^2 + v^2 = u + v + 2uv$. Using the same ideas, other families can be constructed. One can (apparently) always arrange $a$ maps to the “standard element” of order three in the family of the introduction, denote this element by $r$. Then a second non-conjugate family of representations is given by

$$a \mapsto \tau \quad b \mapsto \begin{pmatrix} -5 - 14t - 16t^2 - 6t^3 & 6 + 9t + 4t^2 & 8 + 19t + 18t^2 + 6t^3 \\ 1 - 4t^2 - 6t^3 & -1 + t + 4t^2 & -1 + t + 6t^2 + 6t^3 \\ -2(2 + 3t)(1 + t + t^2) & 5 + 5t + 4t^2 & 6 + 13t + 12t^2 + 6t^3 \end{pmatrix}$$

In addition, there are apparently sporadic examples, like $u = 49$ and $v = 21$ yielding the representation

$$a \mapsto \tau \quad b \mapsto \begin{pmatrix} 1 & 20 & 48 \\ 0 & -8 & -19 \\ 0 & 3 & 7 \end{pmatrix}$$

2. One can also ask the analogous question for other triangle groups (for allowable $p, q$ and $r$’s)
and the same method can be used. For example, a family of faithful integral representations of \( \Delta(3, 3, \infty) \) is given by

\[
\begin{align*}
    a \rightarrow \tau & \quad b \rightarrow \left( \begin{array}{ccc}
        -3(2 + 5k + 4k^2) & -83 - 435k - 744k^2 - 432k^3 & 3(3 + 4k)(16 + 45k + 36k^2) \\
        -3(1 + 2k)^2 & -3(1 + 2k)^2(17 + 36k) & 4(2 + 3k)(11 + 39k + 36k^2) \\
        -2 - 9k - 12k^2 & -3(11 + 75k + 176k^2 + 144k^3) & 3(19 + 109k + 216k^2 + 144k^3)
    \end{array} \right)
\end{align*}
\]

References


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A two dimensional real family of representations of the triangle group $\Delta(3, 3, 4)$ is given by

$$a \mapsto \begin{pmatrix} 1 & 1 & u(-u^3v + 4uv^2 - 2v^3 + u^2(-6v + \sqrt{D})) / \tau \\ 0 & -v/u & \frac{(-u^2 + uv - v^2)/u^2}{1} \\ 0 & (-u^2 + uv - v^2)/u^2 & -1 + v/u \end{pmatrix}$$

$$b \mapsto \begin{pmatrix} -1 + v/u & 0 & u(2u^3 + 2v^3 - u^2v(2 + v) + uv(-2v + \sqrt{D})) / \tau \\ (2u^2 + u^2(2 + v) + u(4v + \sqrt{D})/(2u^2) & 1 & -1 \\ (-2u^3 - 2v^3 + u^2v(2 + v) + uv(2v + \sqrt{D}))/2u^3 & 0 & -v/u \end{pmatrix}$$

where $\tau = 2(u^2 + v^2)(u^2 - uv + v^2)$ and $D = -4u^2(5 + u + 4u(8 + u)v + (-20 + u(4 + u))v^2 - 4v^3$.

This was obtained by using the method of [3] to deform the hyperbolic representation, which occurs at $u = 2\sqrt{2}$, $v = 2\sqrt{2}$. 

3 Appendix

A two dimensional real family of representations of the triangle group $\Delta(3, 3, 4)$ is given by

$$a \mapsto \begin{pmatrix} 1 & 1 & u(-u^3v + 4uv^2 - 2v^3 + u^2(-6v + \sqrt{D})) / \tau \\ 0 & -v/u & \frac{(-u^2 + uv - v^2)/u^2}{1} \\ 0 & (-u^2 + uv - v^2)/u^2 & -1 + v/u \end{pmatrix}$$

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