Computing varieties of representations of hyperbolic 3-manifolds into $SL(4, \mathbb{R})$

Daryl Cooper[1], *Darren Long*[2] *and Morwen Thistlethwaite*

1. Introduction

Following the seminal work of M. Culler and P. Shalen [Culler and Shalen, 1983], and that of A. Casson [Akbulut and McCarthy, 1990], the theory of representation and character varieties of 3-manifolds has come to be recognized as a powerful tool, and has duly assumed an important place in low-dimensional topology. Among the many papers that have appeared in this context, we mention [Culler et al., 1987, Cooper et al., 1994, Boyer and Zhang, 1998]. Most of the work carried out to date is concerned with representations into Lie groups of 2×2 matrices, owing mainly to connections with actions on trees and the isometry groups of hyperbolic space in dimensions 2 and 3, but also owing to the extreme difficulty of computations beyond the realm of such matrices.

This paper was originally motivated by the following question: under what circumstances can one take the hyperbolic structure on a closed hyperbolic 3-manifold and deform it to a real projective structure? In the language of representations, this amounts to beginning with $SO^+(3,1)$-representation ϕ_0 of the fundamental group of the manifold, given by the hyperbolic structure, and then endeavouring to compute the component of the $SL(4, \mathbb{R})$-representation variety containing ϕ_0. Using a computer, it is relatively easy to see that for many closed hyperbolic 3-manifolds there are linear obstructions to deforming, but when these obstructions vanish it is of considerable interest to see whether genuine deformations exist. The purpose of this article, then, is to describe a method for the exact computation of the representation varieties of closed hyperbolic 3-manifolds into $SL(4, \mathbb{R})$. For simplicity we restrict our attention to orientable manifolds with 2-generator fundamental groups.

The technique is sufficiently practical that we have used it to compute 24 varieties of this type exactly, and have investigated numerically the first 4500 closed orientable manifolds with 2-generator groups in the Hodgson-Weeks census [Hodgson and Weeks, 2000]. Numerical evidence strongly suggests that only 52 of these 4500 manifolds admit non-trivial deformations of ϕ_0 into $SL(4, \mathbb{R})$. When these deformations do occur, they lead to a number of interesting constructions, including families of real projective structures on the manifold and families of discrete faithful representations into $PU(3,1)$, the group of orientation-preserving isometries of 3-dimensional complex hyperbolic space. These and other theoretical aspects are considered in more depth in [Cooper et al., 2005]; in this paper we concentrate on the computational aspects of the investigation.

The presence of an embedded totally geodesic surface in the manifold guarantees the ex-

[1]Partially supported by the NSF
[2]Partially supported by the NSF

istence of a well-established type of deformation known as *bending*, but such surfaces cannot exist in the census manifolds, owing to their small volume [Kojima and Miyamoto, 1991]. The underlying reason for the sporadic occurrence of these deformations is still a mystery.

In order to make the paper reasonably self-contained, there now follows a section summarizing the background material.

2. Background

2.1 The Minkowski model for hyperbolic space

A geometric structure on a hyperbolic 3-manifold M corresponds to a discrete faithful representation of the fundamental group of M into the group of isometries of 3-dimensional hyperbolic space $\mathbb{H}^3$. In the case where M is compact, by Mostow rigidity the structure is unique up to isometry (if M is non-compact and of finite volume, we also have uniqueness if we add the requirement that the geometric structure be complete.)

In this article we shall focus our attention on closed orientable hyperbolic 3-manifolds M. Thus the geometric structure on M corresponds to a discrete faithful representation $\phi_0 : \pi_1(M) \to \mathrm{Isom}^+ \mathbb{H}^3$, unique up to conjugacy; here $\mathrm{Isom}^+ \mathbb{H}^3$ denotes the group of orientation-preserving isometries of $\mathbb{H}^3$. In the upper-half space model for $\mathbb{H}^3$, the group $\mathrm{Isom}^+ \mathbb{H}^3$ is naturally identified with $\mathrm{PSL}(2, \mathbb{C})$, the group of Möbius transformations of the boundary, but for us it will be more propitious to work in the *Minkowski model* for $\mathbb{H}^3$, as our intention is to consider $\mathrm{Isom}^+ \mathbb{H}^3$ as a subgroup of $\mathrm{SL}(4, \mathbb{R})$ and search for deformations of the composite homomorphism $\pi_1(M) \xrightarrow{\phi_0} \mathrm{Isom}^+ \mathbb{H}^3 \hookrightarrow \mathrm{SL}(4, \mathbb{R})$.

Minkowski space of dimension $n+1$, denoted $\mathbb{M}^{n+1}$, is the real vector space $\mathbb{R}^{n+1}$ endowed with a quadratic form Q of signature $(n, 1)$, which we take without loss of generality to be

$$Q(x_0, x_1, \dots, x_n) \;=\; -x_0^2 + x_1^2 + \cdots + x_n^2 \;.$$

The group of isometries of $\mathbb{M}^{n+1}$ is the *Lorentz group* $\mathrm{O}(n, 1)$; it consists of those $(n+1) \times (n+1)$ matrices A satisfying $A^{-1} = FA^tF$, where the superscript ^{t} denotes transpose, and where F is the diagonal $(n+1) \times (n+1)$ matrix with entries $-1, 1, \dots, 1$. An excellent reference is [Epstein and Penner, 1988].

Let $\mathbf{x}$ denote $(x_0, x_1, \dots, x_n) \in \mathbb{M}^{n+1}$. In the Minkowski model, *real hyperbolic n-space* is the "upper sheet" of the hyperboloid $\mathbb{H}^n = \{\mathbf{x} : Q(\mathbf{x}) = -1\}$, namely the set of points of this hyperboloid for which $x_0 > 0$. We note that $\mathbb{H}^n$ is asymptotic to the *light cone* $C = \{\mathbf{x} : Q(\mathbf{x}) = 0\}$, and that $Q(\mathbf{x}) > 0$ if and only if the vector $\mathbf{x}$ points in a direction outside C. It is not hard to see that the tangent vectors at any point of $\mathbb{H}^n$ satisfy this condition, whence the $(3,1)$-form of $\mathbb{M}^{n+1}$ is positive definite on the tangent bundle of $\mathbb{H}^n$; indeed it induces the expected Riemannian metric of constant negative curvature. The *boundary* of $\mathbb{H}^n$

in the Minkowski model is the space whose points are rays of the light cone C. The group $\mathrm{Isom}^+ \mathbb{H}^n$ is the subgroup $\mathrm{SO}^+(n, 1)$ of $\mathrm{O}(n, 1)$ consisting of linear transformations that (i) have determinant 1, and (ii) preserve the sheets of the hyperboloid $\{\mathbf{x} : Q(\mathbf{x}) = -1\}$. $\mathrm{SO}^+(n, 1)$ has index 4 in $\mathrm{O}(n, 1)$, and is the component subgroup of the identity.

Let us specialize to the case $n = 3$. The assignment

$$(t, x, y, z) \mapsto \begin{bmatrix} t+z & x+iy \\ x-iy & t-z \end{bmatrix}$$

defines an $\mathbb{R}$-vector space isomorphism from $\mathbb{R}^4$ to the space $\mathcal{H}^2$ of 2×2 Hermitian matrices over the complex numbers; the quadratic form in $\mathcal{H}^2$ corresponding to Q under this isomorphism is simply the negative of the determinant. We therefore have an isomorphism $\mathbb{M}^4 = (\mathbb{R}^4, Q) \approx (\mathcal{H}^2, -\det)$. It is then easy to construct an isomorphism from $\mathrm{PSL}(2, \mathbb{C})$ to the group of isometries of $\mathbb{M}^4$: we simply assign to each matrix $A \in \mathrm{SL}(2,\mathbb{C})$ the map $H \mapsto A^*HA$ ($H \in \mathcal{H}^2$), where $*$ denotes Hermitian transpose. We may construct an explicit isomorphism from $\mathrm{PSL}(2, \mathbb{C})$ to $\mathrm{SO}^+(3,1)$ by choosing a specific basis for $\mathcal{H}^2$, for example

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \right\},$$

and then computing the matrix of the linear transformation $H \mapsto A^*HA$ with respect to this basis.

2.2 Eigensystems of isometries of hyperbolic space

The isomorphism given above may be used to determine the eigenvectors and corresponding eigenvalues of the various kinds of isometry of $\mathbb{H}^3$ in the Minkowski model. The situation may be summarized as follows.

(i) The eigensystem of a loxodromic. Let $g \in \mathrm{Isom}^+ \mathbb{H}^3$ be a loxodromic; then g acts freely on $\mathbb{H}^3$, and fixes two points on the boundary of $\mathbb{H}^3$. In the upper-half space model, the boundary is identified with the extended complex plane, and with suitable choice of coordinates we may assume that the fixed points of g are $0, \infty$. g then corresponds to a dilation $z \mapsto az$ on the boundary, where the complex number a is the *dilation factor* of g (see [Maskit, 1980]). We may think of g as the composition of a "pure translation" with axis the geodesic joining 0 and ∞, together with a rotation (elliptic) about that axis through an angle $\arg(a)$. The isometry g corresponds to the element of $\mathrm{PSL}(2,\mathbb{C})$ represented by the matrix $\begin{bmatrix} a^{1/2} & 0 \\ 0 & 1/a^{1/2} \end{bmatrix}$, where $a^{1/2}$ is either of the square roots of a.

Application of the above isomorphism yields the following matrix in $\mathrm{SO}^+(3,1)$:

$$\begin{bmatrix} p & q & 0 & 0 \\ q & p & 0 & 0 \\ 0 & 0 & r & -s \\ 0 & 0 & s & r \end{bmatrix}, \quad \text{where} \quad \begin{cases} p = \frac{1}{2}\left(|a| + \frac{1}{|a|}\right), \\ q = \frac{1}{2}\left(|a| - \frac{1}{|a|}\right), \\ r = \Re\mathrm{e}\left(\frac{a}{|a|}\right), \\ s = \Im\mathrm{m}\left(\frac{a}{|a|}\right). \end{cases}$$

The eigenvalues of this matrix are $|a|$, $\frac{1}{|a|}$, $e^{i\theta}$, $e^{-i\theta}$, where $\theta = \arg(a)$. The eigenspaces of the real eigenvalues $|a|$, $\frac{1}{|a|}$ are the two rays of the light cone identified with the fixed points of g. Indeed the subspace spanned by these two rays meets $\mathbb{H}^3$ precisely in the axis of g. The pairing of each eigenvalue with its inverse corresponds to the fact that the axis of g admits two orientations. It can happen that $\theta = 0$, in which case the isometry is a pure translation.

(ii) The eigensystem of an elliptic. Let g be an elliptic isometry. Since we may regard g as a degenerate loxodromic, with dilation factor on the unit circle, the corresponding matrix in $\mathrm{SO}^+(3,1)$ will have eigenvalues 1, $e^{i\theta}$, $e^{-i\theta}$, where θ is the angle of rotation, as before. The eigenspace of the eigenvalue 1 is the 2-dimensional subspace of $\mathbb{M}^4$ containing the two rays on the light cone corresponding to the ends of the axis of g.

(iii) The eigensystem of a parabolic. Let g be parabolic; then g has one fixed point on the boundary, and g effects a Euclidean translation along each horosphere centred at this fixed point. If we choose coordinates in the upper-half space model so that the fixed point is ∞, g corresponds to the element of $\mathrm{PSL}(2,\mathbb{C})$ represented by $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, where the complex number a describes the Euclidean translation. The corresponding matrix in $\mathrm{SO}^+(3,1)$ is:

$$\begin{bmatrix} p+1 & -p & r & s \\ p & -p+1 & r & s \\ r & -r & 1 & 0 \\ s & -s & 0 & 1 \end{bmatrix}, \quad \text{where} \quad \begin{cases} p = \frac{1}{2}|a|^2, \\ r = \Im\mathrm{m}\, a, \\ s = \Re\mathrm{e}\, a. \end{cases}$$

This matrix has a single eigenvalue 1 with algebraic multiplicity 4 and geometric multiplicity 2. The eigenspace is spanned by a vector (in this case $(1, 1, 0, 0)$) pointing along the ray on the light cone corresponding to the single fixed point of g, and an orthogonal vector (in this case $(0, 0, -\Im\mathrm{m}\, a, \Re\mathrm{e}\, a)$), whose direction encodes the direction of the Euclidean translation.

3. Trace calculus

The ultimate aim of this paper is to compute representation varieties of $\pi_1(M)$ into $\mathrm{SL}(4,\mathbb{R})$. Since any representation can be composed with an inner automorphism of the target group to produce another, we are content to find just one representation in each such equivalence class. In essence we are computing a variety $\mathcal{V}$ that is an embedded copy of the character variety in the full representation variety.

If the variety $\mathcal{V}$ has dimension n, then it is specified by a single "tautological" representation Ψ into $\mathrm{SL}(4,\mathbb{F})$, where $\mathbb{F}$ is a field of transcendence degree n over $\mathbb{R}$. Individual representations are obtained from Ψ by evaluating at specific points in parameter space. Clearly Ψ depends on a choice of parametrization of $\mathcal{V}$.

Two important fields in this context are (i) the field K generated by the entries of image matrices, and (ii) the subfield T of K generated by the traces of image matrices. By conjugating judiciously, we can guarantee that the field K (hence also T) is algebraic of finite degree over a purely transcendental extension $\mathbb{Q}(u_1, \ldots, u_n)$ of the rationals. The trace field T is independent of the choice of conjugation, and in practice it is often easy to guess T. However, in order to compute the variety $\mathcal{V}$ we shall need to specify generators for K over the base field $\mathbb{Q}(u_1, \ldots, u_n)$; the elementary proposition given below will be helpful in that regard.

In order to state Proposition 1 it will be convenient to introduce some notation. Let $a = (a_{ij})$ be any $n \times n$ matrix over a commutative ring R, and let $\sigma = (n_1, \ldots, n_k)$ be any cyclically ordered sequence of distinct numbers from $\{1, 2, \ldots, n\}$ (thus we regard the sequences $(n_1, n_2, n_3, \ldots, n_k)$, $(n_2, n_3, \ldots, n_k, n_1)$ as being identical). Then we define a_σ to be the following element of R:

$$a_\sigma = a_{n_1 n_2} a_{n_2 n_3} \ldots a_{n_k n_1} .$$

The proof of Proposition 1 is greatly facilitated by having a computer algebra system to hand, for example Mathematica or Maple.

Proposition 1. Let G be a subgroup of $\mathrm{SL}(4,\mathbb{C})$ generated by matrices a, b, where b is a diagonal matrix

$$b = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} .$$

Suppose further that

(i) the λ_i are all distinct;

(ii) $\mathrm{tr}(b) \neq 0$;

(iii) $(\mathrm{tr}(b))^3 \neq \mathrm{tr}(b^3)$.

Let T be the trace field of G, and let K be the field obtained by adjoining $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ to T. Then for each cycle σ of length 1, 2 or 3 in $\{1, 2, 3, 4\}$, $a_\sigma \in K$.

Proof. First we deal with the case where σ has length 1, *i.e.* the diagonal entries a_{ii} $(1 \leq i \leq 4)$. It is readily checked that the four traces $\mathrm{tr}(a)$, $\mathrm{tr}(ab)$, $\mathrm{tr}(ab^2)$, $\mathrm{tr}(ab^3)$ are all linear expressions in the entries a_{ii} over the field K. We therefore have a system of linear equations for the a_{ii} over K, and it is easily verified (using *e.g.* Mathematica) that the determinant of the matrix of coefficients is $\prod_{i<j}(\lambda_i - \lambda_j)$. Therefore, from the hypothesis that the λ_i are distinct, the system has a (unique) solution and the a_{ii} have been shown to lie in the field K.

Next, consider the six products corresponding to cycles of length 2, namely

$$a_{12}a_{21}\,,\ a_{13}a_{31}\,,\ a_{14}a_{41}\,,\ a_{23}a_{32}\,,\ a_{24}a_{42}\,,\ a_{34}a_{43}\ .$$

Let B denote the inverse of b. The trace of any word in the generators a, b involving precisely two occurrences of a (and no occurrences of a^{-1}) is a linear expression in these six products over the field $T(\lambda_1, \lambda_2, \lambda_3, \lambda_4, a_{11}, a_{22}, a_{33}, a_{44})$, which we now know to be equal to K. The six traces

$$\mathrm{tr}(aa)\,,\ \mathrm{tr}(aab)\,,\ \mathrm{tr}(aaB)\,,\ \mathrm{tr}(aabb)\,,\ \mathrm{tr}(abab)\,,\ \mathrm{tr}(aBab)$$

therefore give rise to a linear system in the six products over this field, and one can verify that the determinant of the matrix of coefficients is $4\prod_{i<j}(\lambda_i - \lambda_j)^2$. Again, this determinant is non-zero, and it follows that the six products corresponding to cycles of length 2 are in the field K.

The eight products corresponding to cycles of length 3 are dealt with similarly, using the traces of the words

$$aaa\,,\ aaab\,,\ aaabb\,,\ aaBab\,,\ aabaB\,,\ aaBabb\,,\ abaBabb\,,\ abaBBabbb\ .$$

This time the determinant of the 8×8 matrix of coefficients is
$\left(\prod_{i<j}(\lambda_i - \lambda_j)^4\right) \mathrm{tr}(b) \left((\mathrm{tr}(b))^3 - \mathrm{tr}(b^3)\right)$, and the result follows. $\square$.

Remark 3.1 The choice of words at each stage of the proof of Proposition 1 is certainly not unique, but the eight words used for the last stage need to be chosen quite carefully in order that the matrix of coefficients should have full rank.

Remark 3.2. If M is a closed, orientable, hyperbolic 3-manifold, then each non-trivial element $g \in \pi_1(M)$ is loxodromic. Let $\phi_0 : \pi_1(M) \to \mathrm{SO}^+(3,1)$ be the representation given by the geometric structure; then the eigenvalues of $\phi_0(g)$ are distinct if and only if g is not a pure translation. Therefore, if b is the diagonalization of $\phi_0(g)$ and g has a non-trivial rotational component, then at least condition (i) of Proposition 1 is met. Condition (ii) is automatically met as the trace of $\phi_0(g)$ cannot equal zero; this follows directly from the fact, explained in §2.2, that $\phi_0(g)$ has eigenvalues $|a|$, $\frac{1}{|a|}$, $e^{i\theta}$, $e^{-i\theta}$ with $|a| \neq 1$. Condition (iii) has also been met in all observed cases. Furthermore, these three properties are “open conditions”, so if they are satisfied for $\phi_0(g)$, they are also satisfied for $\phi_1(g)$, given that ϕ_1 is sufficiently close to ϕ_0.

Remark 3.3. It often happens in practice that the characteristic polynomial of the image of $g \in \pi_1(M)$ under the tautological representation Ψ is *reciprocal*, meaning that its roots are in reciprocal pairs; equivalently, the coefficients of the polynomial form a palindromic sequence. If $\phi_0(g)$ is not a pure translation, and if ϕ is an evaluation of Ψ close to the $SO^+(3,1)$ representation ϕ_0, then the two "rotational" eigenvalues λ_3, λ_4 of $\phi(g)$ are non-real. If $\lambda_2 = 1/\lambda_1$ and $\lambda_4 = 1/\lambda_3$, then we have $\lambda_3 + 1/\lambda_3 \in T(\lambda_1)$, whence λ_3, $1/\lambda_3$ are roots of a quadratic over the field $T(\lambda_1)$. Therefore K has degree 2 over $T(\lambda_1)$, and $T(\lambda_1) = K \cap \mathbb{R}$.

The hypotheses of the following Corollary are obviously not best possible, but the Corollary suits our purpose in obtaining representations for which the matrix entry field is algebraic over $\mathbb{Q}(u_1, \ldots, u_n)$.

Corollary 1.1. Let G, K be as in Proposition 1, and let us impose the extra hypothesis that all off-diagonal entries of a are non-zero. Let c be the matrix

$$c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{12} & 0 & 0 \\ 0 & 0 & a_{12}a_{23} & 0 \\ 0 & 0 & 0 & a_{12}a_{23}a_{34} \end{bmatrix} .$$

Then the field generated by the matrix entries of $c\,G\,c^{-1}$ is precisely K.

Proof. $c\,G\,c^{-1}$ is generated by $a' = cac^{-1}$, $b' = cbc^{-1}$. However, since diagonal matrices commute, $b = b'$; also, the $(1,2)$, $(2,3)$, $(3,4)$ entries of the matrix a' are all equal to 1. We now apply Proposition 1 to the group $G' = \langle a', b' \rangle$, and observe that the matrix entries of a' are all expressible as products of the a'_σ and their inverses. □

4. Canonical form for representations

In this section we explain how to conjugate a representation $\phi : \pi_1(M) \to SL(4, \mathbb{R})$ into a convenient "standard" form. As always, we are assuming that M is a closed, orientable, hyperbolic 3-manifold and that its fundamental group is generated by a pair of elements α, β. Let a, b denote the images of α, β respectively under ϕ. Since all non-trivial elements of $\pi_1(M)$ are loxodromic, the matrix b is diagonalizable; also, if ϕ is sufficiently close to the "base" representation $\phi_0 : \pi_1(M) \to SO^+(3,1)$, by continuity, b will have two real eigenvalues λ_1, λ_2 and two other mutually conjugate eigenvalues λ_3, $\overline{\lambda_3}$ (here we are exploiting the fact that the characteristic polynomial of b has real coefficients.) We make the further assumption that the diagonalization of b satisfies conditions (i), (ii), (iii) of Proposition 1. As explained in Remark 2 above, condition (ii) is automatically met for representations ϕ close to ϕ_0, and in all observed cases the other two conditions are also met; should this happen not to be so, one could resort to changing the generating set for $\pi_1(M)$.

Since b is an automorphism of $\mathbb{R}^4$, we may choose real eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ for λ_1, λ_2

respectively, and eigenvectors $\mathbf{v}_3$, $\overline{\mathbf{v}_3}$ for λ_3 , $\overline{\lambda_3}$. Since these four eigenvectors form a linearly independent set over $\mathbb{C}$, the set of real vectors $\{\mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 + \overline{\mathbf{v}_3} , i\,(\mathbf{v}_3 - \overline{\mathbf{v}_3})\}$ is also linearly independent over $\mathbb{C}$, hence also over $\mathbb{R}$. Therefore each vector in $\mathbb{R}^4$ is uniquely expressible as a real linear combination of $\mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 + \overline{\mathbf{v}_3} , i\,(\mathbf{v}_3 - \overline{\mathbf{v}_3})$, or alternatively as a linear combination $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \overline{k_3}\,\overline{\mathbf{v}_3}$ where k_1 , k_2 are real.

Let us now consider the matrices a_1 , $b_1 \in \mathrm{SL}(4,\mathbb{C})$ representing the same linear transformations as a , b respectively, but with respect to the basis of eigenvectors $\{\mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 , \overline{\mathbf{v}_3}\}$. Then b_1 is diagonal, and from the discussion in the previous paragraph, the entries of the upper left 2×2 submatrix of a_1 are real. Assuming that the off-diagonal entries of a_1 are non-zero, we see that we may adjust $\mathbf{v}_2$ by a real scalar so that the $(1,2)$ entry of a_1 is 1 . We may independently adjust the last two eigenvectors by scalars so that the $(2,3)$ and $(3,4)$ entries are also 1 . Then, from Corollary 1.1, all entries of a_1 lie in the field K obtained by adjoining the eigenvalues $\lambda_1 , \lambda_2 , \lambda_3 , \lambda_4$ to the trace field T .

We can improve matters slightly by conjugating a_1 , b_1 to new matrices a_2 , b_2 by means of a further change of basis, namely to $\{\mathbf{v}_1 , \mathbf{v}_2 , a(\mathbf{v}_1) , a(\mathbf{v}_2)\}$. That this set of vectors is linearly independent (for representations within a neighbourhood of ϕ_0) follows from the ergodicity of the action of $\pi_1(M)$ on the boundary of $\mathbb{H}^3$. Note that each of these basis vectors is real, so the resulting matrices have real entries. Indeed they have the convenient form

$$a_2 = \begin{bmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix} \quad , \quad b_2 = \begin{bmatrix} \lambda_1 & 0 & * & * \\ 0 & \lambda_2 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \quad .$$

Since the transition matrix corresponding to this latest change of basis has entries in K , we infer that the entries of a_2 , b_2 are in $K \cap \mathbb{R}$. In particular, if the characteristic polynomial of b is reciprocal, by Remark 3.3 a_2 , b_2 are matrices over the field $T(\lambda_1)$.

Remark 4.1. Matrices with reciprocal characteristic polynomials are particularly desirable, as their eigenvalues admit relatively simple expressions in terms of the coefficients. Indeed, the roots of $1 + px + qx^2 + px^3 + x^4$ are

$$\frac{1}{4}\left(-p - \sqrt{p^2 - 4q + 8} \;\pm\; \sqrt{2\left(p^2 - 2q - 4 + p\sqrt{p^2 - 4q + 8}\right)}\right) \quad ,$$
$$\frac{1}{4}\left(-p + \sqrt{p^2 - 4q + 8} \;\pm\; \sqrt{2\left(p^2 - 2q - 4 - p\sqrt{p^2 - 4q + 8}\right)}\right) \quad ,$$

as is easily verified by substituting $y = x + 1/x$.

5. Infinitesimal and actual deformations

Since a representation of $\pi_1(M)$ into a linear group G is determined by its effect on the generators of $\pi_1(M)$, we may consider the representation variety $\mathrm{Hom}(\pi_1(M) , G)$ as being a subspace of the k-fold product of G, where k is the number of generators. In particular, if $k = 2$ and $G = \mathrm{SL}(4, \mathbb{R})$, we consider the variety to be a subspace of $\mathrm{SL}(4, \mathbb{R}) \times \mathrm{SL}(4, \mathbb{R})$, which in turn embeds naturally into $\mathbb{R}^{32}$ if we use matrix entries as coordinates. Since M is closed, $\pi_1(M)$ has deficiency zero.

Given a manifold M, the first step in deciding whether $\phi_0 : \pi_1(M) \to \mathrm{SO}^+(3,1)$ deforms into $\mathrm{SL}(4, \mathbb{R})$ is to linearize the problem and see whether there exist perturbations $\phi : \pi_1(M) \to \mathrm{GL}(4, \mathbb{R})$ of ϕ_0 that preserve the group relations to first order. Such perturbations are called *infinitesimal deformations* or *deformations to first order*, and form the so-called *Zariski tangent space* at ϕ_0. Calculation of the dimension of the Zariski tangent space by means of a computer algebra program is fairly routine: one constructs a Jacobian matrix J out of the group relators, evaluates it at ϕ_0 and computes its nullity $\nu(J)$. In the present context J will have 32 columns, corresponding to the 32 coordinates, and 32 rows, corresponding to the 32 constraints given by the two group relators.

At this point it is necessary to discuss a fundamental issue regarding the practicalities of computation. A computer has two distinct modes of computation, namely integer and floating-point, each with its advantages and drawbacks. Assuming freedom from programming error, integer computations are exact, and their output may be used directly in a mathematical proof. However, in many situations, for example if one wishes to solve a non-linear equation by means of an iterative method based on analytical principles, integer computations are not appropriate. Moreover, even when one finds oneself in a discrete world, for example a number field, integer computations can be so slow as to be impractical. Floating-point computations, on the other hand, are inherently inexact, however many decimal places of accuracy are used, since the result of any arithmetic operation is rounded off before being stored for further processing. With floating-point calculations it is possible to assert that a result is accurate within certain specified bounds if one keeps track of the propagation of round-off errors. Therefore, for example, with sufficient working accuracy and with due cause, one can assert that the result of a computation is non-zero; however, one can never justifiably assert that the result of a computation is exactly zero.

The computation of the dimension of the Zariski tangent space takes place in floating-point mode. Therefore, strictly speaking, by computing the nullity of J we are only computing an upper bound for the dimension. However, for theoretical reasons we do have a lower bound. There are two kinds of "inessential" infinitesimal deformations (abbreviated in this paragraph as "iid's") that we must exclude from our count. First, composition with inner automorphisms of $\mathrm{GL}(4, \mathbb{R})$ gives rise to $16 - 1 = 15$ dimensions of iid's (we subtract 1 from 16 as the centre of $\mathrm{GL}(4, \mathbb{R})$ has dimension 1.) Secondly, suppose that $H_1(M)$ has a free summand of rank r. Then there exists an epimorphism η of $\pi_1(M)$ to the direct sum of r copies of $\mathbb{Z}$, and we have r independent 1-parameter families of iid's $\phi_{i,\lambda} : g \mapsto \lambda^{(p_i \circ \eta)(g)} \cdot \phi_0(g)$, where the parameter λ is a non-zero real number, and p_i is projection to the ith summand $(1 \le i \le r)$ [3] Manifolds for which $\nu(J)$ is found to equal $15 + r$ are *rigid*, in the sense that they do not admit deformations of the type we are seeking. This assertion is justified by the fact that $15 + r$ is both an upper bound (as a result of the computation) and a lower bound (from the theory.) The excess of $\nu(J)$ over $15 + r$ is the potential dimension of the character variety of deformations of ϕ_0 into $\mathrm{SL}(4, \mathbb{R})$, although quite apart from the uncertainty due to roundoff error, there is no guarantee that an infinitesimal deformation is integrable to an actual deformation (see [Kapovich, 2000], p.71 for a discussion of this issue.)

It is possible in principle to calculate $\nu(J)$ exactly, as J depends only on the base representation $\phi_0 : \pi_1(M) \to \mathrm{SO}^+(3, 1)$. However, in order to be able to paint a broad picture relatively quickly, we have chosen to compute $\nu(J)$ in floating-point mode to a large number (1000) of decimal places. To obtain this accuracy, we start with the representation $\rho : \pi_1(M) \to \mathrm{SL}(2, \mathbb{C})$ given by SnapPea to machine accuracy, and then increase the accuracy of ρ to the required level by means of a few iterations of Newton's method, using the group relations, before converting to a very accurate approximation of $\phi_0 : \pi_1(M) \to \mathrm{SO}^+(3, 1)$. In cases where the computed dimension of the Zariski tangent space exceeds the known dimension of the representation variety, the stated dimension of the Zariski tangent space is therefore not rigorous, albeit almost certainly correct.

This computation was carried out for the first 4500 2-generator closed orientable manifolds in the Hodgson-Weeks census [Hodgson and Weeks, 2000], and it was found that only 61 of these manifolds admit non-trivial infinitesimal deformations. Of these, 21 have been shown rigorously to admit actual deformations, and there is compelling numerical evidence that a further 31 do. Of the remaining 9 manifolds, 3 have been proved to be rigid using a certain third order obstruction explained in [Cooper et al., 2005], and numerical evidence strongly suggests that the remaining 6 are rigid. These results are set out in Table 1. A check mark in the column "rig." indicates that the variety has been computed exactly, and has been shown rigorously to have the stated dimension; absence of a check mark should be interpreted as "compelling numerical evidence only". Details of the computation for the manifold m007(3, 1) are given in section 7 of this paper.

The outcome for v2678(2, 1) is of interest. This manifold apparently has a 5-dimensional space of essential infinitesimal deformations, whereas the $\mathrm{SL}(4, \mathbb{R})$ character variety apparently has two 3-dimensional branches meeting in a 1-dimensional subvariety containing $\chi(\phi_0)$. It would follow that $\chi(\phi_0)$ is not a smooth point of the variety. It is also of some interest that from the first 2000 manifolds in the census, v2678(2, 1) appears to be the only example admitting deformations into $\mathrm{SO}(4, 1)$.

[3] Of course these are not infinitesimal deformations into $\mathrm{SL}(4, \mathbb{R})$, but for computational expediency we first consider all infinitesimal deformations into $\mathrm{GL}(4, \mathbb{R})$ and then take an appropriate subspace.

manifold	volume	inf.	actual	rig.	manifold	volume	inf.	actual	rig.
m007(3,1)	1.014941	1	1	✓	s912(0,1)	4.059766	2	2	
m036(−3,2)	2.029883	1	1	✓	m401(−2,3)	4.059766	2	2	
m034(−4,1)	2.195964	1	1	✓	v825(4,1)	4.059766	1	1	
m160(−3,2)	2.595387	1	1	✓	m358(1,3)	4.059766	1	1	
m082(1,3)	2.786804	1	1	✓	m368(−4,1)	4.059766	1	0	
m078(5,1)	2.816179	1	1	✓	s778(−3,2)	4.059766	2	2	✓
m100(2,3)	2.882494	1	1	✓	s779(1,2)	4.059766	2	2	
m149(−4,1)	3.044824	1	0	✓	m395(−2,3)	4.059766	2	2	
m188(2,3)	3.044824	1	1	✓	s440(−1,3)	4.059766	1	1	
m247(−1,3)	3.044824	1	1	✓	v2678(2,1)	4.116968	5	3	
m159(2,3)	3.044824	1	0	✓	s500(4,1)	4.116968	1	1	
m115(5,2)	3.060334	1	1	✓	v2334(−1,2)	4.116968	2	0	
m121(−4,3)	3.195780	1	1	✓	s490(−4,1)	4.116968	1	1	
m336(−1,3)	3.663862	2	2	✓	s668(4,1)	4.221804	1	1	
m303(−1,3)	3.663862	1	1	✓	s518(−1,4)	4.400901	1	1	✓
s572(1,2)	3.663862	1	1	✓	v2817(−3,1)	4.407345	1	1	
m293(4,1)	3.663862	1	0	✓	s533(1,4)	4.422687	1	1	
s645(−1,2)	3.663862	1	1	✓	m402(2,3)	4.436783	1	1	
m312(−1,3)	3.663862	2	2	✓	v1461(1,3)	4.598034	1	1	
s778(−3,1)	3.663862	1	1	✓	s636(−1,4)	4.598853	1	1	✓
m304(5,1)	3.663862	1	1		s618(1,4)	4.598853	1	1	✓
s682(−3,1)	3.663862	3	2		v1222(−5,1)	4.626243	1	1	
s350(−4,1)	3.663862	1	0		v1251(4,3)	4.686034	1	1	
m294(4,1)	3.663862	1	0		s666(−4,3)	4.686034	1	1	
s495(1,2)	3.663862	1	0		v2413(−3,2)	4.686034	2	0	
s235(−3,4)	3.794090	1	1		v1695(−5,1)	4.834441	1	1	
m290(−3,4)	3.818259	1	1		v1860(2,3)	4.974542	1	1	
m350(−1,3)	3.861814	1	1		v1847(−4,3)	5.016110	1	1	
m360(−2,3)	3.861814	1	1		v1845(−5,2)	5.017640	1	1	
s287(3,4)	3.896345	1	1		v3283(−3,1)	5.171469	1	1	
m346(2,3)	3.933297	1	1						

Table 1: Infinitesimal and actual deformations of closed manifolds

6. Summary of the computational procedure

We are now ready to give an outline of the entire computational procedure. A feature of the method is that the trace field, matrix entry field and exact matrix entries are derived by informed guesswork based on numerical data; it is only at the very last step that the existence of the representation variety is proved, by checking formally that the proposed matrices $\Psi(\alpha)$, $\Psi(\beta)$ satisfy the two relations of $\pi_1(M)$. Initial data needed to get started, *i.e.* generators and relations for $\pi_1(M)$ and ϕ_0 accurate to a few decimal places, can easily be obtained using SnapPea [Weeks, 1990] or Snap [Coulson et al., 2000].

Step 1. Compute the Zariski tangent space at the $SO^+(3,1)$ representation ϕ_0 to a high degree of accuracy, using the method described in §6. If the manifold is found to be rigid, then there are no deformations and we quit.

Step 2. Apply a small random perturbation to ϕ_0 by slightly modifying the matrices $\phi_0(\alpha)$, $\phi_0(\beta)$, and then perform Newton's method to try to converge to a representation $\phi_1 : \pi_1(M) \to \mathrm{SL}(4, \mathbb{R})$ not conjugate to ϕ_0 (the test is to compare characteristic polynomials of matrices $\phi_0(g)$, $\phi_1(g)$ for various elements $g \in \pi_1(M)$.) The 32 unknowns at each stage of the Newton process are the adjustments to the 32 matrix entries needed to cancel out the residuals to first order. The two defining relations of the group provide 32 constraints for these unknowns, but in the case where $H_1(M)$ has a free direct summand it will be necessary to add contraints $\det(\phi_1(\alpha)) = 1$, $\det(\phi_1(\beta)) = 1$. Since there are now more equations than unknowns, we use the QR-decomposition of the matrix of coefficients (*i.e.* the Jacobian matrix) to find a least squares solution to the linear system.

It can happen that the manifold is rigid despite the existence of a non-trivial Zariski tangent; this phenomenon manifests itself here by the Newton process refusing to converge[4]. To prove rigidity one then has to compute a higher order obstruction, for example the third order obstruction described in [Cooper et al., 2005]. Referring to Table 1, in this way it was proved that m149(−4, 1), m159(2, 3), m293(4, 1) are all rigid. The six outstanding cases listed in Table 1 have not been checked rigorously, but there is strong evidence that they are rigid.

If the Newton process converges satisfactorily to a representation ϕ_1 distinguished from ϕ_0 by examination of characteristic polynomials, we move on to Step 3. We note that because we are using floating-point arithmetic, we cannot yet assert definitely that we have found a genuine representation ϕ_1.

Step 3. Find a suitable parametrization of the character variety. The character variety can always be parametrized by means of coefficients of characteristic polynomials, but the aim is to find parameters $u_1, \ldots, u_n$ for which the trace field has small degree over the field $\mathbb{Q}(u_1, \ldots, u_n)$. This usually involves a modest amount of experimentation. From Step 1 we have an upper bound on the number of parameters n.

From Step 2 we already know of at least one trace that varies as one moves away from ϕ_0, say $\mathrm{tr}(\phi(g))$ where $g \in \pi_1(M)$. Add a constraint to the Newton process of Step 2, declaring that this trace is some rational number reasonably close to the value of this trace at ϕ_0. This might make the convergence of the Newton process less robust, in which case it will be necessary to control the step-length, by multiplying the adjustments at each stage by some dynamically controlled scale factor. Once one has achieved convergence, see by means of LLL [Lenstra et al., 1982] (or an alternative, *e.g.* PSLQ [Bailey and Ferguson, 1991]) whether all other traces now appear to be algebraic numbers, and if so, note the degree and complexity of their minimal polynomials. If some trace appears not to be algebraic, select it as an additional parameter, add an extra constraint declaring it to equal an appropriate rational number, and

[4]This is probably because the vanishing of the first order obstruction to the existence of a variety implies the vanishing of the second order obstruction also; see [Cooper et al., 2005]

repeat the process. We note that the proof of Proposition 1 provides a list of traces that is sufficient for this purpose.

Eventually, we should achieve a numerical approximation to a representation where LLL declares that all traces are algebraic. In practice, one would not wish to compute a variety for which $n > 2$, although we have used this method to work out the deformation variety for one 3-dimensional example, the cusped manifold m007. In the last section of this paper, we take the reader in some detail through a 1-dimensional example, namely the closed manifold m007(3, 1) known as Vol3.

Step 4. Compute the trace field T, as an extension of finite degree over the field $\mathbb{Q}(u_1, \ldots, u_n)$. This is best explained by means of a worked example, but broadly speaking the method is to compute generators for the trace field evaluated at each point of a cubic lattice in parameter space, using LLL, and then obtain generators for T over $\mathbb{Q}(u_1, \ldots, u_n)$ using polynomial interpolation. The points of the lattice should be chosen to have rational coordinates, where the denominators are not too large, and should be reasonably close together. If $n > 1$ it will be necessary to interpolate in each of the n coordinate directions, so as to obtain a polynomial in the n variables $u_1, \ldots, u_n$. In practice a row of data points $(x_1, y_1), \ldots, (x_k, y_k)$ for the interpolation may not lie on the graph of the desired polynomial, but $(x_1, \lambda_1 y_1), \ldots, (x_k, \lambda_k y_k)$ will lie on the graph for integers λ_i that are small relative to the y_i. Determining these "multipliers" λ_i is perhaps the trickiest part of the entire process, but skill comes with practice! It is usually self-evident when the correct λ_i have been found, as the degree of the interpolating polynomial is then much smaller than the number of data points.

Once the trace field T has been computed, choose once and for all a basis $\tau_1, \ldots, \tau_m$ for T as a vector space over $\mathbb{Q}(u_1, \ldots, u_n)$.

Step 5. Choose an element of $\pi_1(M)$ whose image is to fulfill the role of the matrix b in §4. If at all possible, to avoid field extensions of uncomfortably large degree, b should have a reciprocal characteristic polynomial. Fortunately such elements have proved to be available for all varieties that we have computed. Once this matrix has been chosen, decide on a basis for the field K generated by the matrix entries over $\mathbb{Q}(u_1, \ldots, u_n)$. This basis will of course be a function of the parameters.

Step 6. Express each trace used in the proof of Proposition 1 as a linear combination $\sum_{i=1}^{n} f_i \tau_i$, where the f_i are rational functions of the parameters $u_1, \ldots, u_n$. Again one uses polynomial interpolation, but for determining the coefficients f_i at each lattice point the required tool is a facility for detecting integer relations. Pari's *lindep* is such a function [Batut et al., 1990]. Neither Mathematica nor Maple has a built-in integer relation facility, but notebooks/worksheets are available that will perform this task.

Step 7. Write a program that uses the method of §4 to compute generating matrices a, b in standard form, to a large number of decimal places, for each point of a lattice in parameter space. The "large number" just referred to depends on the complexity of the situation, but typically 2000 decimal places are appropriate, so that the integer relation detector *lindep* can produce results that are not spurious. The number of points in the lattice depends also on the complexity, but usually we have found that an n-cube of edge-length 50 is sufficient. Fortunately this method of computing numerical approximations to representations is remarkably fast, and even the computation of the 2500 representations for a 50×50 lattice can usually be accomplished in an hour or two.

Step 8. Determine an exact expression for each matrix entry in terms of the parameters, using *lindep* and polynomial interpolation, as in Step 4.

Step 9. Use the formal algebra capabilities of Maple or Mathematica to verify that the exact matrices obtained in the previous step satisfy the group relations.

7. The manifold Vol3

The first manifold in the census to admit $\mathrm{SL}(4,\mathbb{R})$-deformations is the third manifold listed in the census, known as "Vol3". Its denotation in the census is $\mathrm{m007}(3,1)$, meaning that it is obtained by $(3,1)$-surgery on the cusped manifold m007. m007 in turn is the seventh manifold in the census of cusped manifolds obtained by gluing together up to five ideal tetrahedra (for historical reasons, the prefixes "s", "v" are used for 6, 7 ideal tetrahedra respectively.) Surgery coefficients are given relative to the basis $\{[\text{shortest curve}], [\text{second shortest curve}]\}$ for the first homology group of the cusp cross-section.

SnapPea gives the following presentation for the fundamental group of Vol3:

$$\pi_1(\mathrm{Vol3}) \;=\; \langle a\,,\, b \;\mid\; aabbABAbb\,,\, aBaBabaaab\rangle \;,$$

and the following numerical approximation to a lift to $\mathrm{SL}(2,\mathbb{C})$ of the discrete faithful representation into $\mathrm{PSL}(2,\mathbb{C})$:

$$a \mapsto \begin{bmatrix} 0.85323069669636 - 1.25244865807008i & -0.50000000000000 + 0.86602540378443i \\ 0.15937498068339 - 2.13725528220312i & 0.37151417469522 + 1.95955543925663i \end{bmatrix}$$

$$b \mapsto \begin{bmatrix} 0.50000000000000 + 0.86602540378443i & -0.28652283178103 + 0.59441165159384i \\ 0.65803700647625 + 1.36514378766279i & 0.00000000000000 + 0.00000000000000i \end{bmatrix}$$

Here, for notational convenience, we are using the upper-case letters A, B to denote a^{-1}, b^{-1} respectively. Since we shall only be considering representations of $\pi_1(\mathrm{Vol3})$ into linear groups within matrix algebras, we shall take the liberty of considering a, b as matrices and write the group relations as

$$r_1(a,b) := aabb - BBaba = 0 \quad , \quad r_2(a,b) := aBaBa - BAAAB = 0 \ .$$

We begin by using Newton's method to improve the accuracy of the $\mathrm{SL}(2,\mathbb{C})$ representation. The procedure is to compute the residuals y_1, y_2 of r_1, r_2, namely their actual starting values (which are already close to the zero 2×2 matrix), and then solve a linear system to find changes da, db in the matrices a, b that cancel out these residuals to first order. For this we compute formal expressions for the changes dr_1, dr_2 in r_1, r_2 effected by changing a, b to $a+da$, $b+db$; we then solve the system $dr_1 = -y_1$, $dr_2 = -y_2$.

"Differentiating" $mM = I$, we obtain $(dm)M + m(dM) = 0$, whence $dM = -M(dm)M$. Therefore

$$\begin{aligned}
dr_1 = {} & \big((da)abb + a(da)bb + aa(db)b + aab(db)\big) \\
& - \big((-B(db)B)Baba + B(-B(db)B)aba + BB(da)ba + BBa(db)a + BBab(da)\big) \ , \\
dr_2 = {} & \big((da)BaBa + a(-B(db)B)aBa + aB(da)Ba + aBa(-B(db)B)a + aBaB(da)\big) \\
& - \big((-B(db)B)AAAB + B(-A(da)A)AAB + BA(-A(da)A)AB \\
& \quad + BAA(-A(da)A)B + BAAA(-B(db)B)\big) \ .
\end{aligned}$$

The 8 unknowns of the linear system are the entries of the matrices da, db, and entry-by-entry comparison of dr_i with $-y_i$ $(i = 1,2)$ provides 8 equations in these unknowns. All this is easy to program in Mathematica or Maple, and after a small number of iterations we arrive at an $\mathrm{SL}(2,\mathbb{C})$-representation accurate to 1000 decimal places. We note in the case where $H_1(M)$ has a free summand, it is necessary to add constraints $\det(a) = 1$, $\det(b) = 1$ and then use a QR-decomposition on the resulting rectangular matrix of coefficients.

The accurate $\mathrm{SL}(2,\mathbb{C})$-representation that we have just obtained is now converted to an $\mathrm{SO}^+(3,1)$-representation, using the method explained in §2.1. Let a_0, b_0 denote the images in $\mathrm{SO}^+(3,1)$ of the group generators. The Jacobian matrix J, whose nullity we need to compute for Step 1, is obtained using the above expressions for dr_1, dr_2, with a_0, b_0 in place of a, b. Since a_0, b_0 have in total 32 entries, J has size 32×32. We then find that the rank of J is 16, whence the nullity of J is also 16 and from the discussion of §6 there is one dimension's worth of essential infinitesimal deformations[5].

We now proceed to Step 2. We perturb a_0, b_0 very slightly to matrices a_1, b_1, and then try to converge to a representation using Newton's method. The linear system to be solved at each iteration uses the matrix J of the previous step, but with the current matrices a_1, b_1 in place of a_0, b_0. As an example, one can obtain matrices a_1, b_1 giving a representation to 500 decimal places of accuracy, with characteristic polynomials as follows:

[5]Strictly speaking, because we are working in floating-point mode, we may only assert at present that the space of essential infinitesimal deformations has dimension at most 1.

$$
\begin{aligned}
\mathrm{charpoly}(a_1) \;=\;& \;\; 1.0000000000000000000000000000000000000000000000000 \\
& - 2.0000000000000886784402728253059992711608536655069\,x \\
& - 2.0000000000000886784402728253059992711608536655069\,x^3 \\
& + x^4 \;\; , \\
\mathrm{charpoly}(b_1) \;=\;& \;\; 1.0000000000000000000000000000000000000000000000000 \\
& - 1.0000000000000000000000000000000000000000000000000\,x \\
& - 3.0000000000003547137610913090878628538644597935372\,x^2 \\
& - 1.0000000000000000000000000000000000000000000000000\,x^3 \\
& + x^4 \;\; .
\end{aligned}
$$

On the other hand, the characteristic polynomials of a_0 , b_0 are:

$$
\begin{aligned}
\mathrm{charpoly}(a_0) &= 1 - 2x - 2x^3 + x^4 \;\; , \\
\mathrm{charpoly}(b_0) &= 1 - x - 3x^2 - x^3 + x^4 \;\; ,
\end{aligned}
$$

and we are encouraged to try taking the trace of a_1 as parameter, v say. Note the symmetric nature of the characteristic polynomials of a_1 , b_1 ; also, note that the trace of b_1 appears to be constant.

Running the Newton program again with the extra constraint $\mathrm{tr}(A_1) = 2.001$, we obtain a representation (to the same accuracy), with

$$
\begin{aligned}
\mathrm{charpoly}(a_1) \;=\;& \;\; 1.0000000000000000000000000000000000000000000000000 \\
& - 2.0010000000000000000000000000000000000000000000000\,x \\
& - 2.0010000000000000000000000000000000000000000000000\,x^3 \\
& + x^4 \;\; , \\
\mathrm{charpoly}(b_1) \;=\;& \;\; 1.0000000000000000000000000000000000000000000000000 \\
& - 1.0000000000000000000000000000000000000000000000000\,x \\
& - 3.0040010000000000000000000000000000000000000000000\,x^2 \\
& - 1.0000000000000000000000000000000000000000000000000\,x^3 \\
& + x^4 \;\; ,
\end{aligned}
$$

and we suspect strongly that the middle term of the characteristic polynomial of b_1 is $-(v^2-1)$ (A keen observer might have noticed this earlier.)

We now try to identify the trace field. A quick search reveals that the trace of the commutator $a_1 b_1 A_1 B_1$ appears to be irrational, and that, according to LLL, for rational v it appears to be a root of a quadratic over $\mathbb{Q}$. We run the Newton program again for $v = 2 + \frac{1}{100+i}$ $(1 \le i \le 10)$, and print out the minimal polynomials of this trace, as given by LLL. The results are as follows:

i	$p_i(x)$
1	$5181915799 - 1490060879x + 104060401x^2$
2	$5389356866 - 1549614193x + 108243216x^2$
3	$5602964647 - 1610935039x + 112550881x^2$
4	$5822860162 - 1674058049x + 116985856x^2$
5	$6049165607 - 1739018191x + 121550625x^2$
6	$6282004354 - 1805850769x + 126247696x^2$
7	$6521500951 - 1874591423x + 131079601x^2$
8	$6767781122 - 1945276129x + 136048896x^2$
9	$7020971767 - 2017941199x + 141158161x^2$
10	$83041 - 10000x$

We proceed to interpolate the first nine polynomials, assuming that the anomalous degree of the polynomial for $i = 10$ is caused by accidental rationality of $\mathrm{tr}(a_1 b_1 A_1 B_1)$ at $v = 2 + \frac{1}{110}$. The polynomials are only defined up to integer multiples; however, the coefficients appear to lie on a "nice" curve, so it probably will not be necessary to find "multipliers" λ_i.

Indeed, interpolation reveals that the minimal polynomial for this trace is

$$1 \; - \; (v^4 - 2)x \; + \; (2v^4 + 4v^2 + 1)x^2 \; ,$$

with discriminant $v^2(v^2+2)^2(v^2-4)$. We are led to surmise that the trace field is

$$T \; = \; \mathbb{Q}(v)(\alpha) \; \text{,where } \alpha = \sqrt{v^2 - 4} \; ;$$

some support for this conjecture is provided by computation of several other traces. Incidentally, we note that for $v = 2 + \frac{1}{110}$, $\sqrt{v^2 - 4}$ is the rational number $\frac{21}{110}$, explaining the anomalous polynomial of degree 1 for $i = 10$.

We now proceed to Step 5, where we choose the matrix to be diagonalized. Since the characteristic polynomial of b_1 is reciprocal, satisfies the conditions of Proposition 1 and has roots that are relatively simple expressions in v, we select b_1 for this purpose (in fact a_1 would have done equally well.) Two of the eigenvalues of b_1 are real for $v > 2$; they are

$$\lambda_1 = \frac{1}{4}\left(1 + \sqrt{5 + 4v^2} + \sqrt{2\left(-5 + 2v^2 + \sqrt{5 + 4v^2}\right)}\right) \; ,$$
$$\lambda_2 = \frac{1}{4}\left(1 + \sqrt{5 + 4v^2} - \sqrt{2\left(-5 + 2v^2 + \sqrt{5 + 4v^2}\right)}\right) \; .$$

If it should transpire that our guess for the trace field is correct, we can already predict from the discussion of §4 that the field generated by the matrix entries will be $K = \mathbb{Q}(v)(\alpha, \lambda_1)$, and that a vector space basis for K over $\mathbb{Q}(v)$ will be $\{1, \gamma, \beta, \beta\gamma, \alpha, \alpha\gamma, \alpha\beta, \alpha\beta\gamma\}$, where $\alpha = \sqrt{v^2 - 4}$, $\beta = \sqrt{4v^2 + 5}$, $\gamma = \sqrt{2(-5 + 2v^2 + \beta)}$. We note that K is a not a Galois extension of $\mathbb{Q}(v)$, as for $v > 2$ K does not contain the two non-real roots of the minimal polynomial of λ_1. It has an automorphism σ_1 negating α and fixing γ, and an automorphism σ_2 fixing α and negating γ; these automorphisms commute and generate the automorphism group of $K : \mathbb{Q}(v)$, which is a therefore a Klein group of order 4.

Moving on to Step 6, we would now like to produce representations of high numerical accuracy (2000 decimal places to be safe) for a sequence of values of the parameter v, say $v = 2 + \frac{1}{110+i}$ $(1 \le i \le 20)$ (recall that we wish to avoid $v = 2 + \frac{1}{110}$.) In the case of Vol3 it is feasible to obtain these directly from the Newton program; however, typically this is too slow, and a much better approach is to use the method of Proposition 1.

Thus our immediate task is to identify each of the 18 traces used in that Proposition as elements of the trace field $T = \mathbb{Q}(v)(\alpha)$. For this we run the Newton program as we did earlier for the trace of $a_1 b_1 A_1 B_1$, but this time, for each trace t, we run the integer relation detector *lindep* on the vector $(1, \alpha, t)$, and interpolate the resulting coefficients over the data points. In this way we obtain a generic relation $p_0(v) + p_1(v)\alpha + p_2(v)t = 0$, where each p_i is a polynomial in v with integer coefficients, and we record that $t = -\frac{p_0(v)}{p_2(v)} - \frac{p_1(v)}{p_2(v)}\alpha$.

The results for the 18 traces are as follows:

$\mathrm{tr}(a) = v$	$\mathrm{tr}(aa) = v^2$	$\mathrm{tr}(aaa) = 3v + v^3$
$\mathrm{tr}(ab) = v$	$\mathrm{tr}(aab) = 1$	$\mathrm{tr}(aaab) = v$
$\mathrm{tr}(abb) = v$	$\mathrm{tr}(aaB) = -1 + 2v^2$	$\mathrm{tr}(aaabb) = v$
$\mathrm{tr}(abbb) = v^3$	$\mathrm{tr}(aabb) = 1$	$\mathrm{tr}(aaBab) = (1/2)\big(-2v + v^3 + (2 + v^2)\alpha\big)$
	$\mathrm{tr}(abab) = v^2$	$\mathrm{tr}(aabaB) = (1/2)\big(-2v + v^3 - (2 + v^2)\alpha\big)$
	$\mathrm{tr}(aBab) = 1$	$\mathrm{tr}(aaBabb) = (1/2)\big(2v - v^3 + v^5 + (2 - v^2 - v^4)\alpha\big)$
		$\mathrm{tr}(abaBabb) = (1/2)\big(2v - v^3 + v^5 + (2 - v^2 - v^4)\alpha\big)$
		$\mathrm{tr}(abaBBabbb) = -3v + 2v^3 + 2v^5$

These are incorporated into a short program that produces canonical forms a_2, b_2 for the generating matrices, by solving the linear systems given in the proof of Proposition 1 and then conjugating as described in §4.

The penultimate stage of the process is to apply *lindep* and polynomial interpolation to our data, so as to obtain exact expressions for the matrix entries. The $(2, 3)$ entry of the matrix a_2 gives a good idea as to what is involved here. Let us denote this entry x. For each $v = 2 + \frac{1}{110+i}$ $(1 \le i \le 20)$, we apply *lindep* to the 9-component vector $(1, \gamma, \beta, \beta\gamma, \alpha, \alpha\gamma, \alpha\beta, \alpha\beta\gamma, x)$. The output from Pari is a sequence of 20 9-component vectors $\mathbf{w}_i$ $(1 \le i \le 20)$, where each component is an integer of approximately 17 digits. The first component of each vector is the integer y_i in the following list, from which the prospect of fitting a reasonable polynomial looks bleak:

$$
\begin{array}{rlrl}
y_1 = & 9671015960804800 & \lambda_1 = & 1 \\
y_2 = & -510168310445250 & \lambda_2 = & -20 \\
y_3 = & -1195546153052400 & \lambda_3 = & -9 \\
y_4 = & 2835383895633100 & \lambda_4 = & 4 \\
y_5 = & 11949122303635440 & \lambda_5 = & 1 \\
y_6 = & -349544299928850 & \lambda_6 = & -36 \\
y_7 = & -2649179302985600 & \lambda_7 = & -5 \\
y_8 = & -3484248880566300 & \lambda_8 = & -4 \\
y_9 = & -1628653880584800 & \lambda_9 = & -9 \\
y_{10} = & 3852395824372810 & \lambda_{10} = & 4 \\
y_{11} = & 16193135112008400 & \lambda_{11} = & 1 \\
y_{12} = & -94497839870700 & \lambda_{12} = & -180 \\
y_{13} = & -17860109073398800 & \lambda_{13} = & -1 \\
y_{14} = & 4686438400061250 & \lambda_{14} = & 4 \\
y_{15} = & 2185299690910560 & \lambda_{15} = & 9 \\
y_{16} = & -5156780125192300 & \lambda_{16} = & -4 \\
y_{17} = & -4325046517516800 & \lambda_{17} = & -5 \\
y_{18} = & -629535323374650 & \lambda_{18} = & -36 \\
y_{19} = & 23742505309322800 & \lambda_{19} = & 1 \\
y_{20} = & 6216057796580460 & \lambda_{20} = & 4
\end{array}
$$

But we have to bear in mind that the vectors output by *lindep* are homogeneous, and that Pari will reduce each vector so that the GCD of its components is 1. Note that y_1, y_5, y_{11}, y_{13}, y_{19} look (up to sign) as if they stand a fair chance of lying on the correct polynomial curve, so we begin by fitting a polynomial $p(x)$ to these five data points. Our desired polynomial will probably have higher degree than $p(x)$, but we hope that $p(x)$ will be a close approximation. This is given support by the fact that $p(i)/y_i$ is very close to being an integer for all data points, so we take these integers as our multipliers λ_i. The list of the 20 multipliers λ_i is given above, alongside the y_i.

We now interpolate the components of the vectors $\lambda_i \mathbf{w}_i$ at all 20 data points, obtaining the following 9-component vector of polynomials in v:

$$
\begin{array}{rr}
p_1 = & 10v(v^2-1)(v^2-4)(4v^2+5) \\
p_2 = & -v(v^2+2)(v^2-4)(4v^2+5) \\
p_3 = & -2v(v^2-1)(v^2-4)(2v^2+15) \\
p_4 = & -v(v^2+2)(v^2-10) \\
p_5 = & 2v^2(v^2-1)(4v^2+5) \\
p_6 = & -v^2(v^2+2)(4v^2+5) \\
p_7 = & -2v^2(v^2-1)(2v^2-5) \\
p_8 = & (v^2+2)(7v^2-10) \\
p_9 = & -32(v^2-1)(v^2-5)(4v^2+5)
\end{array}
$$

Since the degree of each of these polynomials is much smaller than the degree of 19 that one would obtain generically, we are confident that we have the right answer, although this will not be proved until the final formal check of the two group relations.

Therefore we record that $x = -\frac{1}{p_9}\sum_{i=1}^{8} p_i v_i$, where $\{v_1, \ldots, v_8\}$ is the basis we have chosen for the matrix entry field over $\mathbb{Q}(v)$, and we are done. The other entries of a_2, b_2 are computed similarly.

Once exact expressions for all entries of the generators a_2, b_2 have been computed, it remains for Maple or Mathematica to check that they are correct by verifying formally that the group relations are satisfied, *i.e.* that $r_1(a_2, b_2) = r_2(a_2, b_2) = 0$. This can take a moderate amount of time; the key is to simplify aggressively at each stage of the computation.

Remark 8.1. The real hyperbolic representation occurs at $v = 2$. There are in fact two non-conjugate $SL(4, \mathbb{C})$ representations for each $v \in \mathbb{C}$, $v \neq 2$, corresponding to the two square roots of $\alpha^2 = v^2 - 4$. There is a symmetry of the variety interchanging these two representations, induced by the *contragredient* automorphism of $GL(4, \mathbb{C})$ that assigns to each matrix the transpose of its inverse.

Remark 8.2. Further experimentation can sometimes result in a conjugate family of representations with simpler expressions for the matrix entries. In the case of Vol3, a significant improvement may be obtained by exploiting symmetries of the manifold. The symmetry group of Vol3 is semihedral of order 16 [Hodgson and Weeks, 1994], and in particular there is a rotational symmetry ρ of order 4, fixing the axis of $aBaba$ and and inducing an automorphism ρ_* of $\pi_1(\text{Vol3})$ sending a to BA and b to aba. Thus we may form the orbifold fundamental group $\pi_1(\text{Vol3}/\langle\rho\rangle)$ as a supergroup of $\pi_1(\text{Vol3})$ by adding a new generator u together with relations $u^4 = 1$, $Uau = BA$, $Ubu = aba$. It follows easily that u^2a has order 2, and that $\pi_1(\text{Vol3}/\langle\rho\rangle)$ is generated by u, u^2a. Starting from the above family of representations and considering $\pi_1(\text{Vol3}/\langle\rho\rangle)$ as a carefully chosen group of isometries of 3-dimensional complex hyperbolic space, for a suitable range of values of the parameter v (see [Cooper et al., 2005]), the following much simpler curve of representations was found:

$$\Psi(u) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{v^2-4}{v^2+8}} & \sqrt{\frac{-2v^2-4}{v^2+8}} \\ 0 & 0 & \sqrt{\frac{-2v^2-4}{v^2+8}} & -\sqrt{\frac{v^2-4}{v^2+8}} \end{bmatrix},$$

$$\Psi(u^2a) = \begin{bmatrix} p & 0 & q & 0 \\ 0 & p^* & 0 & q^* \\ q & 0 & -p & 0 \\ 0 & q^* & 0 & -p^* \end{bmatrix},$$

where

$$p = \frac{1}{4}\left(v + \sqrt{v^2+8}\right) \quad , \quad q = -\frac{1}{2\sqrt{2}}\sqrt{4 - v^2 - v\sqrt{v^2+8}} \quad ,$$

$$p^* = \frac{1}{4}\left(v - \sqrt{v^2+8}\right) \quad , \quad q^* = -\frac{1}{2\sqrt{2}}\sqrt{4 - v^2 + v\sqrt{v^2+8}} \quad .$$

Remark 8.3. Earlier, we showed that the trace field contained $\mathbb{Q}(v)(\alpha)$, but did not prove equality. However, we can show equality as follows. Let K be the matrix entry field used in the main part of this section, and let K' be the matrix entry field of Remark 8.2. Since the trace field is invariant under conjugation, it must lie in $K \cap K' = \mathbb{Q}(v)(\alpha)$, QED.

References

[Akbulut and McCarthy, 1990] Akbulut, S. and McCarthy, J. (1990). Casson's invariant for oriented homology spheres: an exposition. *Mathematical Notes, Princeton University Press*, 36.

[Bailey and Ferguson, 1991] Bailey, D. and Ferguson, H. (1991). A new polynomial time algorithm for finding relations among real numbers. Technical report, SRC Technical Report SRC-TR-92-066.

[Batut et al., 1990] Batut, C., Belabas, K., Bernardi, D., Cohen, H., and Olivier, M. (1990). Pari-gp calculator. Computer program, available from http://pari.math.u-bordeaux.fr.

[Boyer and Zhang, 1998] Boyer, S. and Zhang, X. (1998). On Culler-Shalen seminorms and Dehn filling. *Annals of Math.*, 148:737–801.

[Cooper et al., 1994] Cooper, D., Culler, M., Gillet, H., Long, D., and Shalen, P. (1994). Plane curves associated to character varieties of 3-manifolds. *Inventiones Math.*, 118:47–84.

[Cooper et al., 2005] Cooper, D., Long, D., and Thistlethwaite, M. (2005). Flexing closed hyperbolic 3-manifolds. *preprint*.

[Coulson et al., 2000] Coulson, D., Goodman, O. A., Hodgson, C. D., and Neumann, W. D. (2000). Computing arithmetic invariants of 3-manifolds. *Experimental Mathematics*, 9(1):127–152.

[Culler et al., 1987] Culler, M., Gordon, C. M., Luecke, J., and Shalen, P. (1987). Dehn surgery on knots. *Annals of Math.*, 125(2):237–300.

[Culler and Shalen, 1983] Culler, M. and Shalen, P. (1983). Varieties of group representations and splittings of 3-manifolds. *Annals of Math.*, 117:109–146.

[Epstein and Penner, 1988] Epstein, D. and Penner, R. (1988). Euclidean decompositions of noncompact hyperbolic manifolds. *J. Differential Geometry*, 27:67–80.

[Hodgson and Weeks, 1994] Hodgson, C. and Weeks, J. (1994). Symmetries, isometries and length spectra of closed hyperbolic three-manifolds. *Experimental Mathematics*, 3 No. 4:261–274.

[Hodgson and Weeks, 2000] Hodgson, C. and Weeks, J. (2000). A census of closed hyperbolic 3-manifolds. *Preprint*.

[Kapovich, 2000] Kapovich, M. (2000). *Hyperbolic Manifolds and Discrete Groups*, volume 183 of *Progress in Mathematics*. Birkhäuser, Boston, Massachusetts.

[Kojima and Miyamoto, 1991] Kojima, S. and Miyamoto, Y. (1991). The smallest hyperbolic 3-manifolds with totally geodesic boundary. *J. Differential Geom.*, 34:175–192.

[Lenstra et al., 1982] Lenstra, A., Lenstra, H., and Lovasz, L. (1982). Factoring polynomials with rational coefficients. *Math. Annalen*, 261:515–534.

[Maskit, 1980] Maskit, B. (1980). *Kleinian Groups*. Springer.

[Weeks, 1990] Weeks, J. (1990). Snappea, a program for creating and studying hyperbolic 3-manifolds, available from http://www.geometrygames.org/snappea/.