

## Fields of definition of canonical curves

D. D. Long and A. W. Reid

ABSTRACT. This note studies the question of which number fields can arise as canonical components.

### 1. Introduction

Let  $k \subset \mathbf{C}$  be a field. A complex algebraic set  $V \subset \mathbf{C}^n$  is *defined over*  $k$  if the ideal of polynomials  $I(V)$  vanishing on  $V$  is generated by a subset of  $k[x_1, \dots, x_n]$ . We say that a field  $k$  is *the field of definition* of  $V$  if  $V$  is defined over  $k$ , and if for any other field  $K \subset \mathbf{C}$  with  $V$  defined over  $K$ , then  $k \subset K$  (for the existence of the field of definition, see [14], Chapter III). Note that the field of definition of an algebraic variety depends on the embedding in a particular  $\mathbf{C}^n$ . By a curve we will mean an irreducible algebraic curve unless otherwise stated.

Now let  $M$  be an orientable finite volume hyperbolic 3-manifold with cusps, and let  $X(M)$  (resp.  $Y(M)$ ) denote the  $\mathrm{SL}(2, \mathbf{C})$ -character variety (resp.  $\mathrm{PSL}(2, \mathbf{C})$ -character variety) associated to  $\pi_1(M)$  (see for example [7] and [2] for definitions). In [7] and [2] it is shown that  $X(M)$  and  $Y(M)$  are defined over  $\mathbf{Q}$ . However, the fields of definition of irreducible components of  $X(M)$  and  $Y(M)$  may be defined over other number fields; i.e. subfields of  $\mathbf{C}$  which are finite extensions of  $\mathbf{Q}$ .

If we restrict  $M$  to have a single cusp, then the work of Thurston [23] shows that a component of  $X(M)$  and  $Y(M)$  containing the character of a faithful discrete representation of  $\pi_1(M)$  is a curve. There may be two such curves in  $Y(M)$ , related by complex conjugation (which corresponds to change of orientation of  $M$ ) and several in  $X(M)$ , arising from the different lifts of  $\pi_1(M)$  from  $\mathrm{PSL}(2, \mathbf{C})$  to  $\mathrm{SL}(2, \mathbf{C})$  (see [2] and [9] §2.7 for more on this). Throughout this paper we will usually simply fix one of these curve components, and denote it by  $X_0(M)$  (resp.  $Y_0(M)$ ) or  $X_0$  (resp.  $Y_0$ ) if no confusion will arise. These are called *canonical components*. Notice that, if the field of definition of  $Y_0$  is a real field, then  $Y_0$  will be fixed by complex conjugation and so there will be precisely one component in  $Y(M)$  containing the character of a faithful discrete representation. If the field of definition of  $Y_0$  is a non-real field, there is possibly a second component defined

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over the field obtained by applying complex conjugation (see §5 for more on this). A similar discussion applies to  $X_0$ .

It is known that there are examples of  $M$  with a single cusp for which the field of definition is not  $\mathbf{Q}$ . For example, in [10], there is an example of a once punctured torus bundle  $M$  where  $Y_0$  is defined over  $\mathbf{Q}(i)$ . Little else seems known, and so a natural question is the following.

**Question:** *Which number fields can arise as fields of definition of the curves  $X_0$  and  $Y_0$ ?*

In this paper we will restrict attention to  $Y_0$ . As mentioned, little seems known about what number fields can arise as fields of definition for canonical curve components. In particular, to our knowledge, there is no known obstruction to a number field being a field of definition.

In this note we provide some constructions of examples of one cusped hyperbolic 3-manifolds for which the fields of definition of  $Y_0$  are extensions of  $\mathbf{Q}$  of degree  $\geq 2$ . To give an indication of the type of result we shall prove we introduce some notation. Let  $p \geq 2$  be an integer, and  $\ell_p$  be the totally real number field  $\mathbf{Q}(\cos \pi/p)$ .

**THEOREM 1.1.** *For every odd integer  $p \geq 5$ , there is a one cusped finite volume orientable hyperbolic 3-manifold  $M_p$  for which  $Y_0(M_p)$  has, as field of definition, a number field  $k_p$  containing  $\ell_p$ .*

It is a standard fact that  $[\ell_p : \mathbf{Q}]$  goes to infinity with  $p$  and so the fields  $k_p$  will be distinct on passage to a subsequence. Moreover, the methods also allow us to construct examples for which the field of definition has arbitrarily large degree over any  $\ell_p$  ( $p$  odd), and examples for which the field of definition is a non-real number field. The methods of proof exploit certain “rigidity” phenomena, in particular the work in [19].

The discussion and results described above are similar in spirit to those concerning the question as to which number fields arise as (invariant) trace-fields for finite co-volume Kleinian groups. We refer the reader to the recent survey article [20] (in these proceedings), and [22] for more on this. Beyond the obvious fact that the invariant trace-field of a finite volume hyperbolic 3-manifold cannot be a real field, no other obstructions are known. Some obstructions are known for certain classes of manifolds; for example once punctured torus bundles [4]. In §5 we prove a theorem that gives obstructions for a number field to be the field of definition of  $Y_0$  for a hyperbolic knot complement in  $S^3$ . Modulo a conjecture about characters of real representations, we prove that the field of definition in this case has to be either real or contain a real subfield of index 2 (see Theorem 5.1).

We conclude the Introduction by remarking that, in §6 we construct examples of hyperbolic knots in  $S^3$  that have invariant trace-fields with class numbers at least 2 (see §6 for more discussion of this topic).

**Remark:** Much of this paper was basically written in 1998 and remained stubbornly unfinished. In conversations at a recent workshop on character varieties at Banff International Research Station, it became clear that there is some interest in the fields of definition of character varieties. It was this, together with the invitation of the organizers/editors of the conference/proceedings “Interactions Between Hyperbolic Geometry, Quantum Topology and Number Theory” that took place in

Columbia in 2009 to submit a paper that prompted us to finish off the paper.

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### 2. Fields of definition

It will be convenient to describe some additional background material on fields of definition. In this section,  $k \subset \mathbf{C}$  will be a number field.

Suppose that  $V \subset \mathbf{C}^m$  is a complex algebraic variety defined over  $k$ : recall that this means that

$$I(V) = \{f \in \mathbf{C}[X_1, \dots, X_m] : f(z) = 0 \forall z \in V\}$$

is generated by polynomials  $f_1, \dots, f_r \in k[X_1, \dots, X_m]$ . If  $k$  is the field of definition then  $k$  is the smallest such field for which this can be achieved.

We now fix  $k$  to be the field of definition, and  $\bar{k} \subset \mathbf{C}$  the algebraic closure of  $k$ . Then notice that  $V$  is also the vanishing set of the  $\bar{k}[X_1, \dots, X_m]$ -ideal

$$\{f \in \bar{k}[X_1, \dots, X_m] : f(z) = 0 \forall z \in V\}.$$

It will be convenient to work with this ideal in the following discussion and for convenience we will just refer to this ideal as  $I(V)$ . Now, as is easy to see, if  $G = \text{Gal}(\bar{k}/k)$  denotes the Galois group of the extension  $\bar{k}/k$ , then  $G$  acts on  $\bar{k}[X_1, \dots, X_m]$  by application of  $\sigma \in G$  to the coefficients of a polynomial in  $\bar{k}[X_1, \dots, X_m]$ . Since  $k$  is fixed by any  $\sigma \in G$ , it follows that  $I(V)$  is preserved by  $\sigma$ . Briefly, any polynomial in  $I(V)$  is a sum of terms  $g_i(X_1, \dots, X_m)f_i(X_1, \dots, X_m)$  with  $g_i(X_1, \dots, X_m) \in \bar{k}[X_1, \dots, X_m]$ . Applying  $\sigma$  to this product fixes the coefficients of  $f_i$ , and thereby determines a term  $g_i^\sigma(X_1, \dots, X_m)f_i(X_1, \dots, X_m) \in I(V)$ . With this observation we prove the following lemma.

**LEMMA 2.1.** *Let  $V$  be as above, and assume further that for some fixed  $j$  with  $1 \leq j \leq m$ , there exists an algebraic number  $t$  such that every point of  $V$  has  $x_j$  co-ordinate equal to  $t$ . Then  $t \in k$ .*

**Proof:** We will assume that  $j = 1$  for convenience. By assumption the polynomial  $X_1 - t$  vanishes on  $V$ , and hence  $X_1 - t \in I(V)$ . Assume to the contrary that  $t \notin k$ . Since  $t$  is algebraic we can find an element  $\sigma \in G$  such that  $\sigma(t) \neq t$ .

As noted above,  $\sigma$  preserves  $I(V)$ , and so  $\sigma(X_1 - t) \in I(V)$ , that is to say  $X_1 - \sigma(t) \in I(V)$ . Then  $t - \sigma(t) \in I(V)$ , which is a non-zero constant. In particular, this does not vanish on  $V$ , which contradicts that all elements of  $I(V)$  must vanish on  $V$ .  $\square$

### 3. A lemma

Throughout this section,  $M$  will denote a cusped orientable hyperbolic 3-manifold of finite volume.

**3.1.** It will be convenient to recall some of the construction of  $X(M)$  and  $Y(M)$  from [7] and [2]. We begin with  $X(M)$ . Recall that given a representation  $\rho : \pi_1(M) \rightarrow \mathrm{SL}(2, \mathbf{C})$ , this determines a character  $\chi_\rho : \pi_1(M) \rightarrow \mathbf{C}$  by  $\chi_\rho(\gamma) = \mathrm{tr}(\rho(\gamma))$ .

It is shown in [7] there exists a finite collection of elements  $\{\gamma_1, \dots, \gamma_m\}$  of  $\pi_1(M)$  such that for each  $\gamma \in \pi_1(M)$ ,  $\chi_\rho(\gamma)$  is determined by the collection  $(\chi_\rho(\gamma_i))_{1 \leq i \leq m}$ . The complex algebraic structure on  $X(M)$  is then determined by the embedding:

$$\chi_\rho \rightarrow (\chi_\rho(\gamma_1), \chi_\rho(\gamma_2), \dots, \chi_\rho(\gamma_m)).$$

The collection of elements  $\{\gamma_i\}_{1 \leq i \leq m}$  can be taken to be any generating set for  $\pi_1(M)$  and all double and triple products of these generators ([7], [12]).

We will say that  $\gamma \in \pi_1(M)$  has *constant trace* on an irreducible component  $X \subset X(M)$ , if  $\chi_\rho(\gamma) = \mathrm{tr}(\rho(\gamma))$  is constant for all  $\chi_\rho \in X$ .

This can be extended to  $Y(M)$  in the following way. Recall from [2] §3, that  $Y(M)$  is constructed from  $X(M)$  as follows. The group  $H^1(\pi_1(M); \mathbf{Z}/2\mathbf{Z}) = \mathrm{Hom}(\pi_1(M); \{\pm I\})$  acts on  $X(M)$  via:

$$\epsilon(\chi_\rho)(\gamma) = \chi_{\epsilon(\rho)}(\gamma) = \epsilon(\gamma)\chi_\rho(\gamma),$$

where  $\epsilon \in H^1(\pi_1(M); \mathbf{Z}/2\mathbf{Z})$ ,  $\chi_\rho \in X(M)$  and  $\gamma \in \pi_1(M)$ . This action is algebraic and  $Y(M)$  is the quotient of  $X(M)$  by this action.

Using this map, as shown in [2] §3, co-ordinates for  $Y(M)$  can be constructed in a similar way to that described above for  $X(M)$ . We can therefore talk about an element  $\gamma \in \pi_1(M)$  having constant trace on a component of  $Y \subset Y(M)$ .

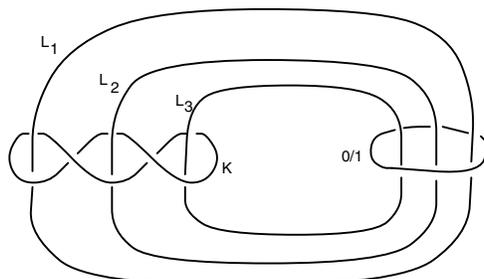
**3.2.** As remarked in the Introduction, the field of definition of an algebraic variety is an invariant of an algebraic variety that depends on the embedding in affine space. In particular, in the context of the character variety,  $X(M)$  (resp.  $Y(M)$ ) is computed with respect to a generating set, which determines (via the finite collection of characters described in §3.1) an embedding in some  $\mathbf{C}^m$ . However notice that if  $\langle S \rangle$  and  $\langle S' \rangle$  are finite generating sets for  $\pi_1(M)$ , and  $g \in S$ , then  $\chi_\rho(g)$  is a  $\mathbf{Z}$ -polynomial in the finite collection of characters described in §3.1 arising from  $S'$  and similarly, if  $g' \in S'$  then  $\chi_\rho(g')$  is a  $\mathbf{Z}$ -polynomial in the finite collection of characters described in §3.1 arising from  $S$ . These integral maps determine an isomorphism of the different embeddings of  $X(M)$  (resp.  $Y(M)$ ), and also for components of the character variety. We summarize this in the following proposition.

**PROPOSITION 3.1.** *Let  $M$  be a cusped hyperbolic 3-manifold, and  $X$  (resp.  $Y$ ) an irreducible component of  $X(M)$  (resp.  $Y(M)$ ). Then the field of definition of  $X$  (resp.  $Y$ ) does not depend on the generating set used to compute  $X(M)$  (resp.  $Y(M)$ ).*

**3.3.** Our constructions depend on the following lemma which is a simple consequence of Lemma 2.1 and Proposition 3.1.

**LEMMA 3.2.** *Let  $\{\gamma_1, \dots, \gamma_m\}$  be as above, and assume for some  $j$ ,  $\gamma_j$  has constant trace on an irreducible component  $Y \subset Y(M)$ , say with value  $t$ . Then the field of definition of  $Y$  contains  $t$ .*

In Lemma 3.2, we dealt with the case of a generator having constant trace. However, there is no loss in generality in doing this, since if  $\gamma$  is any element of



constant trace, then we can simply adjoin  $\gamma$  to a generating set and work with this. The discussion in §3.2 and Proposition 3.1 shows that this does not effect the field of definition.

**3.4.** It is easy to construct hyperbolic knots in  $S^3$  that have many curve components in  $Y(M)$ , all defined over an extension of  $\mathbf{Q}$ , and for which none of these components contains the character of a faithful discrete representation.

For example, [2] Example 3.2 shows that the free product of two non-trivial finite cyclic groups of orders  $p$  and  $q$  has a  $\mathrm{PSL}(2, \mathbf{C})$ -character variety with  $[p/2][q/2]$  curve components. By Lemma 3.2, it follows that these components are defined over fields containing  $\cos \pi/p$  and  $\cos \pi/q$ . Considering these groups as the base orbifold groups of appropriate torus knot exteriors (assuming  $p$  and  $q$  are relatively prime), it is easy to construct examples of hyperbolic knots whose fundamental groups surject these torus knots groups. In some cases these hyperbolic knots can be made 2-bridge and so can only have curve components in their character varieties (by [5] Theorem 4.1 for example).

#### 4. Applications

Lemma 3.2 gives a method to construct canonical components which are defined over extensions of  $\mathbf{Q}$ . We now describe settings where this can be achieved.

**4.1.** The examples which prove Theorem 1.1, come from [19]. These examples exploit a “rigid” totally geodesic surface. We briefly recall the one cusped hyperbolic 3-manifolds  $M_p$  constructed in §2 of [19]. These manifolds are  $p$ -fold cyclic covers of the orbifolds obtained by  $(p, 0), (p, 0), (p, 0)$  Dehn filling on the components  $L_1, L_2, L_3$  of the manifold shown in the figure (this is Figure 2 of [19]). The odd  $p$  assumption was used in [19] to easily arrange a manifold cover with a single cusp.

By construction,  $M_p$  contains an embedded non-separating totally geodesic surface  $\Sigma_p$  of genus  $(p - 1)/2$ . The arguments of [19] show that because  $\Sigma_p$  covers a rigid orbifold (namely  $\mathbf{H}^2$  modulo the  $(p, p, p)$ -triangle group), then  $\Sigma_p$  remains rigid under all generalized hyperbolic Dehn surgeries. In particular, we deduce from this that  $\mathrm{tr}(\rho(\gamma))$  is constant on the component  $Y_0(M_p)$  for all  $\gamma \in \pi_1(\Sigma_p)$ . Now since  $p$  is odd, the trace-field and the invariant trace-field of the rigid orbifold coincide, and equals  $\ell_p$ . Hence it follows that the trace-field of  $\rho(\pi_1(\Sigma_p))$  is  $\ell_p$ , and so Lemma 3.2 now implies Theorem 1.1.  $\square$

**4.2.** It is easy to modify the construction so that for any fixed odd  $p$ , the field  $k_p$  has arbitrarily large degree over  $\ell_p$ . To see this, let  $B_p$  denote the 3-manifold obtained by cutting  $M_p$  along  $\Sigma_p$ . So  $B_p$  has a single cusp and a pair of totally geodesic boundary components, both isometric to  $\Sigma_p$ . This manifold covers a hyperbolic 3-orbifold with a single cusp and a pair of totally geodesic orbifolds isometric to  $\mathbf{H}^2$  modulo the  $(p, p, p)$ -triangle group.

Performing a genuine hyperbolic  $r$ -Dehn filling on the cusp of  $B_p$  gives a hyperbolic 3-manifold  $B_p(r)$  with geodesic boundary, which by the aforementioned rigidity is a pair of surfaces isometric to  $\Sigma_p$ . An adaptation of an argument of Hodgson (see also [15]) shows that as we vary  $r$  the degree of the trace of the core curve (denoted by  $t_r$ ) of the  $r$ -Dehn filling goes to infinity. Now form 1-cusped hyperbolic 3-manifolds  $N_{p,r}$  by gluing  $B_p$  to  $B_p(r)$ . Since  $t_r$  will remain constant on  $Y_0(N_{p,r})$ , Lemma 3.2 and the construction in §4.1 implies the field of definition of  $Y_0(N_{p,r})$  contains both  $\cos \pi/p$  and  $t_r$ . This proves the claim.

**4.3.** Examples where one does not have a rigid surface can also be constructed using the methods of [19]. Theorem 2 of [19] provides an example of a 2 cusped hyperbolic 3-manifold  $M$  whose cusps were geometrically isolated from each other. This manifold was conjectured to have strongly geometrically isolated cusps, in the sense that performing a genuine topological Dehn filling produced a 1-cusped hyperbolic 3-manifold containing a closed geodesic  $\gamma$  (the core of the attached solid torus) with  $\text{tr}(\rho(\gamma))$  being constant on  $Y_0$ . This was proved in [3]. Both [19] and [3] exploit a certain 2-cusped hyperbolic orbifold  $Q$  that arises as a 2-fold quotient of  $M$  (see [19] Fig 7 and [3]). Indeed the strong geometric isolation established in [3] is proved via the orbifold  $Q$ .

As was pointed out in [10], the example of [10] mentioned in §1 is constructed as a filling on the manifold  $M$ , which shows why this example has field of definition  $\mathbf{Q}(i)$ . We describe below an example that is built in a similar way. This example seems interesting as it appeared in [6] in connection with Bloch group computations.

**Example:** Let  $N$  denote the manifold  $v3066$  of the SnapPea (or the recent updated version SnapPy [8]) census. This is a 1-cusped hyperbolic 3-manifold of volume  $6.2328329776455849\dots$ . Using SnapPea (or SnapPy) the manifold  $N$  can be seen to arise from Dehn filling on a double cover of  $Q$ . This double cover is distinct from  $M$ .

The core of the Dehn filling has trace  $t \notin \mathbf{R}$  with  $\mathbf{Q}(t)$  a non-real embedding of the unique cubic field  $k$  of signature  $(1, 1)$  and discriminant  $-59$  ( $t$  is a root of the polynomial  $x^3 + 2x^2 + 1$ ). This trace will be constant on  $Y_0$  and so the field of definition contains  $k$ .

**Remark:** Note that  $k$  is also a subfield of the invariant trace-field since  $t^2$  generates  $k$ . It is amusing to play with Snap to construct invariant trace fields of surgeries on  $N$  (or the example of [10]) where one can check (using Pari [21]) inclusions of number fields to “see” the field  $k$  as a subfield.

For example, the invariant trace-field of  $N$  is a field generated by a root of the polynomial  $x^6 - 4x^4 + 4x^2 + 1$  (which has 3 complex places). We can test inclusion of  $k$  using the Pari command `nfsincl`; this produces an output of 0 if there is no embedding or returns an embedding of fields we find:

$$\text{nfisincl}(x^3 + 2x^2 + 1, x^6 - 4x^4 + 4x^2 + 1) = [x^2 - 2].$$

This can be repeated for many small filling coefficients. For example, doing (1, 2)-Dehn filling has  $k$  as invariant trace-field.

### 5. Obstructions on fields for knots in $S^3$

We now focus on fields of definition of  $Y_0$  for the case when  $M$  is a hyperbolic knot complement in  $S^3$ . As far as the authors are aware no example of a non-real field of definition for  $Y_0$  is known in this case. We make the following observations in this regard.

By [13], every knot in  $S^3$  admits a curve of characters of infinite non-abelian (non-faithful) representations into  $SU(2)$  and  $SO(3) \cong PSU(2)$ . For the case of a hyperbolic knot it is as yet unknown as to whether the canonical component contains such characters. However, this seems plausible. Indeed the following conjecture is a weak version of this.

**Conjecture:** *Let  $K \subset S^3$  be a hyperbolic knot. Then  $Y_0$  contains infinitely many characters which are the characters of irreducible real representations.*

Given this we have the following obstruction to certain fields being fields of definition.

**THEOREM 5.1.** *Let  $K \subset S^3$  be a hyperbolic knot, and suppose that the above conjecture holds. Then the field of definition of  $Y_0 = Y_0(S^3 \setminus K)$  is either real or contains a real subfield of index 2.*

**Proof:** Let  $Y'_0$  denote the canonical component that is obtained by applying complex conjugation to  $Y_0$ . The conjecture asserts the existence of infinitely many characters of irreducible real representations. Let  $C \subset Y_0$  denote this set. Now  $C$  is fixed by complex conjugation and so  $Y_0$  and  $Y'_0$  meet along  $C$ . However, these are curves and so it follows that  $Y_0 = Y'_0$ ,

Let  $k$  denote the field of definition of  $Y_0$ . In particular, generators for the ideal  $I(Y_0)$  are elements of  $k[x_1, \dots, x_m]$ . Applying complex conjugation, the coefficients of these generators lie in  $k'$  (the field obtained by applying complex conjugation to  $k$ ). Thus  $Y'_0$  is defined over  $k'$ . Since  $Y_0 = Y'_0$ , it follows that  $Y_0$  is defined over  $k'$ . Now  $k$  is the field of definition of  $Y_0$  so we must have  $k \subset k'$ , and therefore  $k = k'$ . A field that is fixed by complex conjugation is either real, or contains a real subfield of index 2.  $\square$

**Remark:** By way of contrast with §4, we know of no example of a hyperbolic knot in  $S^3$  (or even an integral homology 3-sphere) for which the canonical component is defined over an imaginary field.

### 6. Class numbers and knot groups

In this section we make an observation regarding class numbers of invariant trace-fields of hyperbolic knots. We begin with some motivation for this.

Let  $k$  be a number field with ring of integers  $R_k$ . An enormous amount of research has been devoted to the study of the *class group* of  $R_k$ . This group measures the extent to which  $R_k$  fails to be a principal ideal domain. It is a classical

theorem that the class group is finite (see [24] for example); its order is called the *class number* of  $k$  (and denoted by  $h_k$ ). It is an open problem dating back to the time of Gauss whether there are infinitely many number fields of class number one, or even if given a constant  $C$ , there are infinitely many number fields with class number at most  $C$ .

We now relate this to questions about subgroups of  $\mathrm{PSL}(2, \mathbf{C})$ . Thus, suppose that  $\Gamma < \mathrm{PSL}(2, \mathbf{C})$  (not necessarily discrete). The fixed points of parabolic elements of  $\Gamma$  are the cusps of  $\Gamma$ . By positioning three of these cusps at  $0, 1$  and  $\infty$  in  $\mathbf{C} \cup \{\infty\}$ , one computes easily that all the cusps now lie in the invariant trace-field of  $\Gamma$ . In the special case that  $\Gamma = \mathrm{PSL}(2, R_k)$  one sees that the cusps are precisely the elements of  $k \cup \{\infty\}$  and the action of the group by fractional linear transformations gives an action on the field  $k$ . It is a theorem of Bianchi and Hurwitz (see [11] Chapter 7.2) that the number of equivalence classes for this action is  $h_k$ . Note that the groups  $\mathrm{PSL}(2, R_k)$  are discrete only when  $k$  is  $\mathbf{Q}$  or imaginary quadratic.

If now  $\Gamma$  is a non-cocompact Kleinian group of finite covolume with  $\Gamma < \mathrm{PSL}(2, R_{k\Gamma})$  (e.g any small hyperbolic knot in  $S^3$ ), if one can prove that every element of  $k \cup \{\infty\}$  is a parabolic fixed point of  $\Gamma$ , then it follows from the Bianchi-Hurwitz result that  $h_{k\Gamma} = 1$ .

With this as background, following a good deal of experimental work by the authors, it seems plausible that infinitely many hyperbolic knot complements in  $S^3$  have invariant trace-fields with class number 1 (for example every 2-bridge twist knot).

The purpose of this section is to prove the following result in the opposite direction.

**THEOREM 6.1.** *There are hyperbolic knots in  $S^3$  for which the invariant trace-field has class number at least 2.*

**6.1.** To prove Theorem 6.1 we make use of periodic knots. Recall that a knot  $K \subset S^3$  is said to have *period*  $q > 1$  if there is an orientation-preserving homeomorphism  $h : S^3 \rightarrow S^3$  of order  $q$  mapping  $K$  to itself and with fixed point set a circle disjoint from  $K$ . The character variety technology for a 1-cusped orientable hyperbolic 3-orbifold with a torus cusp is exactly as in the manifold setting (see for example [16] §2.2).

**LEMMA 6.2.** *Let  $Q = \mathbf{H}^3/\Gamma$  be a 1-cusped orientable hyperbolic 3-orbifold with a torus cusp, and  $\gamma \in \Gamma$  an element of order  $q > 1$ . Then the field of definition of  $Y_0(Q)$  contains  $\ell_q$ .*

**Proof:** As discussed §3.1, we can assume without loss of generality that  $\gamma$  is part of a generating set for  $\Gamma$ . Since under infinitely many hyperbolic Dehn surgeries on  $Q$ ,  $\gamma$  remains an element of order  $q$ , it follows that  $\gamma$  has constant trace on  $Y_0(Q)$ , namely  $\pm 2 \cos \pi/q$ . Applying Lemma 3.2 implies the result.  $\square$

We now describe a particular family of periodic knots and identify the component  $Y_0(Q)$  (as in Lemma 6.2) for the orbifold quotient that we will make use of below.

**Example:** Let  $q$  be an odd positive integer  $> 1$  and relatively prime to 3. The  $(3, q)$  Turks heads knot  $K_q$  is the closure of the 3-braid  $(\sigma_1 \sigma_2^{-1})^q$ , and is a knot with period  $q$ . Let  $Q_q$  denote the orbifold quotient obtained by quotienting out by this symmetry. The  $\mathrm{PSL}(2, \mathbf{C})$  character variety of  $Q_q$  was basically computed

in [18]; this paper deals only with representations, but these representations are normalized by a choice of conjugacy. We find it more convenient to work with more traditional trace co-ordinates. The orbifold groups of  $Q_q$  can be presented as (see [18] Theorem 5)

$$\langle x, \lambda | \lambda^q = 1, [\lambda, x\lambda x\lambda^{-1}x^{-1}\lambda^{-1}x\lambda x] = 1 \rangle .$$

In this presentation,  $x$  is the image of a meridian of  $K_q$ . Using the co-ordinates  $P = \text{tr}(x)$ ,  $R = \text{tr}(x\lambda)$ , a simple mathematica computation shows that the algebraic set  $X_1$  which contains the characters of all irreducible  $\text{SL}(2, \mathbf{C})$ -representations is defined by the vanishing locus of

$$F(P, R) = 1 - (2 \cos \pi/q)PR - 3R^2 + P^2R^2 + (2 \cos \pi/q)^2R^2 - (2 \cos \pi/q)PR^3 + R^4.$$

As is evident this is defined over  $\ell_q$ . To determine  $Y_0(Q)$  from this it will be convenient to make the following observations.

Note that  $H_1(Q_q; \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}/q\mathbf{Z}$  (see the presentation given above). Now the discussion in §3 regarding the construction of the  $\text{PSL}(2, \mathbf{C})$ -character variety still applies to the orbifold group of  $Q_q$ . Since  $q$  is odd, we deduce that  $H^1(Q_q; \mathbf{Z}/2\mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}$  and moreover, that  $\mathbf{Z}/2\mathbf{Z}$  acts on  $X(Q_q)$  via the image of  $x$ . Applying this to the traces we find that in the  $(P, R)$ -co-ordinates this involution (which we will denote by  $\tau$ ) acts by

$$\tau : (P, R) \mapsto (-P, -R).$$

Note that  $PR$  is invariant under  $\tau$ , and so to describe the quotient under the action of  $\tau$ , we will switch co-ordinates to  $(X, R)$  where  $X = PR$ . This gives the polynomial (still defined over  $\ell_q$ ):

$$G(X, R) = 1 - (2 \cos \pi/q)X - 3R^2 + X^2 + (2 \cos \pi/q)^2R^2 - (2 \cos \pi/q)XR^2 + R^4.$$

Taking the quotient by  $\tau$  describes the  $\text{PSL}(2, \mathbf{C})$ -image of  $X_1$  to be the vanishing locus of

$$\overline{G}(X, Y) = 1 - (2 \cos \pi/q)X - 3Y + X^2 + (2 \cos \pi/q)^2Y - (2 \cos \pi/q)XY + Y^2.$$

We now establish that the plane curve  $\overline{G}(X, Y)$  is irreducible and so its vanishing locus will define  $Y_0(Q)$  (which is therefore defined over  $\ell_q$ ).

To that end, notice that,  $\overline{G}(X, Y)$  is quadratic in both  $X$  and  $Y$ . Solving this quadratic for  $Y$  in terms of  $X$  gives the following roots (we let  $D = (2X \cos(\frac{\pi}{q}) - 2 \cos(\frac{2\pi}{q}) + 1)$ ):

$$\frac{1}{2} \left( \pm \sqrt{D^2 - 4 \left( -2X \cos \left( \frac{\pi}{q} \right) + X^2 + 1 \right) + D} + D \right).$$

Thus if  $\overline{G}(X, R)$  is reducible, it follows that the term

$$P_q(X) = \left( 2X \cos \left( \frac{\pi}{q} \right) - 2 \cos \left( \frac{2\pi}{q} \right) + 1 \right)^2 - 4 \left( -2X \cos \left( \frac{\pi}{q} \right) + X^2 + 1 \right)$$

is a square of a polynomial in  $X$ . This can be readily checked by seeing whether  $P_q(X)$  and its derivative have a common zero. However, a simple mathematical calculation computes the appropriate resultant to be:

$$-64 \sin^2 \left( \frac{\pi}{q} \right) \left( 2 \cos \left( \frac{2\pi}{q} \right) - 3 \right)$$

which is non-zero since  $q \neq 1$ . This completes the identification of  $Y_0(Q)$ .  $\square$

**6.2.** We now prove Theorem 6.1.

Consider the knots  $K_q$  from §6.1, and for convenience we again assume that  $q$  is odd. As noted in §6.1,  $H_1(Q_q; \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}/q\mathbf{Z}$ , and so the invariant trace-field of  $Q_q$  (and therefore  $K_q$ ) coincides with the trace field of  $Q_q$  (see [17] Theorem 4.2.1). This is only stated for manifolds but the same proof holds in the current situation).

Let  $L_q$  denote the invariant trace-field of  $K_q$ . This can be obtained by specializing  $P = 2$  in  $F(P, R)$ , and this gives

$$(-1 - R + (2 \cos \pi/q)R - R^2)(-1 + R + (2 \cos \pi/q)R - R^2) = 0.$$

It follows that  $L_q$  is generated over  $\ell_q$  by a root of one of the factors above, and in either case this determines an imaginary quadratic extension of  $\ell_q$ .

As we discuss below, the proof is completed by applying Theorem 10.1 of [24] (which is stated below for convenience) and the fact that there are values of  $q$  for which the class number of  $\ell_q$  is greater than 1. For example, [1] shows that the smallest prime  $q$  for which the class number is greater than 1 is 257.

We now recall the theorem referred to above.

**THEOREM 6.3.** *Suppose that the extension of number fields  $L/K$  contains no unramified abelian subextensions  $F/K$  with  $K \neq F$ . Then  $h_K | h_L$ .*

In our context,  $L_q$  is a quadratic extension of  $\ell_q$  and so the condition on subextensions is vacuous. Moreover, since  $L_q$  is necessarily imaginary it is a ramified extension of  $\ell_q$ ; in the case of embeddings of the field, this simply means that the identity embedding of  $\ell_q$  lifts to a pair of complex conjugate embeddings. Thus we can apply Theorem 6.3 to  $L_q/\ell_q$  to deduce that  $h_{L_q}$  is greater than one for certain values of  $q$ .  $\square$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CA 93106  
E-mail address: [long@math.ucsb.edu](mailto:long@math.ucsb.edu)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS, AUSTIN, TX 78712  
E-mail address: [areid@math.utexas.edu](mailto:areid@math.utexas.edu)