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On Subgroup Separability in Hyperbolic Coxeter Groups

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Abstract. We prove that certain hyperbolic Coxeter groups are separable on their geometrically finite subgroups.

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1. Introduction

Recall that a subgroup H of a group G is *separable in* G if, given any $g \in G \setminus H$, there exists a subgroup $K < G$ of finite index with $H < K$ and $g \notin K$. G is called *subgroup separable* (or *LERF*) if G is H-subgroup separable for all finitely generated $H < G$. This powerful property has attracted a good deal of attention in the last few years, largely motivated by questions which arise in low dimensional topology (see [1], and [15] for example). In that context, and in the context of negatively curved groups it makes most sense to restrict to subgroups which are geometrically finite (or quasiconvex in the negatively curved case) and to this end, the notion of GFERF was introduced in [1] and [12]. Since the property of being geometrically finite is preserved by passage to sub- and super-group of finite index, as in the case of subgroup separability (see [15]), it follows that GFERF is a commensurability invariant.

This paper studies GFERF in the context of arithmetic groups and certain Coxeter groups which we discuss further below. However, this paper should really be viewed as a broad generalization of the geometric and algebraic methods used in [1], in which the theory of quadratic forms was used to help control properties of discrete groups.

Throughout this paper we will use the term *hyperbolic simplex group* to refer to those Coxeter groups which arise as groups generated by reflections in the faces of a non-compact geodesic hyperbolic simplex of finite volume. Thus, these

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hyperbolic simplex groups are finite co-volume but non-cocompact discrete subgroups of $\mathrm{Isom}(\mathbf{H}^n)$ for some n. The general definition of a Coxeter group is given in Section 2.

As is well-known, there are very few hyperbolic simplex groups, and these are completely classified; they exist only in dimensions $3 \leqslant n \leqslant 9$ (see [7], pp. 142–144). A list of the Coxeter diagrams of such groups in dimensions $4 \leqslant n \leqslant 8$ is given in the Appendix. In dimension 3, there are 23 such hyperbolic simplex groups, and all but 6 are arithmetic. The arithmetic ones are commensurable with either of the Bianchi groups $\mathrm{PSL}(2, \mathbf{Z}[i])$ or $\mathrm{PSL}(2, \mathbf{Z}[\omega])$ where ω is a cube root of unity (see [8] or [14], for example). Thus by [1] these arithmetic hyperbolic simplex groups are GFERF. In dimensions $\geqslant 4$ all the commensurability classes with one exception (in dimension 5) are arithmetic (see below and [17]). This paper shows that the commensurability classes of arithmetic hyperbolic simplex groups in dimensions $\leqslant 8$ are GFERF.

THEOREM 1.1. *Let Γ be an arithmetic finite volume non-cocompact hyperbolic simplex group of dimension $\leqslant 8$. Then Γ is GFERF.*

The main calculation is summed up by the following result—see Section 2 for notation and definitions.

THEOREM 1.2. *Let Γ be an arithmetic hyperbolic simplex group of dimension $4 \leqslant n \leqslant 9$ and $F(\Gamma)$ the rational form constructed in Section 5. Suppose that the determinant of the form $F(\Gamma)$ is $-k$.*

Then $F(\Gamma)$ is equivalent over $\mathbf{Q}$ to $\langle 1, \dots, 1, -k\rangle$.

In particular, either $F(\Gamma)$ (in the case $k = 1$) or $F(\Gamma) \oplus \langle k\rangle$ (otherwise) is equivalent over $\mathbf{Q}$ to the standard form $\langle 1, \dots, 1, -1\rangle$. □

The method of proof for Theorem 1.1 is closely related to that of [1]; as in that paper we observe that there are hyperbolic simplex groups in dimensions 6, 7, 8 which are commensurable with a group generated by reflections in an all right polyhedron. Theorem 3.1 of [1] now shows that this latter group is GFERF, so that the hyperbolic simplex group in the appropriate dimension is GFERF. It is then shown from certain arithmetic considerations that every hyperbolic arithmetic simplex group, is already commensurable with one of these groups, and so is GFERF, or can be embedded into one of these groups, hence inherits the GFERF property. There are arithmetic hyperbolic simplex groups in dimension 9, but we do not know if these are commensurable with all right reflection groups, and hence our results concerning GFERF are limited to dimensions $\leqslant 8$.

As will be apparent, a central role in our methods is played by the groups $\mathrm{SO}_0(f_n; \mathbf{Z})$; where f_n denotes the form $\langle 1, \dots, 1, -1\rangle$ of signature $(n, 1)$. Indeed, it appears that the family of groups $\mathrm{SO}_0(f_n; \mathbf{Z})$ form a potentially important ‘universal family’ of groups in the following sense.

THEOREM 1.3. $\mathrm{SO}_0(f_m; \mathbf{Z})$ *is GFERF for all* $m \geqslant 2$ *if and only if every non-cocompact arithmetic hyperbolic group of dimension n is GFERF.*

In the final section of the paper we discuss the limitations of the methods of [1] used here. In particular, we give a simple construction of some right angled (abstract) Coxeter groups (see Section 2) that are not LERF.

2. Algebraic Preliminaries

We need to recall some standard facts about quadratic forms and orthogonal groups of such forms; [10] is a standard reference.

2.1. If f is a quadratic form in $n+1$ variables with coefficients in K and associated symmetric matrix F, let

$$\mathrm{O}(f) = \{X \in \mathrm{GL}(n+1, \mathbf{C}) | X^t F X = F\}$$

be the *Orthogonal group* of f, and

$$\mathrm{SO}(f) = \mathrm{O}(f) \cap \mathrm{SL}(n+1, \mathbf{C}),$$

the *Special Orthogonal group* of f. These are algebraic groups defined over K.

DEFINITION. Two n-dimensional quadratic forms f and q defined over a field K (with associated symmetric matrices F and Q) are *equivalent* over K if there exists $P \in \mathrm{GL}(n, K)$ with $P^t F P = Q$.

If $K \subset \mathbf{R}$ is a number field, and R_K its ring of integers, then $\mathrm{SO}(f; R_K)$ is an arithmetic subgroup of $\mathrm{SO}(f; \mathbf{R})$, [3] or [2]. The following is well-known and proved in [1] for example.

LEMMA 2.1. *Let* $K \subset \mathbf{R}$ *be a number field and* R_K *its ring of integers. Let f and q be n-dimensional quadratic forms with coefficients in* R_K *which are equivalent over K.*

- $\mathrm{SO}(f; \mathbf{R})$ *is conjugate to* $\mathrm{SO}(q; \mathbf{R})$ *and* $\mathrm{SO}(f; K)$ *is conjugate to* $\mathrm{SO}(q; K)$.
- $\mathrm{SO}(f; R_K)$ *is conjugate to a subgroup of* $\mathrm{SO}(q; K)$ *commensurable with* $\mathrm{SO}(q; R_K)$. □

There is a converse to the second part of Lemma 2.1 which we record here (see [6] or [19]). Note that if $f' = \lambda f$, for $\lambda \in K$ (non-zero), then $\mathrm{SO}(f'; K) = \mathrm{SO}(f; K)$. With notation as above,

LEMMA 2.2. *Suppose* $\mathrm{SO}(f; R_K)$ *and* $\mathrm{SO}(q; R_K)$ *are commensurable. Then f is equivalent to* λq *for some non-zero* $\lambda \in K$. □

In order to apply Lemma 2.1, we will need a criterion for when two forms are equivalent over a field. To this end, let K denote either a number field or a completion thereof, and q a non-singular quadratic form defined over K. Let the associated symmetric matrix be Q. We define the *determinant* of q as the element $d(q) = \det(Q)\dot{K}^2$, where $\dot{K}$ are the invertible elements in K. It is not hard to see that $d(q)$ is an invariant of the equivalence class of q.

The *Hasse invariant* (see [10], p. 122) of a non-singular diagonal form $\langle a_1, a_2, \ldots, a_n\rangle$ with coefficients in K is an element in the Brauer group $B(K)$, namely

$$s(q) = \prod_{i<j}\left(\frac{a_i, a_j}{K}\right),$$

where $((a_i, a_j)/K)$ describes a quaternion algebra over K, and the multiplication is that in $B(K)$, see [10], Chapter 4.

Every non-singular form over K is equivalent over K to a diagonal one, and the definition of the Hasse invariant is extended to non-diagonal forms by simply defining it to be the Hasse invariant of a diagonalization (that this is well-defined follows from [10], p. 122) The following theorem, the 'Weak Hasse–Minkowski Principle' (see [10], p. 168) is the criterion we shall use:

THEOREM 2.3. *Let q_1 and q_2 be non-singular quadratic forms of the same dimension, defined over K with the property that if σ is a real embedding of K the forms q_1^σ and q_2^σ have the same signature over* $\mathbf{R}$. *Then q_1 is equivalent to q_2 over K if and only if $d(q_1) = d(q_2)$ and $s(q_1) = s(q_2)$ over all non-Archimedean completions of K.* □

Note that if $d(q_1) = d(q_2)$ (resp. $s(q_1) = s(q_2)$) then the same holds at all non-Archimedean completions.

2.2. We introduce some notation that will be convenient. If q is a diagonal quadratic form defined over $\mathbf{Q}$ and $k \in \mathbf{Q}$ we shall refer to the process of forming the orthogonal sum $q \oplus \langle k\rangle$ as *stabilization by* $\langle k\rangle$. The following lemma is clear.

LEMMA 2.4. *In the notation above, the group* $\mathrm{SO}(q; \mathbf{Z})$ *is a subgroup of* $\mathrm{SO}(q \oplus \langle k\rangle; \mathbf{Z})$. □

A sequence of stabilizations by $\langle a_i\rangle$ will also be referred to as stabilization. The motivation for the terminology will become clear in Sections 5 and 6.

2.3. We recall some basic statements about Coxeter groups, see [5] and [7] for details.

Suppose that W is a group and S is a set of generators all of order 2. Then (W, S) is a *Coxeter system* if W admits a presentation:

$$\langle S \mid (s \cdot t)^{m(s,t)} = 1\rangle$$

where $m(s, t)$ is the order of $s \cdot t$ and there is one relation for each pair s, t with $m(s, t) < \infty$.

We refer to W as a *Coxeter group*. The *Coxeter diagram* of this presentation consists of a vertex for each element of S together with an edge connecting distinct vertices s, t whenever $m(s, t) \neq 2$ and the edge is labelled by $m(s, t)$. It is also standard practice in the case when $m(s, t) = 3$ to leave the edge unlabelled, and we follow that convention here. Since the generators have order 2, this means that if two vertices are not connected by an edge then the generators corresponding to the vertices commute. Thus, if the diagram is not connected, the Coxeter group is the direct sum of the subgroups given by the connected components. A Coxeter group (W, S) is called *reducible* if its diagram is not connected. Otherwise the Coxeter group is irreducible.

A Coxeter group is called an *all right Coxeter group* if $m(s, t) = 2$ or $m(s, t) = \infty$ for all $s \neq t \in S$.

3. Arithmetic Subgroups of Isom($\mathbf{H}^n$)

In this section we expand somewhat on arithmetic subgroups of Isom($\mathbf{H}^n$). For more details, see [3], [2] and [19]. We also prove some results on embedding all non-cocompact arithmetic groups in $\mathrm{SO}_0(f_n; \mathbf{R})$ into a *fixed* $\mathrm{SO}_0(f_m; \mathbf{R})$, with $m > n$. This generalizes one of the key ideas in [1].

3.1. Let f_n be the $(n+1)$-dimensional quadratic form $\langle 1, 1, \ldots, 1, -1\rangle$. The connected component of the identity in $\mathrm{O}(f_n; \mathbf{R})$ will be denoted $\mathrm{O}_0(f_n; \mathbf{R})$. This group preserves the upper sheet of the hyperboloid $f_n(x) = -1$ but contains reflections so reverses orientation. We identify $\mathrm{O}_0(f_n; \mathbf{R})$ with Isom($\mathbf{H}^n$). Passing to the connected component of the identity in $\mathrm{SO}(f_n; \mathbf{R})$, denoted $\mathrm{SO}_0(f_n; \mathbf{R})$ (which has index 4 in $\mathrm{O}(f_n; \mathbf{R})$), gives a group may be identified with $\mathrm{Isom}_+(\mathbf{H}^n)$; it preserves the upper sheet of the hyperboloid $f_n(x) = -1$ and the orientation. Given a (discrete) subgroup G of $\mathrm{O}(f_n; \mathbf{R})$, $G \cap \mathrm{SO}_0(f_n; \mathbf{R})$ has finite index in G.

A particular class of arithmetic subgroups of $\mathrm{O}_0(f_n; \mathbf{R})$ (resp. $\mathrm{SO}_0(f_n; \mathbf{R})$) are those that are called *arithmetic groups of simplest type* in [19] Chapter 6.

Assume that $k \subset \mathbf{R}$ is totally real, and let f be a form in $n+1$-variables with coefficients in k, and be equivalent over $\mathbf{R}$ to the form f_n. Furthermore, if $\sigma\colon k \to \mathbf{R}$ is a field embedding, then the form f^σ obtained by applying σ to f is defined over the real number field $\sigma(k)$. We insist that for embeddings $\sigma \neq id$, f^σ is equivalent over $\mathbf{R}$ to the form in $(n+1)$-dimensions, of signature $(n+1, 0)$. Since f is equivalent over $\mathbf{R}$ to f_n, it from follows Lemma 2.1 that $\mathrm{O}(f; \mathbf{R})$ is conjugate, by a matrix P say in $\mathrm{GL}(n+1, \mathbf{R})$ to $\mathrm{O}(f_n; \mathbf{R})$. From [3] (or [2]) $P\mathrm{SO}_0(f; R_k)P^{-1}$ defines an arithmetic subgroup in $\mathrm{Isom}_+(\mathbf{H}^n)$, and so necessarily of finite co-volume. In what follows we will abuse notation, and suppress the conjugating matrix, and simply identify $\mathrm{SO}_0(f; R_k)$ as an arithmetic subgroup of $\mathrm{Isom}_+(\mathbf{H}^n)$.

The group $\mathrm{SO}_0(f; R_k)$ is cocompact if and only if the form f does not represent 0 non-trivially with values in k, see [3]. Whenever $n \geqslant 4$, the arithmetic groups constructed above are non-cocompact if and only if the form has rational coefficients, since it is well known every indefinite quadratic form over $\mathbf{Q}$ in at least 5 variables represents 0 non-trivially, see [10].

The following theorem summarizes what we shall need (see [19] Chapter 6 and page 365 of [11]).

THEOREM 3.1. *If* Γ *is a non-cocompact arithmetic subgroup of* $\mathrm{SO}_0(f_n;\mathbf{R})$ *then* Γ *is commensurable (up to conjugacy) with a group* $\mathrm{SO}_0(f;\mathbf{Z})$ *where* f *is a diagonal quadratic form with rational coefficients and signature* $(n,1)$. □

3.2. The main result of this subsection is.

THEOREM 3.2. *Let* f *be a quadratic form with rational coefficients and signature* $(n,1)$. *Then there is an* m *such that* $\mathrm{SO}_0(f;\mathbf{Z})$ *contains subgroup of finite index conjugate to a subgroup of* $\mathrm{SO}_0(f_m;\mathbf{Z})$.

Theorem 3.2 will follow from

THEOREM 3.3. *Let* f *be a quadratic form with rational coefficients and signature* $(n,1)$. *Then there exists an* m *such that* f *can be stabilized to a form rationally equivalent to* f_m.

Assuming Theorem 3.3 for the moment we complete the proof of Theorem 3.2. Theorem 3.3 yields a form q such that $f\oplus q$ is equivalent over $\mathbf{Q}$ to f_m. Thus by Lemma 2.1 there is a $g\in\mathrm{GL}(m+1,\mathbf{Q})$ such that $g\mathrm{SO}_0(f\oplus q;\mathbf{Z})g^{-1}$ is commensurable with $\mathrm{SO}(f_m;\mathbf{Z})$. Since $g\mathrm{SO}_0(f;\mathbf{Z})g^{-1}<g\mathrm{SO}_0(f\oplus q;\mathbf{Z})g^{-1}$ we obtain the required subgroup. □

Proof of Theorem 3.3. Without loss of generality we may assume that f is diagonal, say $f=\langle a_1,a_2,\dots,a_n,-e\rangle$, where a_i and e are all square free positive integers. Consider the form defined by stabilizing f by summing with

$$q=\langle a_1,a_2,\dots,a_n\rangle\oplus\langle a_1,a_2,\dots,a_n\rangle\oplus\langle a_1,a_2,\dots,a_n\rangle.$$

Then $f\oplus q$ has signature $(4n,1)$, and determinant $-e$. Since the form $\langle a,a,a,a\rangle$ is equivalent over $\mathbf{Q}$ to $\langle 1,1,1,1\rangle$ (same determinant and Hasse invariant), by rearranging $f\oplus q$ we see that it is equivalent to:

$$\langle 1,1,1,\dots,1,-e\rangle,$$

where there are $4n$ ones. Now do a further stabilization by $\langle e\rangle$, to give

$$\langle 1,1,1,\dots,1,-e\rangle\oplus\langle e\rangle$$

and observe that this form has signature $(4n+1,1)$ and determinant -1. Furthermore $((-e,e)/\mathbf{Q})$ is the matrix algebra. By Theorem 2.3, this final form is equivalent over **Q** to the standard form f_{4n+1}, that is we have stabilized the form to f_{4n+1} as needed. □

The following corollary puts [1] in a broader context. There we showed $\mathrm{SO}_0(f_6; \mathbf{Z})$ is GFERF, and injected all the Bianchi groups up to finite index.

COROLLARY 3.4. $\mathrm{SO}_0(f_m; \mathbf{Z})$ *is GFERF for all* $m \geqslant 2$ *if and only if every non-cocompact arithmetic subgroup of* $\mathrm{SO}_0(f_n; \mathbf{R})$ *is GFERF for every* $n \geqslant 2$.

Proof. Since $\mathrm{SO}_0(f_m; \mathbf{Z})$ is an arithmetic subgroup of $\mathrm{SO}(f_m; \mathbf{R})$, one direction is clear. Thus assume that $\mathrm{SO}_0(f_m; \mathbf{Z})$ is GFERF for all m. By Theorem 3.1 every non-cocompact arithmetic subgroup of $\mathrm{SO}_0(f_n; \mathbf{R})$ is commensurable with a group $\mathrm{SO}_0(f; \mathbf{Z})$ where f is a form of signature $(n, 1)$ defined over **Q**. Thus it suffices to prove that the group $\mathrm{SO}_0(f; \mathbf{Z})$ is GFERF. By Theorem 3.2 we can arrange that $\mathrm{SO}_0(f; \mathbf{Z})$ contains a subgroup Γ of finite index which is (conjugate to) a subgroup of some $\mathrm{SO}_0(f_m; \mathbf{Z})$. By assumption these are assumed GFERF and the result now follows. □

4. Arithmetic Hyperbolic Simplex Groups

We specialize some of Section 3.1 to arithmeticity of Coxeter groups. This was investigated by Vinberg [17], who gave a beautiful characterization theorem in terms of the Gram matrix. Since we will only be concerned with simplices that have at least one ideal vertex, we will consider only this case. We begin with some general considerations.

4.1. Let Λ^n denote the Lobachevskii space model of hyperbolic n-space. Let P be a polyhedron in Λ^n bounded by finitely many linear hyperplanes through the origin. For each hyperplane choose an outward pointing normal $\mathbf{e_i}$ and normalise such that $\langle \mathbf{e_i}, \mathbf{e_i} \rangle = 1$ (where $\langle\,,\rangle$ is the bilinear form associated to the quadratic form f_n in Section 2.2).

The *Gram matrix* $G(P)$ of this system of vectors is $G(P) = [a_{ij}]$ where $a_{ij} = \langle \mathbf{e_i}, \mathbf{e_j} \rangle$. If the dihedral angle between the planes H_i, H_j is α then $a_{ij} = -\cos\alpha$. When the planes do not intersect then the entry is a function of the hyperbolic distance between the hyperplanes. Note that when P is a simplex then all faces meet, so all off diagonal entries of $G(P)$ will be of the form $-\cos\alpha$.

Note that the rank of the Gram matrix of P is $n+1$, and in the case of a hyperbolic simplex $G(P)$ is an $(n+1) \times (n+1)$-matrix.

4.2. For details of what is described below see [17] or [19]. Let P be a hyperbolic Coxeter simplex. That is, P has all dihedral angles of the form π/k for $k \in \{2, 3, \ldots\}$. Let Γ be the Coxeter group generated by the reflections $\{r_i\}_{1 \leqslant i \leqslant n+1}$ in the faces of P; the reflection r_i in the hyperplane H_i is given by

$$r_i(\mathbf{v}) = \mathbf{v} - 2 < \mathbf{v}, \mathbf{e_i} > \mathbf{e_i},$$

and so r_i can be represented by an $m \times m$ matrix X such that $X^t G(P) X = G(P)$.

For any

$$\{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, n\}$$

define the cyclic products

$$b_{i_1 i_2 \cdots i_k} = 2^k a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k} a_{i_k i_1}$$

and let $K(P) = \mathbf{Q}(\{b_{i_1 i_2 \cdots i_k}\})$. It is not difficult to see that it suffices to assume in the definition of $K(P)$ that the suffices $\{i_1, i_2, \dots, i_k\}$ are distinct. Vinberg's theorem ([17]) is then:

THEOREM 4.1. *In the notation above, the finite volume non-cocompact simplex group Γ is arithmetic if and only if:*

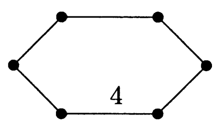
- $K(P) = \mathbf{Q}$
- *all the cyclic products $b_{i_1 i_2 \cdots i_k}$ are rational integers.* □

Indeed more is true from the proof of Vinberg's theorem. Suppose the simplex group Γ is arithmetic. Consider the following vectors

$$\mathbf{v}_{i_1 i_2 \cdots i_k} = 2^k a_{1 i_1} a_{i_1 i_2} \cdots a_{i_{k-1} i_k} \mathbf{e}_{i_k}$$

with $\mathbf{v}_1 = 2\mathbf{e_1}$ and the suffices defined as above. The $\mathbf{Q}$-module $V(P)$ spanned by $\{\mathbf{v}_{i_1 i_2 \cdots i_k}\}$ is a Γ module which has dimension $n+1$ over $\mathbf{Q}$. With the restriction of the inner product, $V(P)$ is a quadratic space over $\mathbf{Q}$ and so if we let $F(\Gamma)$ denote the diagonal form obtained by diagonalizing over $\mathbf{Q}$ we obtain a representation of Γ into the orthogonal group $O(F(\Gamma); \mathbf{Q})$ commensurable with $\mathrm{SO}_0(F(\Gamma); \mathbf{Z})$. The following theorem is due to Vinberg [17].

THEOREM 4.2. *Let P be a non-compact hyperbolic Coxeter simplex in $\mathbf{H}^n$ where $n \geqslant 4$. Let Γ denote the Coxeter group generated by reflections in faces of P. Then Γ is arithmetic unless P is in dimension 5, and has Coxeter diagram:*



5. Proof of Theorem 1.1

We now begin the proof of the main theorem.

THEOREM 5.1. *Let Γ be an arithmetic hyperbolic Coxeter group of dimension $4 \leqslant n \leqslant 9$ and $F(\Gamma)$ the rational form constructed above. Suppose that $d(F(\Gamma)) = -k$. Then $F(\Gamma)$ is equivalent over the rationals to $\langle 1, \dots, 1, -k \rangle$.* □

Deferring the proof of this for the moment, note that if $F(\Gamma)$ is a form in $(n+1)$-variables, then if $k = 1$, $F(\Gamma)$ is just the form f_n. Otherwise, $F(\Gamma) \oplus \langle k \rangle$ is equivalent over $\mathbf{Q}$ to the form f_{n+1} (the proof of this being an application of Theorem 2.3, and is contained in the proof of Theorem 3.3). These observations complete the proof of Theorem 1.2. As a corollary of these discussions we have,

COROLLARY 5.2. *Let Γ be an arithmetic hyperbolic Coxeter group of dimension $4 \leqslant n \leqslant 9$. Then Γ contains a subgroup of finite index which is conjugate to a subgroup of $\mathrm{SO}(f_m; \mathbf{Z})$ for $m = n$ or $n+1$. This latter index may be infinite.*

Proof. From the discussion preceding the Corollary, in the case when $k = 1$, a direct application of Lemma 2.1 provides a commensurability between Γ and $\mathrm{SO}(f_n; \mathbf{Z})$ for an appropriate n. In the case when $k \neq 1$, Lemma 2.1 implies Γ is commensurable with $\mathrm{SO}_0(\langle 1, 1, \ldots, 1, -k \rangle; \mathbf{Z})$. By stabilization of $F(\Gamma)$ by $\langle k \rangle$, Theorem 3.2 implies that Γ contains a subgroup of finite index conjugate to a subgroup of infinite index (stabilizing a co-dimension 1 totally geodesic submanifold in $\mathbf{H}^{n+1}$) in $\mathrm{SO}_0(f_{n+1}; \mathbf{Z})$. This proves the corollary. □

Theorem 1.1 will follow, using the technique of [1] applied to a slightly wider class of Coxeter group.

Proof of Theorem 1.1. Let Γ be an arithmetic non-cocompact hyperbolic simplex group acting on $\mathbf{H}^n$ where $4 \leqslant n \leqslant 8$. By Corollary 5.2, Γ contains a subgroup of finite index contained in $\mathrm{SO}_0(f_n; \mathbf{Z})$ or $\mathrm{SO}_0(f_{n+1}; \mathbf{Z})$, $(4 \leqslant n \leqslant 8)$. Since $\mathrm{SO}_0(f_n; \mathbf{Z})$ is a subgroup of $\mathrm{SO}_0(f_{n+1}; \mathbf{Z})$ for all n, we deduce that if $4 \leqslant n \leqslant 7$, Γ contains a subgroup of finite index conjugate to a subgroup of $\mathrm{SO}_0(f_8; \mathbf{Z})$. In dimension $n = 8$, the simplex groups corresponding to the first three Coxeter diagrams in the Appendix are all commensurable with $\mathrm{SO}(f_8; \mathbf{Z})$. The final Coxeter diagram in dimension 8 gives a form with determinant $= -2$. We show below that the form is rationally equivalent to a scalar multiple of f_8.

Theorem 1.1 will follow by establishing $\mathrm{SO}_0(f_8; \mathbf{Z})$ is GFERF. This is done analogously to Section 3.3 of [1] (see Lemma 3.4.4). We recall this briefly. The Coxeter diagram

corresponds to a hyperbolic 8-simplex Σ with one ideal vertex. Vinberg [18] showed that the hyperbolic simplex group generated by reflections in the faces of Σ is the group $G = \mathrm{O}_0(f_8; \mathbf{Z})$. Now, the vertex opposite the plane corresponding to the rightmost node of the diagram is stabilised by the finite group E_8, so that we may assemble $|E_8|$ copies of Σ and form an all right ideal polyhedron Q in $\mathbf{H}^8$. The reflection group generated from the faces of Q is GFERF by Theorem

3.1 of [1] and by construction it is commensurable with the group G above, which is therefore GFERF.

For the final form in dimension 8, from Theorem 5.1 the form is rationally equivalent to

$$\langle 1, 1, 1, 1, 1, 1, 1, 1, -2\rangle.$$

By Lemma 2.2 we can multiply this form by 2 and not change the group. This gives a form rationally equivalent to

$$\langle 2, 2, 2, 2, 2, 2, 2, 2, -1\rangle$$

which, as is easy to check by the methods above, has determinant -1 and trivial Hasse invariant. This completes the proof. □

Proof of Theorem 5.1. The calculation needs to be done on a case by case basis, so we indicate a typical example which contains all the essential ideas.

It will be helpful to recall that for $d \in \mathbf{Z}$, all the Hilbert symbols

$$\left(\frac{1, 1}{\mathbf{Q}}\right), \left(\frac{1, d}{\mathbf{Q}}\right), \left(\frac{1, -d}{\mathbf{Q}}\right), \left(\frac{d, -d}{\mathbf{Q}}\right)$$

are quaternion algebras of 2×2 matrices over $\mathbf{Q}$, and in particular make a trivial contribution to the Hasse invariant. (See [10] Chapter 3, p. 60)

We discuss the example in dimension 7, with $d(q) = -7$. The Gram Matrix in this case is

$$\begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

Changing basis to

$$\begin{aligned} f_1 &= \{1, 0, 0, 0, 0, 0, 0, 0\}, \\ f_2 &= \{0, 1, 0, 0, 0, 0, 0, 0\} + 1/2f_1, \\ f_3 &= \{0, 0, 1, 0, 0, 0, 0, 0\} + 0f_1 + 2/3f_2, \\ f_4 &= \{0, 0, 0, 1, 0, 0, 0, 0\} + 0f_1 + 0f_2 + 3/4f_3, \\ f_5 &= \{0, 0, 0, 0, 1, 0, 0, 0\} + 0f_1 + 0f_2 + 0f_3 + 4/5f_4, \\ f_6 &= \{0, 0, 0, 0, 0, 1, 0, 0\} + 0f_1 + 0f_2 + 0f_3 + 0f_4 + 5/6f_5, \\ f_7 &= \{0, 0, 0, 0, 0, 0, 1, 0\} + 0f_1 + 0f_2 + 0f_3 + 0f_4 + 0f_5 + 6/7f_6, \\ f_8 &= \{0, 0, 0, 0, 0, 0, 0, 1\} + 0f_1 + 2/3f_2 + 1/2f_3 + 2/5f_4 + 1/3f_5 + 2/7f_6 + 9/8f_7, \end{aligned}$$

one finds that the form determined by the Gram Matrix is rationally equivalent to

$$F = \langle 1, 3, 6, 10, 15, 21, 7, -7\rangle.$$

One could compute the Hasse invariant directly from the form, but it is slightly easier to make some preliminary reductions. For example, $\langle -7, 7\rangle$ has signature $(1, 1)$, determinant -1 (recall this is modulo squares) and the Hilbert symbol $((-7, 7)/\mathbf{Q})$ is a matrix algebra. This is the same as the invariants for the quadratic form $\langle -1, 1\rangle$ so that they are rationally equivalent and we deduce that F is rationally equivalent to $\langle 1, 3, 6, 10, 15, 21, 1, -1\rangle$. Letting $\sim$ denote rational equivalence of quadratic forms, we get $\langle 3, 6\rangle \sim \langle 1, 2\rangle$ since the determinants are the same and $((3, 6)/\mathbf{Q})$ and $((1, 2)/\mathbf{Q})$ both represent the matrix algebra over $\mathbf{Q}$. A similar argument applies to give $\langle 10, 15\rangle \sim \langle 1, 6\rangle$ so that

$$F \sim \langle 1, 3, 6, 10, 15, 21, 1, -1\rangle \sim \langle 1, 1, 2, 6, 1, 21, 1, -1\rangle.$$

Since $\langle 2, -1\rangle \sim \langle -1, 2\rangle$ and $\langle 6, -2\rangle \sim \langle 1, -3\rangle$ we get a further simplification

$$F \sim \langle 1, 1, 1, 6, 1, 21, 1, -2\rangle \sim \langle 1, 1, 1, 1, 1, 21, 1, -3\rangle$$

Finally, $\langle 21, -3\rangle \sim \langle 1, -7\rangle$ and we have

$$F \sim \langle 1, 1, 1, 1, 1, 1, 1, -7\rangle$$

as required. □

Remarks. (1) In the Appendix we list forms by dimension and determinant. By Theorem 5.1 these forms are equivalent over the rationals to a form $\langle 1, \dots, 1, -k\rangle$. As is easy to see this form has trivial Hasse invariant, and this is implicit in these lists.

(2) One can actually simplify some of the considerations required in stabilizing the forms. For example, in dimension 5, all the simplex groups giving the quadratic forms of determinant -1 are commensurable. This is also noted in [8] and [9].

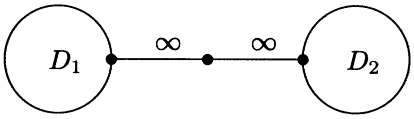
In dimension 4, it can be deduced that simplex groups associated to the Coxeter diagrams shown in (4.1), (4.4), (4.5), (4.6), (4.8) and (4.9) are all commensurable with $O_0(f_4; \mathbf{Z})$, and so are already GFERF. Some of this can be seen by directly checking equivalence, and applying Lemma 2.1. The forms obtained from the Gram matrix in the cases (4.5), (4.8) and (4.9) are $2f_4$, and so by Lemma 2.2, the unit groups obtained in these cases are again just $\mathrm{O}_0(f_4; \mathbf{Z})$. (4.6) can be handled in a similar way. These commensurability relationships are discussed further in [9].

6. Final Remarks

The methods developed above all hinge on [1] (following [15]), on achieving GFERF by getting groups commensurable with Coxeter groups arising as groups generated by reflections in all right polyhedra in $\mathbf{H}^n$. As a word of caution on how far one could generalize such a construction we prove the following (see Section 2.3 for notation).

PROPOSITION 6.1. *There is an irreducible all right Coxeter group that is not LERF.*

Proof. Let G be the hyperbolic Coxeter group generated by reflections in an all right pentagon in $\mathbf{H}^2$, and D the Coxeter diagram of G. Take a pair of disjoint copies D_1 and D_2 of D. Then the Coxeter diagram of the disjoint union is that of the reducible group $G_1 \times G_2$ with $G_i \cong G$ for $i = 1, 2$. Note that since G_1 contains a free group F of rank 2, so $G_1 \times G_2$ contains $F \times F$. This group is well-known not to be LERF, since it does not have a positive solution to the generalized word problem [13] pp. 193–194. Hence $G_1 \times G_2$ is not LERF. To get an irreducible Coxeter group connect the Coxeter diagrams D_1 and D_2 by a pair of edges each labelled ∞, incident at the vertex v as shown.



This diagram D_0 determines an irreducible Coxeter system (W, S) say (see [5]). We claim W is not LERF. To see this, if we delete the reflection r_v (associated to the vertex v) from the set S then by definition of the Coxeter relations, we obtain a subgroup of W isomorphic to $G_1 \times G_2$, and in particular from the remarks above cannot be LERF. This completes the proof. □

Remark. The use of the all right pentagon is not important, the above construction works when any compact all right polyhedron in $\mathbf{H}^n$ is used.

In the language of geometric group theory, the role of geometrically finite is played by quasi-convex (see [4]). The obvious generalization of [1] and [15] is the following:

QUESTION. Let W be an all-right Coxeter group. If H is a quasi-convex subgroup of W, is W H-separable?

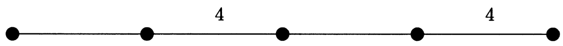
We do not know if the non-separable subgroups constructed in the example are quasi-convex or not. The methods of [1] seemed best suited to generalize to the case when the all right Coxeter group is word hyperbolic(see [4]).

QUESTION. Let W be a word hyperbolic all-right Coxeter group. If H is a quasi-convex subgroup of W, is W H-separable?

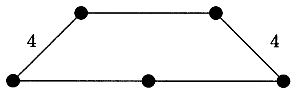
Note the example above is not word hyperbolic since it contains $\mathbf{Z} \oplus \mathbf{Z}$.

Appendix

Dimension $= 4$, $d(q) = -1$

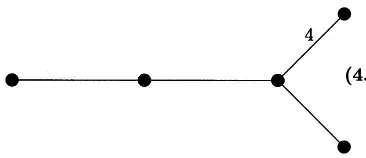


(4.1)

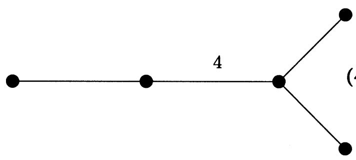


(4.2)

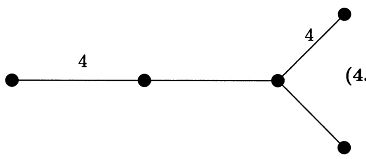
Dimension $= 4$, $d(q) = -2$



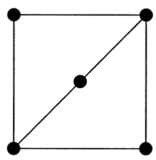
(4.3)



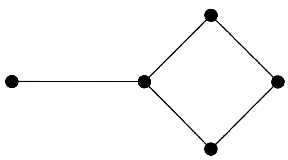
(4.4)



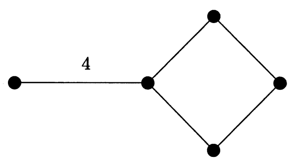
(4.5)



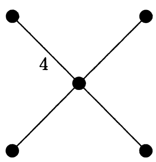
(4.6)



(4.7)

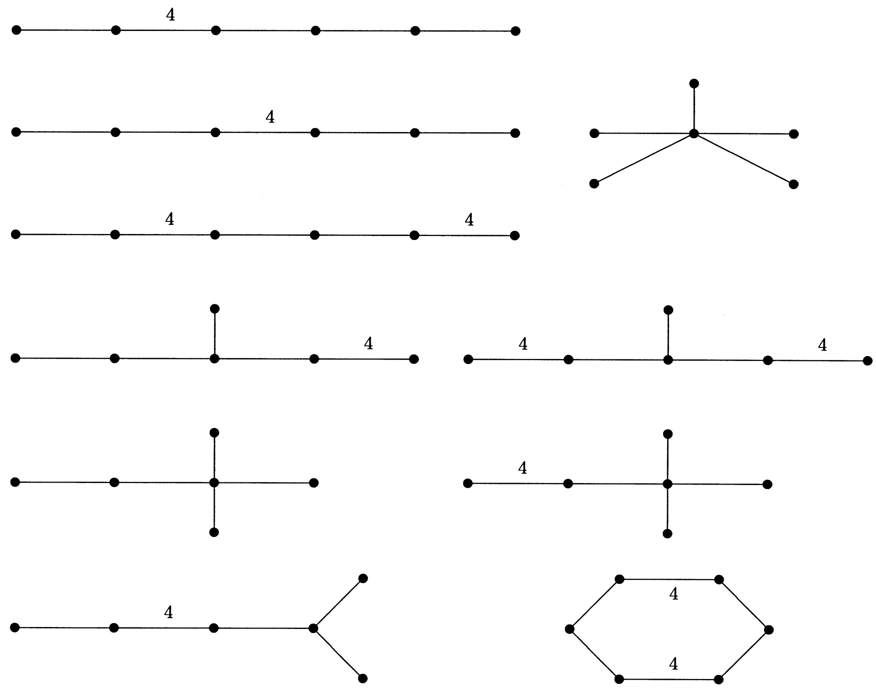


(4.8)

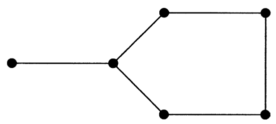


(4.9)

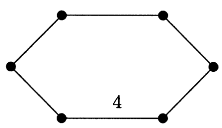
Dimension $= 5$, $d(q) = -1$



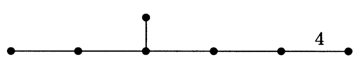
Dimension $= 5$, $d(q) = -5$



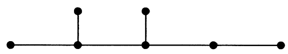
Dimension $= 5$, Nonarithmetic



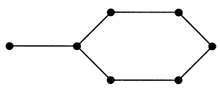
Dimension $= 6$, $d(q) = -1$



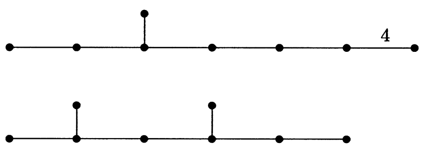
Dimension $= 6$, $d(q) = -2$



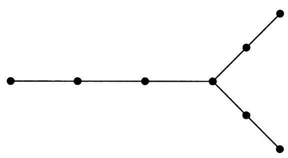
Dimension $= 6$, $d(q) = -3$



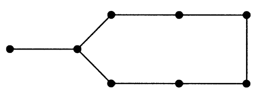
Dimension $= 7$, $d(q) = -1$



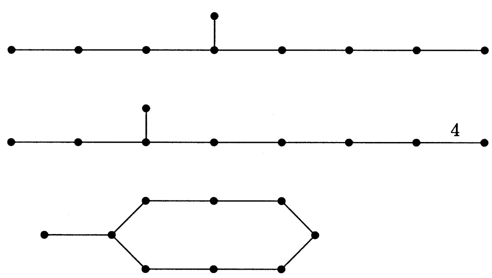
Dimension $= 7$, $d(q) = -3$



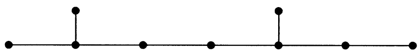
Dimension $= 7$, $d(q) = -7$



Dimension $= 8$, $d(q) = -1$



Dimension $= 8$, $d(q) = -2$



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