Zariski dense surface groups on the Hitchin component.

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September 4, 2019

0.1 Introduction.

The work of Labourie [2] has emphasised the importance of understanding the so-called Hitchin component $X_{Hit}(\Sigma, SL(n, \mathbf{R}))$ of representations of a closed hyperbolic surface group Σ into $SL(n, \mathbf{R})$. The fact that all the representations on this component are discrete and faithful has proved to be a powerful tool for exhibiting thin groups (see, for example [8] and [5]) and in particular, it's of considerable interest to understand the Zariski closure of images of representations which are not in the Teichmüller locus. In this note we prove:

Theorem 0.1. *For any closed orientable hyperbolic surface Σ, there is a Zariski closed proper subset $\mathcal{B} \subset X_{Hit}(\Sigma, SL(n, \mathbf{R}))$ with the property that any representation $\rho \notin \mathcal{B}$ is Zariski dense in $SL(n, \mathbf{R})$.*

In fact our proof also applies to certain other components of $Hom(\pi_1(\Sigma), SL(n, \mathbf{R}))$ but their consideration is deferred to some closing remarks.

The power of our approach is that it is based upon p-adic methods pioneered by Lubotzky and in particular, his beautiful theorem of [6] that if for a single prime $p \geq 5$, the reduction of Γ mod p is all of $SL(n, \mathbf{Z}/p)$, then the Zariski closure of Γ is automatically all of the algebraic group $SL(n)$. We sketch the (very mild) generalization of his argument below. We can then bring to bear powerful theorems from the theory of finite groups [3] to prove certain representations surject $SL(n, \mathbf{Z}/p)$ and thereby deduce that they are Zariski dense.

0.2 Some preliminaries.

We briefly recall some generalities concerning Hitchin components. We recall that the (finite dimensional) representation theory of $SL(2, \mathbf{R})$ is well understood: For each n, there is a *Weyl representation* $SL(2, \mathbf{R}) \longrightarrow SL(n, \mathbf{R})$ given by extending the obvious action of $SL(2, \mathbf{R})$ on two variables X and Y to an action on homogeneous polynomials of degree $d = n - 1$ in those variables. This representation is absolutely irreducible. Given a hyperbolic structure on a closed surface Σ, the associated holonomy map can be composed with one of these representations $\rho : \pi_1(\Sigma) \longrightarrow SL(2, \mathbf{R}) \longrightarrow SL(n, \mathbf{R})$ and the component

*Partially supported by the NSF

of $Hom(\pi_1(\Sigma), SL(n, \mathbf{R}))$ containing this representation is the Hitchin component, which we denote $X_{Hit}(\Sigma, SL(n, \mathbf{R}))$. It contains the lifts of all the discrete faithful representations which is usually called the Teichmüller locus. Nothing in the Teichmüller locus can be Zariski dense in $SL(n, \mathbf{R})$ since the images lie in $SO(k, k-1)$ or $Sp(2k, 2k)$ depending on where n is even or odd.

A key role for us is played by the following theorem of Lubotzky [6]; in this statement the p-adic integers are denoted by $\mathbf{Z}_p$ and the cyclic group of order p by $\mathbf{Z}/p$. We have not quite stated the theorem in the way that Lubotzky does, as his hypothesis is that the matrices in question lie in $SL(n, \mathbf{Z})$; we do not restrict ourselves to this case but the proof extends verbatim.

Theorem 0.2. *Let S be a subset of $SL(n, \mathbf{Z}_p)$. There is a natural map $\pi_p : SL(n, \mathbf{Z}_p) \longrightarrow SL(n, \mathbf{Z}/p)$ and suppose that $\pi_p(S)$ generates $SL(n, \mathbf{Z}/p)$. If $n = 2$ assume that $p \neq 2, 3$ and if $n = 3, 4$ assume that $p \neq 2$. Let Γ be the abstract group in $SL(n, \mathbf{Z}_p)$ generated by S.*

Then Γ is Zariski dense in the algebraic group $SL(n)$.

Proof. It is known (see [10]) that other than the listed exceptional cases, the kernel $ker(\pi_p)$ is the Frattini subgroup of $SL(n, \mathbf{Z}_p)$ and it follows that S generates $SL(n, \mathbf{Z}_p)$ topologically, in other words, (in the notation of the statement of the theorem) that Γ is a dense subgroup of the p-adic analytic group $SL(n, \mathbf{Z}_p)$. It now follows that Γ is Zariski dense in the algebraic group $SL(n)$ since the Zariski topology is weaker than the usual topology on $SL(n, \mathbf{Z}_p)$. $\square$

To apply this result, we appeal to the remarkable result of [3], see Theorem 1.7.

Theorem 0.3. *Let $G \neq PSp_4(q)$ be a finite simple classical group.*

Then the probability that two randomly chosen elements x, y of order 3 in G generate G tends to 1 as $|G| \longrightarrow \infty$.

Furthermore, the same holds if we choose x, y from a largest conjugacy class of elements of order 3 in G.

To utilize this result requires some understanding of the size of conjugacy classes of elements of order 3 in $SL(n, \mathbf{Z}/p)$. To this end, we recall that the conjugacy class of largest size has the smallest centralizer. Suppose then that $p >> 0$ and M is a matrix in $SL(n, \mathbf{Z}/p)$ which has order 3. The characteristic polynomial has the form $(Q^2+Q+1)^k.(Q-1)^l$ where $2k + l = n$ and after an $GL(n, \mathbf{Z}/p)$ conjugacy we can regard M as block diagonal with k 2×2 blocks of the shape $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ together with an $l \times l$ identity matrix.

We claim that for large p, the centralizer of this matrix has order $O(p^{2k^2+l^2-1})$. There are two cases, depending on whether $(Q^2 + Q + 1)$ factors in $\mathbf{Z}/p$ or not. If it does, then if w denotes a primitive cube root of unity in $\mathbf{Z}/p$, the matrix is equivalent to a diagonal matrix with k w's, k $1/w$'s and l 1's so that the centralizer of this matrix is $GL(k, \mathbf{Z}/p) \times GL(k, \mathbf{Z}/p) \times GL(l, \mathbf{Z}/p)$ intersected with $SL(n, \mathbf{Z}/p)$, proving the result in this case. The case where (Q^2+Q+1) is irreducible over $\mathbf{Z}/p$ is similar and yields the same estimate. Now one sees easily that $O(p^{2k^2+l^2-1})$ is minimized for k (and hence l) chosen to be around $n/3$.

With a view to the upcoming argument, we note that if one takes the image of a diagonal matrix in $SL(2, \mathbf{C})$ with eigenvalues $\{\lambda, 1/\lambda\}$ under the Weyl representation defined above, then the image matrix has eigenvalues $\{\lambda^d, \lambda^{d-2}, ..., \lambda^{-d}\}$ and so in the particular case that λ is primitive cube root of unity, the multiplicites of the eigenvalues distribute as closely as they can to $(n/3, n/3, n/3)$. In particular, reducing any such representation modulo any prime of norm p, the image of an element of order 3 will lie in the conjugacy class of maximal size.

Finally, we need to observe the following. Let $K \subset \mathbf{R}$ be a number field which is not the rationals, and denote the ring of integers of K by $\mathcal{O}$. Let $\wp$ be a prime of $\mathcal{O}$ of norm p, this defines a $\pi_p : SL(n, \mathcal{O}) \longrightarrow SL(n, \mathbf{Z}/p)$ which is surjective, since $SL(n, \mathcal{O}) > SL(n, \mathbf{Z})$ and it is well known that the latter group surjects. Then we claim:

Lemma 0.4. *For any element $g \in SL(n, \mathbf{Z}/p)$, the set $\pi_p^{-1}(g)$ is dense in $SL(n, \mathbf{R})$*

Proof. Since K is not the rationals and does have a real embedding, $\mathcal{O}$ has rank at least two and is therefore dense in $\mathbf{R}$. It follows that the group $SL(n, \mathcal{O}) < SL(n, \mathbf{R})$ is not discrete since (for example) it contains all elements which are the identity on the diagonal, zero below the diagonal and have arbitrary elements of $\mathcal{O}$ above the diagonal.

Set G to be the topological closure of $SL(n, \mathcal{O})$ in $SL(n, \mathbf{R})$. Then G is a closed subgroup of a Lie group and therefore a Lie group. Moreover, the remarks of the previous paragraph show that G contains the upper triangular subgroup of $SL(n, \mathbf{R})$. By the same token, G contains the lower triangular group and it follows G contains all elementary matrices. Therefore $G = SL(n, \mathbf{R})$, i.e. $SL(n, \mathcal{O})$ is topologically dense in $SL(n, \mathbf{R})$.

We may also apply the same argument to the normal subgroup $\pi_p^{-1}(I)$, since $\wp.\mathcal{O}$ is also dense in the reals, so that $G_0 = \overline{\pi_p^{-1}(I)} = SL(n, \mathbf{R})$. Given $g \in SL(n, \mathbf{Z}/p)$, there is an element $\gamma \in SL(n, \mathbf{Z}) < SL(n, \mathcal{O})$ with $\pi_p(\gamma) = g$. Then $\gamma.G_0 = \overline{\gamma.\pi_p^{-1}(I)} = \overline{\pi_p^{-1}(g)} = SL(n, \mathbf{R})$, i.e. $\pi_p^{-1}(g)$ is dense in $SL(n, \mathbf{R})$. $\square$

0.3 Proof of Theorem 0.1.

Guided by the considerations of the previous section, we consider the hyperbolic orbifold $S^2(3, 3, 3, 3)$, we denote its fundamental group by Γ. One sees easily that

$$\Gamma =< d_1, d_2, d_3, d_4 \ | \ d_j^3 = I \quad 1 \leq j \leq 4 \quad d_1.d_2.d_3.d_4 = I >$$

Fix some n and set X_{Hit} to be the Hitchin component of Γ in $SL(n, \mathbf{R})$. Fix some ρ on X_{Hit} with the properties that it does not lie on any of the other components of $Hom(\Gamma, SL(n, \mathbf{R}))$, has algebraic entries in some number field $K \neq \mathbf{Q}$ and with the further property that ρ is absolutely irreducible when restricted to the subgroup $< d_3, d_4 >$.

With regard to the last condition, we note that since there are no interesting Lie subgroups in $SL(2, \mathbf{R})$, the image of $< d_3, d_4 >$ under any hyperbolic structure is Zariski dense in $SL(2, \mathbf{R})$. Since the Weyl representation of $SL(2, \mathbf{R})$ in $SL(n, \mathbf{R})$ is absolutely irreducible, it will be absolutely irreducible when restricted to $< d_3, d_4 >$, so that (for example) any representation on the Teichmüller locus has this property. Moreover, by the

Burnside lemma, this is a Zariski open condition and it follows that any algebraic representation in X_{Hit} away from a Zariski closed subset and away from all other components of $Hom(\Gamma, SL(n, \mathbf{R}))$ has all the above properties.

Let A be the subring of $K \subset \mathbf{R}$ generated by the entries of $\rho(d_j)$ for $1 \leq j \leq 4$ together with $\mathcal{O}_K$; this is a finitely generated integral domain in characteristic zero and we have $\rho : \Gamma \longrightarrow SL(n, A)$. Then it is well known (for example we refer to Lemma 3(b) in [7]) that for infinitely many primes p there is an embedding of $A \longrightarrow \mathbf{Z}_p$, which in turn gives rise to a representation $\rho : \Gamma \longrightarrow SL(n, \mathbf{Z}_p)$. By composing with the natural map, we obtain a homomorphism $\pi_p : \Gamma \longrightarrow SL(n, \mathbf{Z}/p)$. We choose the prime p to be sufficiently large that we can apply Theorem 0.3, i.e. $|SL(n, \mathbf{Z}/p)|$ is sufficiently large so that the probability that two maximal conjugacy class elements of order 3 generates $SL(n, \mathbf{Z}/p)$ is > 0.

This choice means that there is *some* pair of elements x, y of order 3 lying in the maximal conjugacy class so that x, y generates all of $SL(n, \mathbf{Z}/p)$. As noted above, both $\pi_p(d_1)$ and $\pi_p(d_2)$ lie in this maximal conjugacy class, so without loss of generality $x = \pi_p(d_1)$ and further, there is an element $\eta \in SL(n, \mathbf{Z}/p)$ with the property that $< \pi_p(d_1), \pi_p(d_2)^\eta >$ generates all of $SL(n, \mathbf{Z}/p)$.

By Lemma 0.4, there is an element $\tau \in SL(n, \mathcal{O}_K)$ which is very close to the identity matrix and for which $\pi_p(\tau) = \eta$. Then this choice of τ ensures that the group $< \rho(d_1), \rho(d_2)^\tau >$ surjects $SL(n, \mathbf{Z}/p)$, whence by Lubotzky's result is Zariski dense in $SL(n, \mathbf{R})$.

Now we appeal to the following result of [4] (cf Proposition 4.1)

Proposition 0.5. *Suppose that* $< d_3, d_4 >$ *is an absolutely irreducible subgroup of* $SL(n, \mathbf{R})$*. Denoting the* $SL(n, \mathbf{R})$ *conjugacy class of* a *by* $[a]$*, we define*

$$\mathcal{S}' = \{a' \, b' \; : \; a' \in [d_3] \; , \; b' \in [d_4]\}$$

Then $\mathcal{S}'$ *contains a neighbourhood in* $SL(n, \mathbf{R})$ *of* $d_3.d_4$*.*

We may apply this result to $< d_3, d_4 >$ to find ξ_1, ξ_2 very close to the identity so that $\rho(d_1).\rho(d_2)^\tau.\rho(d_3)^{\xi_1}.\rho(d_4)^{\xi_1} = I$ and in this way we obtain a representation ρ^* of Γ very close to ρ (i.e. the generators map to $\rho(d_1), \rho(d_2)^\tau, \rho(d_3)^{\xi_1}, \rho(d_4)^{\xi_1}$) which is clearly Zariski dense in $SL(n, \mathbf{R})$. Since we chose ρ away from all the other components of $Hom(S^2(3, 3, 3, 3), SL(n, \mathbf{R}))$ it follows that $\rho^* \in X_{Hit}$.

To summarise, to this point we have shown that away from a Zariski closed subset of $X_{Hit}(\Gamma)$ (for example, in a classically open neighbourhood of the Teichmüller locus in $X_{Hit}(\Gamma)$), there is classically dense set of representations of Γ which have Zariski dense image in $SL(n, \mathbf{R})$.

We need the following lemma.

Lemma 0.6. *Suppose that* ρ *is any Zariski dense representation on* $X_{Hit}(\Gamma)$*.*

Then ρ *restricted to any subgroup of finite index in* Γ *is Zariski dense in* $SL(n, \mathbf{R})$*.*

Proof. If H is any subgroup of finite index in $\rho(\Gamma)$ and suppose that $H \subset G$, some algebraic subgroup of $SL(n, \mathbf{R})$. The component of G which contains the identity has finite index, so replacing H by $H \cap G_0$ and G by G_0, we may suppose that G is connected.

Let $N \subset H$ be the intersection of the $\rho(\Gamma)$ conjugates of H so that N is normal in $\rho(\Gamma)$. Clearly $N \subset G$, so if $\gamma_1, ...\gamma_k$ is a set of coset representatives for N in $\rho(\Gamma)$, then $G^* = \bigcap_i \gamma_i.G.\gamma_i^{-1}$ is a (possibly disconnected) algebraic group containing N and normalised by $\rho(\Gamma)$. Therefore $< G^*, \rho(\Gamma) >$ is a finite disjoint collection of cosets of G^* and is an algebraic group containing $\rho(\Gamma)$. By hypothesis, the only algebraic group with this property is the connected group $SL(n, \mathbf{R})$, so $(G^*)_0 = SL(n, \mathbf{R})$, and since clearly $(G^*)_0 \subset G$, it follows that $G = SL(n, \mathbf{R})$ as was required. $\square$

Now any closed hyperbolic surface $\Sigma(g)$ of genus $g \geq 2$ covers $S^2(3,3,3,3)$ and we may restrict any $\rho \in X_{Hit}$ to a subgroup corresponding to $\Sigma(g)$. We claim this gives a map from X_{Hit} to a subset of $X_{Hit}(\Sigma(g))$. The reason is this. There is a result of Guichard [1] (proving a converse to a result due to Labourie cf [2]) which says a representation lies in the Hitchin component of a surface group if and only if the image group preserves a hyperconvex curve in $\mathbf{RP}^{n-1}$. Since [2] implies that ρ preserves a hyperconvex curve and clearly any subgroup of finite index preserves the same curve, it follows from [1] that the restricted representation lies in $X_{Hit}(\Sigma(g))$. Appealing to Lemma 0.6, we have produced Zariski dense representations in $X_{Hit}(\Sigma(g))$.

The proof of Theorem 0.1 is now completed by the following:

Theorem 0.7. *The set of Zariski dense representations on $X_{Hit}(\Sigma(g))$ is Zariski open.*

Proof. It follows from a result of [9] (see Proposition 17) that there are finitely many (absolutely) irreducible non-trivial representations $r_i : SL(n, \mathbf{R}) \longrightarrow GL(V_i)$ such that any proper connected subgroup G of $SL(n, \mathbf{R})$ fixes a line in at least one of those representations. Let $\rho \in X_{Hit}(\Sigma(g))$ be any representation with Zariski dense image, then for each i, $r_i \circ \rho$ is absolutely irreducible so by the Burnside lemma that one can find a finite collection of elements of $\Sigma(g)$ whose image spans $End(V_i)$. Not spanning $End(V_i)$ is a Zariski-closed condition, so that away from a Zariski closed subset of $X_{Hit}(\Sigma(g))$, any representation composed with r_i remains absolutely irreducible. Repeating this argument for each i, the Golsefidy-Varjú theorem ensures that any representation of $X_{Hit}(\Sigma(g))$ chosen away from a Zariski closed set does not lie in any proper algebraic subgroup of $SL(n, \mathbf{R})$, as required. $\square$

Remarks. The role of the Hitchin component in this argument was that it guarantees the correct eigenvalue structure for the elements of order 3. However there are other components of $Hom(S^2(3,3,3,3), SL(n, \mathbf{R}))$ which have this property. For example there is a Weyl representation $SL(3, \mathbf{R}) \longrightarrow SL(6, \mathbf{R})$ coming from the obvious action on homogeneous polynomials of degree two in three variables, and one can check that if one takes an Anosov representation of $S^2(3,3,3,3)$ into $SL(3, \mathbf{R})$ and composes with this representation the resulting representation has an eigenvalue structure to which this method applies. It follows that the component (which is not the Hitchin component) containing this representation has a Zariski open subset consisting of Zariski dense representations and this finds other components of surface group representations by restriction.

Acknowledgement. The authors are indebted to Alex Kontorovich for some useful guid-

ance concerning algebraic groups.

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