Surface subgroups and subgroup separability in 3-manifold topology

D. D. Long (UC Santa Barbara)
A. W. Reid (UT Austin)

Chapter 1

Introduction.

The picture that has emerged of the structure of closed 3-manifolds over the last thirty years is that they are geometric which roughly speaking means that any 3-manifold admits a canonical decompositions into pieces and each piece is (in a certain sense which need not really concern us here) modeled on one of eight geometries. A reasonable picture to bear in mind is the situation for closed surfaces, in this case there are three models and any surface is either spherical, flat or hyperbolic depending on the sign of the Euler characteristic.

Of the eight geometries in dimension three, all but the hyperbolic geometry are actually quite well understood, but the structure of hyperbolic 3-manifolds (or indeed hyperbolic n-manifolds) remains rather mysterious. This class has proved to be a magnet for research not only in topology, but also in other fields, including number theory and geometric group theory. This course will be concerned with aspects of the central question of whether hyperbolic 3-manifolds always contain surface groups as well as the related (but presumably deeper) questions of whether such immersed surfaces can be promoted in a finite sheeted covering to an embedded surface.

Such questions can be approached in many ways, here we shall introduce many of the classical facts related to such problems, and in particular highlight the connection with group theory. In this setting, the surface group question is closely related to various separability properties of the fundamental groups of hyperbolic manifolds.

These notes are organized as follows. We begin with some introductory material about the basics of hyperbolic manifolds. The main consequence of the existence of a hyperbolic structure from this point of view is that one obtains a canonical representation of the fundamental group of the manifold into the group of isometries of the relevant hyperbolic space. We then recall some classical facts about 3-manifolds, in particular we list four open conjectures which motivate all that follows.

In §3, we begin discussions of separability properties of groups and introduce the notions of residually finite and the much more powerful idea of subgroup separability. It turns out that these are intimately related to the questions about surface groups that we wish to pursue, and we show that these purely geometric notions have natural topological definitions. We go on to prove Scott's theorem which was historically the first real breakthrough in the area of subgroup separability. We also describe certain surface groups which are known to be separable, the so-called totally geodesic surfaces.

This leads naturally to the notion of an arithmetic hyperbolic 3-manifold and these are briefly explored; one consequence of this technology is that one can show there are closed hyperbolic 3-manifolds which contain no totally geodesic surface. An important class of arithmetic manifolds is the so-called Bianchi groups and we sketch the recent proof that the Bianchi groups are subgroup separable.

Finally we close with a discussion of new directions recently opened in the attacks on these problems - in particular we discuss local retractions over cyclic groups and show that Bianchi groups and (most) Coxeter groups admit such retractions.

1.1 Preliminaries.

We begin with reviewing very briefly the notion of a hyperbolic n-manifold. We refer to [4] and [25] for standard facts about hyperbolic space, many of which we will use without proof. Since our focus here is largely algebraic, many of the more basic geometric aspects can be easily ignored in this exposition.

Let $\mathbf{H}^n$ denote the unique connected simply connected Riemannian manifold all of sectional curvatures are -1. Suppose that M is a closed topological manifold, that is to say it is a compact Hausdorff (and paracompact!) topological space with the property that every point lies in an open neighbourhood homeomorphic to Euclidean n-space. Then a *hyperbolic structure* on M is an atlas of charts $\phi_U : U \to \mathbf{H}^n$ with the property that on the overlaps we have

$$\phi_V \circ \phi_U^{-1} : \phi_U(U \cap V) \longrightarrow \phi_V(U \cap V) \subset \mathbf{H}^n$$

is the restriction of a hyperbolic isometry from hyperbolic space to itself. Such isometries are real analytic and a standard construction shows that a hyperbolic structure as defined above produces a homeomorphism

$$D : \tilde{M} \longrightarrow \mathbf{H}^n$$

where $\tilde{M}$ is the universal covering of M. This identification, in turn, yields a way of transferring the canonical action of $\pi_1(M)$ on $\tilde{M}$ to an action of $\pi_1(M)$ by isometries of $\mathbf{H}^n$, that is to say, we obtain a faithful representation

$$\rho : \pi_1(M) \longrightarrow \mathrm{Isom}(\mathbf{H}^n)$$

called the *holonomy representation.* As is discussed in the standard texts, this representation has many properties, the most important for us being that the action of $\rho(\pi_1(M))$ on $\mathbf{H}^n$ is properly discontinuous, so the projection $\mathbf{H}^n \to \mathbf{H}^n/\pi_1(M)$ is a covering map and we have a homemorphism $M \cong \mathbf{H}^n/\rho(\pi_1(M))$.

The existence of a hyperbolic structure imposes many topological constraints on the manifold. For example, the above discussion shows that M has contractible universal covering, so that M is a $K(\pi_1(M), 1)$; from which it follows, for example, that $\pi_1(M)$ is torsion free. Perhaps the most important basic theorem about this situation is *Mostow's rigidity theorem* which says that for $n \geq 3$, the representation ρ we have constructed above is the unique one (up to conjugacy in $Isom(\mathbf{H}^n)$) with the properties of being discrete and faithful. This means that many geometrical properties of M (for example, its volume when equipped with a topological metric) are actually topological invariants, however, we shall make scant use of these facts, at least at the outset.

1.1.1 Models of hyperbolic space and groups of isometries

It will be convenient to recall two models of $\mathbf{H}^n$ that will be useful in what follows. The upper half space model for $\mathbf{H}^n$ is defined to be:

$$U = \{(x_1, x_2, \ldots x_n) \in \mathbf{R}^n : \ x_n > 0\}$$

equipped with the metric defined by

$$\mathrm{ds}^2 = \frac{x_1^2 + x_2^2 \ldots + x_n^2}{x_n^2}.$$

This is particularly useful in low dimensions (i.e. $n = 2, 3$), since it affords a description of the groups of orientation-preserving isometries of these models as $\mathrm{PSL}(2, \mathbf{R})$ and $\mathrm{PSL}(2, \mathbf{C})$ respectively. The full groups of isometries of these models can be identified with $\mathrm{PGL}(2, \mathbf{R})$ and $< \mathrm{PSL}(2, \mathbf{C}), \tau >$ where $\tau : U \to U$ is the isometry that is a reflection in the (x_1, x_3)-plane.

A description of the groups of isometries as linear groups is universally obtained by using the hyperboloid model. Let $V = \mathbf{R}^{n+1}$, and equip V with the $(n+1)$-dimensional quadratic form $< 1, 1 \ldots, 1, -1 >$, which we denote by f_n throughout. Consider the upper sheet of the hyperboloid $f_n(x) = -1$, which we denote for now by $\mathcal{H}^n$. Associated to the quadratic form f_n is the bilinear form $B_n : V \times V \to \mathbf{R}$, and this can be used to define a metric

$$d : \mathcal{H}^n \times \mathcal{H}^n \to \mathbf{R}$$

by decreeing that d is the function that assigns to each pair $(x, y) \in \mathcal{H}^n \times \mathcal{H}^n$ the unique number $d(x, y)$ such that

$$\cosh(d(x, y)) = -B_n(x, y).$$

$(\mathcal{H}^n, d)$ defines a metric space that is isometric to $\mathbf{H}^n$ with the metric described above, and we henceforth we make no distinction. The group of isometries of $\mathbf{H}^n$ can be identified as follows. The orthogonal group of the form f_n is

$$\mathrm{O}(n, 1; \mathbf{R}) = \{X \in \mathrm{GL}(n + 1, \mathbf{R}) \ : \ X^t F_n X = F_n\},$$

where F_n is the diagonal matrix associated to the quadratic form f_n. This is a lie group and has four connected components. The subgroup of index 2 which preserves $\mathcal{H}^n$ is denoted $\mathrm{O}_0(n,1;\mathbf{R})$ and is identified with $\mathrm{Isom}(\mathbf{H}^n)$. Passing to the subgroup consisting of elements of determinant 1, denoted $\mathrm{SO}(n,1;\mathbf{R})$, it follows that $\mathrm{SO}_0(f_n;\mathbf{R})$ may be identified with $\mathrm{Isom}^+(\mathbf{H}^n)$.

1.1.2

The non-trivial elements acting on $\mathbf{H}^n$ can divided into 3 classes; *parabolic elements*, *elliptic elements*, and *hyperbolic elements*. In terms of the action on $\mathbf{H}^n \cup \mathbf{S}^{n-1}_\infty$, an element γ is parabolic if it has precisely one fixed point on $\mathbf{S}^{n-1}_\infty$, elliptic if it has no fixed points on $\mathbf{S}^{n-1}_\infty$, but has a fixed point in $\mathbf{H}^n$, and hyperbolic if it has two fixed points on $\mathbf{S}^{n-1}_\infty$. In this last case, if we denote the fixed points by α_+ and α_-, there is a geodesic A_γ (called *the axis* of γ) in $\mathbf{H}^n$ with endpoints α_+ and α_-, and the element γ acts by translating by some distance along A_γ, and possibly rotating through some angle.

In the case of $n = 2, 3$, one can make use of the description of elements as 2×2 matrices (up to sign) and get an algebraic characterization. In this setting a non-trivial element γ is parabolic if $\mathrm{tr}^2(\gamma) = 4$, elliptic if $\mathrm{tr}^2(\gamma) < 4$ and hyperbolic otherwise. In the last case, one can distinguish the class of purely hyperbolic elements, i.e. those with $\mathrm{tr}^2(\gamma) > 4$.

1.1.3 3-Manifold topology and surface subgroups

The case of most interest is when $n = 3$. Closed hyperbolic 3-manifolds admit an amazing number of ways to be studied; we shall take the point of view here which is not only of interest for itself, but is connected to other mainstream problems in areas of mathematics other than topology.

A primary tool for understanding 3-manifolds (not at this stage, necessarily hyperbolic) emerged in the sixties in the seminal work of Waldhausen [32]. To describe this work, we need to introduce some definitions - for simplicity, we shall restrict attention to the case of closed 3-manifolds. A closed 3-manifold is said to be *irreducible* if every embedded 2-sphere in M bounds a ball in M. For example, the Schoenflies theorem shows that the 3-sphere S^3 is irreducible and it is not hard to show that this implies that S^3/F where F is some finite group acting freely is also irreducible. In contrast, if M is irreducible and has infinite fundamental group, then one sees easily that its universal covering is contractible, so that it is a $K(\pi_1(M), 1)$; for the purposes of these notes, a good example to bear in mind is that of a closed hyperbolic 3-manifold.

We say a closed 3-manifold is *sufficiently large* if there is an embedding of a closed orientable 2-manifold $i : F \to M$ with the property that either F is a 2-sphere and the image of F does not bound a 3-ball in M, or F has genus at least one, and the induced map $i_* : \pi_1(F) \to \pi_1(M)$ is injective. We often identify the surface with its image and refer to it as an *incompressible surface* in M. Notice that this rules out the possibility of manifolds covered by 3-sphere. We also point out that with some fairly elementary 3-manifold topology, if we assume that M is irreducible, then it can be shown that an essential map $M \to S^1$ can be homotoped so that a generic point preimage is incompressible and so in particular, if $H_1(M)$ is infinite, then it is sufficiently large.

Manifolds which are irreducible and sufficiently large are often referred to as *Haken*. The results of [32] give a marvelous array of techniques for understanding Haken manifolds and a good deal more is known about this class than for the general irreducible 3-manifold, even if this manifold has infinite fundamental group. For example, Waldhausen proves that such manifolds are determined up to *homeomorphism* by their fundamental groups, and that homotopic homeomorphisms are in fact isotopic.

Historically speaking, there was a feeling in the late 60's and early 70's that being Haken was rather common, but the work of Thurston in the middle 70's suggested otherwise and nowadays it is felt that perhaps Haken 3-manifolds are in some sense rather rare. So one needs to develop techniques which suffice even if the manifold is not Haken, this being especially important for the case that the manifold is hyperbolic. There has been an enormous amount of work in this direction in low dimensional topology, we shall focus on just one aspect, the notion of being virtually Haken and related ideas.

Definition 1.1.1. *An irreducible* 3*-manifold* M *is defined to be* virtually Haken *is there is a finite sheeted covering* $M_F \longrightarrow M$ *for which* M_F *is Haken.*

In contrast to the situation for Haken manifolds, it's generally expected that every irreducible 3-manifold with infinite fundamental group will in fact be virtually Haken and the key case (in fact the only case left assuming the truth of Perelman) is that of a closed hyperbolic 3-manifold. In fact there is evidence that much stronger conjectures hold, in particular all the following conjectures are generally expected to be true:

Surface Conjecture *Every closed hyperbolic* 3*-manifold contains the fundamental group of a closed orientable surface, (necessarily of genus* > 1*).*

Conjecture 0 *Every closed hyperbolic* 3*-manifold has a finite sheeted covering which is Haken.*

Conjecture 1 *Every closed hyperbolic* 3*-manifold has a finite sheeted covering which has* $H_1(M)$ *infinite.*

Conjecture ∞ *Given a* $K > 0$*, every closed hyperbolic* 3*-manifold has a finite sheeted covering which has* $H_1(M)$ *infinite and of rank* $> K$*.*

These are, of course, ordered so that each conjecture implies all the conjectures above it.

The question of whether every closed hyperbolic 3-manifold contains a surface group has received a good deal of attention and while there are some results known ([13] and [23]) this is not the aspect of the problem that we shall concentrate on in these notes, rather we shall ask the question, suppose that one is given a surface subgroup, how can it be used to address the other problems? We note that a hyperbolic manifold is a $K(G,1)$ so standard obstruction theory guarantees that the inclusion map $i : \pi_1(F) \to \pi_1(M)$ is induced by a continuous map $f : F \to M$ which we may suppose to be an immersion. It is the desire to take this non-embedded π_1- injective surface and the attempt to promote it to an embedding in a finite sheeted covering that guides us in what follows, and relates to *separability properties* of groups.

1.1.4 An example

A good example of a Haken hyperbolic 3-manifold to keep in mind is that of a *surface bundle over the circle*. These are constructed as follows. Let Σ_g be a closed orientable surface of genus $g \geq 2$, and $\phi : \Sigma_g \to \Sigma_g$ a homeomorphism. The *Mapping Torus* of ϕ is the closed 3-manifold, denoted M_ϕ that is formed by taking

$$\Sigma_g \times [0,1]/\{(x,0) \equiv (\phi(x),1)\}.$$

From this description, M_ϕ is seen to be a fiber bundle over S^1 and the fibers are embedded incompressible surfaces homeomorphic to Σ_g. In particular these manifolds are Haken; indeed the first Betti number of M_ϕ is positive.

Part of proof Thurston's hyperbolization theorem shows that M_ϕ is hyperbolic if and only if ϕ is a pseudo-Anosov map. This construction fits with the conjectures discussed above. A strengthening of Conjecture 1 above asks:

Conjecture $1'$: *Let M be a closed hyperbolic 3-manifold. Then M has a finite sheeted cover which is a surface bundle over the circle.*

Chapter 2

Separability properties of groups

Motivated by the discussion in §3.2 our point of view is that one wishes to try and understand a complicated infinite group, namely $\pi_1(M)$, and one way of doing this is to attempt to understand the finite quotients of this group; equivalently, the standard theory of covering spaces shows that one might wish to gain insights into M by thinking about its finite covering spaces. This raises an immediate problem: On the face of it, there is no *a priori* reason to think that the group $\pi_1(M)$ has any subgroups of finite index at all. This is the first issue that we shall address and is the motivation for the next section

2.1 Residual finiteness

Definition 2.1.1. *A group G is said to be* residually finite*, if, given any nonidentity element $g \in G$, there is a subgroup of finite index H in G with $g \notin H$.*

By using the action of G on the left cosets of H by left translation, we obtain a permutational representation $p : G \longrightarrow S_{[G:H]}$ from which it follows that H contains the normal subgroup of finite index $\ker(p)$

and there is therefore a homomorphism

$$\phi : G \longrightarrow A \cong G/\ker(p)$$

where it is visible that $|A| < \infty$ and $\phi(g) \neq I$.
Conversely, if such a homomorphism exists, then, $g \notin \ker(\phi)$ which has finite index, so that these conditions are equivalent. We shall use them both. We also note that the restriction to one element is not necessary, one sees easily that it is equivalent replace the element g by any finite set of elements in the above definition.

Residual finiteness guarantees a large supply of subgroups of finite index in G, indeed it shows that the intersection of all the subgroups of finite index in G yields only the identity element I. For future reference we notice that the algebraic condition above is equivalent to the following geometric condition (which we do not attempt to state in the most general form possible)

Lemma 2.1.2. *Suppose that M is a closed manifold and $G = \pi_1(M)$. Then G is residually finite if and only if the following condition holds:*

For every compact subset C of $\tilde{M}$, there is a finite sheeted covering M_F of M so that the natural map $\tilde{M} \to M_F$ is an embedding when restricted to C.

Proof: We recall that the natural action of $\pi_1(M)$ on its universal covering is properly discontinuous, that is to say, given any compact set C, the number of group elements for which $gC \cap C \neq \emptyset$ is finite.

If we assume the group G is residually finite, we can find a subgroup of finite index H in G which excludes this finite number of group elements, and elementary covering space theory now shows that the set C embeds in the finite covering M_F corresponding to H.

Conversely, suppose the geometric condition holds, and we are given a nontrivial element of the fundamental group g. We may represent g by a based map $[0, 1] \to M$ and nontriviality is equivalent to the preimage of this map being an arc in the universal covering which is to say the endpoints are distinct. Taking these two endpoints as the compact set C, we see that there is a finite sheeted covering of M in which this arc fails to close up, that is to say a finite sheeted covering M_F to which the loop g does not lift as a loop. Covering space theory shows that the element g does not lie in the subgroup corresponding to the covering M_F.

It will also be useful to note the following simple group theoretic facts

Lemma 2.1.3. *Let G be a group and H a subgroup of G*
(i) If G is residually finite, then so is H.
(ii) If H is residually finite and $[G : H] < \infty$, then G is residually finite.

Proof: (i) Take any nonidentity element h of H; G is residually finite so there is a homomorphism to a finite group $\phi : G \to A$ which does not kill g; restrict this homomorphism to H.
(ii) Given a $g \in G$, either $g \notin H$ in which case we are done, or we can find a subgroup of finite index in H which excludes g; this subgroup also has finite index in G.

While it is by no means true that all groups are residually finite (see the example given at the end of this section, and indeed there are infinite groups with no subgroups of finite index at all, see for example [28]), many of the groups which arise in nature are, and, in fact, this is a fairly *soft* property in the sense that there are quite general results which guarantee that a group is residually finite. The most famous of these, and the most useful for us, is Mal'cev's theorem:

Theorem 2.1.4. *Let R be a finitely generated integral domain. Then for any nonidentity element $g \in \mathrm{GL}(n, R)$ there is a finite field K and a homomorphism*

$$\phi : \mathrm{GL}(n, R) \to \mathrm{GL}(n, K)$$

so that $\phi(g) \neq I$.

Proof: The key ingredient is the following purely algebraic result:

Lemma 2.1.5. *Let R be a finitely generated integral domain. Then*
(i) The intersection of all the maximal ideals of R is the zero ideal.
(ii) If $\mathcal{M}$ is any maximal ideal of R, then $R/\mathcal{M}$ is a finite field.

If we assume this result, we may prove Mal'cev's theorem: The given element g is not the identity element, so that at least one of the elements of the difference $g - I$ is nonzero; fix such an element an denote it by r. By (i) of the Lemma, there is a maximal ideal $\mathcal{M}$ which does not contain r and by (ii) the quotient $R/\mathcal{M}$ is a finite field. The map induced from projection

$$\phi : \mathrm{GL}(n, R) \longrightarrow \mathrm{GL}(n, R/\mathcal{M})$$

has the required properties, since by choice of r, the element $\phi(g)$ is not the same in the quotient as the element $\phi(I)$.

This result is used in the following fashion. As discussed in §3.1, the groups $\mathrm{Isom}(\mathbf{H}^n)$ are all subgroups of $\mathrm{GL}(n+1, \mathbf{R})$. If M is a hyperbolic n-manifold, then although the canonical representation $\rho : \pi_1(M) \to \mathrm{Isom}(\mathbf{H}^n) < \mathrm{GL}(n+1, \mathbf{R})$ apparently takes its values in the real field, we can form a finitely generated integral domain by taking a generating set for $\pi_1(M)$ and looking at the subring of $\mathbf{R}$ generated by the entries of the ρ-images of these generators - together with 1 say. Denoting this ring by R we see that $\rho : \pi_1(M) \to \mathrm{GL}(n+1, R)$. Since ρ is injective we have proved:

Theorem 2.1.6. *The fundamental group of a closed hyperbolic n-manifold is residually finite.*

Indeed, Mal'cev's theorem is a rich source of residually finite groups, since many commonly occurring groups have faithful linear representations, free groups and surface groups being the most obvious examples.

Assuming Perelman's solution to geometrization in dimension 3 it follows that all compact 3-manifolds have residually finite fundamental groups. Indeed, for Haken manifolds this can be established without geometrization (see [15]), and the proof assuming geometrization builds on [15].

However, the following is still an interesting open problem.

Question 2.1.7. *Let M be a Haken 3-manifold. Does $\pi_1(M)$ admits a faithful representation into $\mathrm{GL}(n, \mathbf{C})$ for some n?*

Example of a non-residually finite group:

The most famous class of examples of finitely generated non-residually finite groups are the *Baumslag-Solitar groups* defined as follows. Let p and q be natural numbers and define the group

$$\mathrm{BS}(p,q) =< a, b|\ ba^p b^{-1} = a^q > .$$

If neither p or q is 1, it was shown in [3] that the group $\mathrm{BS}(p,q)$ is not residually finite. Note that when $p = 1$ the group $BS(1,q)$ is linear and hence residually finite. The linear representation is described as follows; take $a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $a = \begin{pmatrix} \sqrt{q} & 0 \\ 0 & 1/\sqrt{q} \end{pmatrix}$.

2.2 Subgroup separability

We now introduce a property, that while being related to residual finiteness is a good deal stronger.

Definition 2.2.1. *Let G be a group and H a finitely generated subgroup, G is called H-*subgroup separable *if given any $g \in G \setminus H$, there exists a subgroup $K < G$ of finite index with $H < K$ and $g \notin K$.*

G is called subgroup separable *(or* LERF*) if G is H-subgroup separable for all finitely generated $H < G$.*

In contrast to the situation for residually finite groups, one cannot reduce to the case that K is normal in G; since conjugates of K will not in general contain H. It is left to the reader to verify that the equivalent condition which refers to homomorphisms in this case is: The subgroup H separable in the group G if and only if for element $g \notin H$, there is a homomorphism

$$\phi : G \longrightarrow A$$

where $A < \infty$ and $\phi(g) \notin \phi(H)$.

It is also left to the reader to verify that the following group theoretic properties continue to hold

Lemma 2.2.2. *Let G be a group and H a subgroup of G*
(i) If G is subgroup separable, then so is H.
(ii) If H is subgroup separable and $[G : H] < \infty$, then G is subgroup separable.

Subgroup separability is an extremely powerful property, and it is much stronger than residual finiteness. In particular, there is no theorem analogous to Mal'cev's - more or less every example must be treated on its individual merits. The class of groups for which subgroup separability is known for *all* finitely generated subgroups is extremely small; abelian groups, free groups, surface groups and carefully controlled amalgamations of these.

Examples.
(i) (Hall, [14]) Free groups are subgroup separable.
(ii) (Scott, [27]) Surface groups are separable.
(iii) (Folklore) If A and B are subgroup separable then so is $A * B$.
(iv) If A, B and C are finite, then $A *_C B$ is subgroup separable. (It contains a free subgroup of finite index [26])

The powerful nature of this property is underlined by listing some apparently well-behaved groups which fail to be subgroup separable. We note that $\mathrm{SL}(2, \mathbf{Z})$ contains a free subgroup of finite index, so that it follows from Hall's theorem together with (iv) above that it is subgroup separable. However, as we discuss in §5 this fails for $\mathrm{SL}(n, \mathbf{Z})$ when $n \geq 3$. Another interesting example is the following. Let F_n denote the free group of rank n. Then $F_n \times F_n$ is not subgroup separable although it is the direct product of subgroup separable groups. This failure can be attributed to the lack of a solution to the generalized word problem for these groups, see [21] for instance. It is known that LERF (like residual finiteness for the word problem) implies a solution to the generalized word problem. The groups $\mathrm{BS}(1, q)$ are residually finite, but not LERF. This can be seen by checking that the the cyclic subgroup $< a >$ is not separable. This is a special case of a more general result of Blass and Neumann [5] that shows that if G is a group and $H < G$ with the property that H is conjugate into a proper subgroup of itself, then G is not H-separable.

From the point of view of 3-dimensional topology, it is known that there compact 3-manifolds whose fundamental groups are not LERF. These examples are all closely related to an example in [11] of a 1-punctured torus bundle over the circle whose fundamental group is not LERF.

There is also an analogue of a geometric equivalence.

Lemma 2.2.3. *Suppose that M is a closed manifold and $G = \pi_1(M)$. Then G is subgroup separable if and only if the following condition holds:*

For every finitely generated subgroup $H < \pi_1(M)$ and every compact subset C of $\tilde{M}/H$, there is a finite sheeted covering $M_F = \tilde{M}/K \to M$ subordinate to $\tilde{M}/H$, (i.e. $H \leq K$) so that the natural map $\tilde{M}/H \to \tilde{M}/K$ is an embedding when restricted to C.

It is this geometric equivalence which fuels much of the interest in this property in low dimensional topology and is discussed in detail in the next section.

Chapter 3

Subgroup separability and Scott's theorem

As described in §2.3, one of the central problems we are interested in solving is that if we are given an immersed π_1-injective surface F, can we promote it to an embedding in a finite sheeted covering of the ambient 3-manifold M. Restricting attention to the hyperbolic case, it is known that the manifold $\mathbf{H}^3/\pi_1(F)$ is a topological product $F \times \mathbf{R}$ and therefore contains an embedding of closed orientable surface homeomorphic to F. Taking this surface to be the compact set C in the geometric version of the subgroup separability property described by Lemma 2.2.3, we see that there is a subgroup of finite index $K > \pi_1(F)$ in $\pi_1(M)$, so that the surface F embeds in $\mathbf{H}^3/K$.

There is an important reduction which we introduce at this point. We first need a definition. We refer to [25] for some of the details that we omit. Suppose that H is a subgroup of a discrete group of hyperbolic isometries G. For technical reasons, we need to exclude subgroups H which are very small, in the sense that it contains a soluble subgroup of finite index. Then associated to H is a canonical set, its *limit set* which we define as the closure in the sphere at infinity of hyperbolic space of the union of all the fixed points of hyperbolic elements of H. We denote the limit set by $\Lambda(H)$. It is clear from this definition that the limit set is H invariant and provided we exclude small subgroups, it contains infinitely many points. We can use this set to construct an H invariant subset lying inside $\mathbf{H}^n$. To this end we define a *totally geodesic hyperplane* to be any co-dimension one totally geodesic submanifold of $\mathbf{H}^n$. The closed set lying to one side of a totally geodesic hyperplane is a *closed half space*.

We define the *convex hull* of the limit set, $C(\Lambda(H))$, to be the intersection of all those half spaces which contain the limit set. One can see easily from the definition that the convex hull is the smallest convex closed set which contains all the geodesic axes of elements of H. The set $C(\Lambda(H))$ is visibly H invariant and we may form the quotient $C(\Lambda(H))/H$. One defines H to be *geometrically finite* if this set is compact. (Or finite volume in the case that H is a subgroup of a finite volume hyperbolic group) If H is not geometrically finite it is called *geometrically infinite*.

It turns out that in the case of subgroups of $\mathrm{Isom}(\mathbf{H}^2)$ that geometrically finite is equivalent to finitely generated, but in general the situation is more complicated. It is not entirely elementary, but not hard, that if H is a normal subgroup of G, then $\Lambda(H) = \Lambda(G)$. In particular, in the case of a hyperbolic surface bundle over the circle (as described in §2.4), the limit set of the fibre surface is the same as the limit set of the whole 3-manifold group, i.e. the whole 2-sphere at infinity.

Somewhat amazingly, the separability situation in this apparently more complicated context can actually be resolved. The recent solution of the Tameness conjecture by Agol [1] and independently by Calegari and Gabai [12] shows that any finitely generated geometrically infinite subgroup Δ of the fundamental group of a finite volume hyperbolic 3-manifold M is a *virtual fiber*; that is to say, M has a finite sheeted cover that is a hyperbolic surface bundle over the circle and the fiber group is Δ. Combining this with prior work of Thurston and Bonahon [29] and [6] we summarize what is important for us in the following theorem

Theorem 3.0.4. *Suppose that M is a finite volume hyperbolic* 3-*manifold. Then the finitely generated geometrically infinite subgroups of $\pi_1(M)$ are separable in $\pi_1(M)$.*

It follows from this result that we may restrict attention to the geometrically finite surface groups and indeed, guided by this and other considerations, the general direction of the theory has been trying to separate geometrically finite subgroups, or in the case of negatively curved groups in the sense of Gromov, the quasi-convex subgroups. The results of Hall and Scott fit into this pattern since coincidentally in those settings, finitely generated and geometrically finite turn out to be equivalent.

3.1 Scott's theorem.

The proof that free groups are subgroup separable given by Hall was essentially algebraic, but can be made geometric fairly easily. However little progress was made for many years until a new technique was introduced by Scott [27]. We shall now give an exposition of this technique, recast in modern language and somewhat refined, as described in [2].

Suppose that P is a finite volume polyhedron in $\mathbf{H}^n$ all of whose dihedral angles are $\pi/2$. Henceforth we call this an *all right polyhedron.* Then the Poincaré polyhedron theorem implies that the group generated by reflections in the co-dimension one faces of P is discrete and a fundamental domain for its action is the polyhedron P, that is to say, we obtain a tiling of hyperbolic n-space by tiles all isometric to P. Let the group so generated be denoted by $G(P)$.

Theorem 3.1.1. *The group $G(P)$ is H-subgroup separable for every finitely generated geometrically finite subgroup $H < G(P)$.*

Proof: As in Lemma 2.2.3 the separability of H is equivalent to the following

Suppose that we are given a compact subset $X \subset \mathbf{H}^n/H$. Then there is a finite index subgroup $K < G(P)$, with $H < K$ and with the projection map $q : \mathbf{H}^n/H \longrightarrow \mathbf{H}^n/K$ being an embedding on X.

We sum up the strategy which achieves this geometric condition. The group H is geometrically finite and one can enlarge its convex hull in $\mathbf{H}^n/H$ so as to include the compact set X in a convex set contained in $\mathbf{H}^n/H$; this convex set lifts to an H-invariant convex set inside $\mathbf{H}^n$. One then defines a coarser convex hull using only the hyperbolic half-spaces bounded by totally geodesic planes which come from the P-tiling of $\mathbf{H}^n$; this hull is denoted by $H_P(C^+)$. This hull is H-invariant and the key point is to show that $H_P(C^+)/H$ only involves a finite number of tiles. The remainder of the proof follows [27] and is an elementary argument using the Poincaré polyhedron theorem and some covering space theory. We now give the details.

Let C be very small neighbourhood of the convex hull of H, regarded as a subset of $\mathbf{H}^n$. In our setting, the group $G(P)$ contains no parabolic elements so that the hypothesis implies that C/H is compact.

The given set X is compact so that there is a t with the property that every point of X lies within a distance t of C/H. Let C^+ be the $10t$ neighbourhood of C in $\mathbf{H}^n$. This is still a convex H-invariant set and C^+/H is a compact convex set containing X.

As discussed above, take the convex hull $H_P(C^+)$ of C^+ in $\mathbf{H}^n$ using the half spaces coming from the P-tiling of $\mathbf{H}^n$. By construction $H_P(C^+)$ is a union of P-tiles, is convex and H-invariant. The crucial claim is:

Claim. $H_P(C^+)/H$ involves only a finite number of such tiles.

To see this we argue as follows.

Fix once and for all a point in the interior of a top dimensional face of the tile and call this its *barycentre*. The tiles we use actually often have a geometric barycentre (i.e. a point which is equidistant from all of the faces) but such special geometric properties are not used; it is just a convenient reference point.

Our initial claim is that if the barycentre of a tile is too far away from C^+, then it cannot lie in $H_P(C^+)$.

The reason for this is the convexity of C^+. If a is a point in $\mathbf{H}^n$ not lying in C^+ then there is a unique point on C^+ which is closest to a. Moreover, if this distance is R, then the set of points distance precisely R from a is a sphere touching C^+ at a single point p on the frontier of C^+ and the geodesic hyperplane tangent to the sphere at this point is the (generically unique) supporting hyperplane separating C^+ from a.

Suppose then that P^* is a tile whose barycentre is very distant from C^+. Let a^* be the point of P^* which is closest to C^+ and let

p be a point on the frontier of C^+ which is closest to P^*. As noted above, there is a geodesic supporting hyperplane $\mathcal{H}_p$ through p which is (generically) tangent to C^+ and separates C^+ from a^*. Let the geodesic joining a^* and p be denoted by γ. Note that since p is the point of C^+ closest to a^*, γ is orthogonal to $\mathcal{H}_p$.

If a^* happens to be in the interior of a tile face of P^*, then this tile face must be at right angles to γ, since a^* was closest. Let $\mathcal{H}_{a^*}$ be the tiling plane defined by this tile face. Since in this case γ is at right angles to both $\mathcal{H}_{a^*}$ and $\mathcal{H}_p$, these planes are disjoint and so the tiling plane separates P^* from C^+ as required.

If a^* is in the interior of some smaller dimensional face, σ, then the codimension one faces of P^* which are incident at σ cannot all make small angles with γ since they make right angles with each other. The hyperplane $\mathcal{H}$ which makes an angle close to $\pi/2$ plays the role of $\mathcal{H}_{a^*}$ in the previous paragraph. The reason is that since a^* and p are very distant and the planes $\mathcal{H}_p$ and $\mathcal{H}$ both make angles with γ which are close to $\pi/2$, the planes are disjoint and we see as above that P^* cannot lie in the tiling hull in this case either.

The proof of the claim now follows, as there can be only finitely many barycentres near to any compact subset of $\mathbf{H}^n/H$.

The proof of subgroup separability now finishes off as in [27]. Let K_1 be the subgroup of $G(P)$ generated by reflections in the sides of $H_P(C^+)$. The Poincaré polyhedron theorem implies that $H_P(C^+)$ is a (noncompact) fundamental domain for the action of the subgroup K_1. Set K to be the subgroup of $G(P)$ generated by K_1 and H, then $\mathbf{H}^n/K = H_P(C^+)/H$ so that K has finite index in $G(P)$. Moreover, the set X embeds as required.

3.2 Totally geodesic surfaces.

As described above, one of the interests for low-dimensional topology in the subgroup separability condition is to pass from a surface subgroup to an embedded non-separating surface in a finite sheeted covering. However, this makes clear that we have no real need to separate all finitely generated subgroups, certain special classes suffice.

In particular, there is a very restricted class of subgroup which can always be separated.

Definition 3.2.1. *A closed surface group $\pi_1(F) < \pi_1(M)$ in a closed hyperbolic manifold is said to be* totally geodesic *if the discrete faithful representation of $\pi_1(M)$ can be conjugated so that the image of $\pi_1(F)$ lies inside* $\mathrm{PSL}(2, \mathbf{R})$.

We have stated this condition algebraically since it is in this form that we shall use it, but it has a natural interpretation in the context of differential geometry. As usual we can construct an immersion $i : F \hookrightarrow M$ realising the surface group and the totally geodesic condition means that the metric induced on the surface F from the ambient 3-manifold M can be arranged to be a constant curvature hyperbolic metric. In this sense they are at the opposite end of the spectrum from geometrically infinite surfaces.

In the language introduced above, the universal covering of F is a totally geodesic hyperplane in $\mathbf{H}^3$. The importance of totally geodesic surfaces for us is the following theorem:

Theorem 3.2.2. *Let M be a closed hyperbolic 3-manifold containing a closed totally geodesic surface F. Then there is a finite sheeted covering of M which contains an embedded closed orientable totally geodesic surface.*

To prove this theorem, we first establish:

Lemma 3.2.3. *Let $\mathcal{C}$ be a circle or straight line in $\mathbf{C} \cup \infty$, and $M = \mathbf{H}^3/\Gamma$ a closed hyperbolic 3-manifold. Let*

$$\mathrm{stab}(\mathcal{C}, \Gamma) = \{\gamma \in \Gamma : \gamma\mathcal{C} = \mathcal{C}\}.$$

Then $\mathrm{stab}(\mathcal{C}, \Gamma)$ *is separable in* Γ.

Proof: Let H denote $\mathrm{stab}(\mathcal{C}, \Gamma)$. We may assume without loss of generality that H is nontrivial since Γ is residually finite. Note that H is either a Fuchsian group or a $\mathbf{Z}_2$-extension of a Fuchsian group. To prove the Lemma we need to show that given $g \notin H$ then there is a finite index subgroup of Γ containing H but not g. By conjugating, if necessary, we can assume that H stabilizes the real line. Denoting complex conjugation by τ, then τ extends to $\mathrm{SL}(2, \mathbf{C})$ and is well-defined on $\mathrm{PSL}(2, \mathbf{C})$. The stabilizer of $\mathbf{R}$ in $\mathrm{PSL}(2, \mathbf{C})$ is then characterized as those elements γ such that $\tau(\gamma) = \gamma$. The rest of the argument is similar in spirit to the proof of Mal'cev's theorem. Let Γ be generated by matrices $g_1, g_2, \ldots, g_t$. Let R be the subring of $\mathbf{C}$ generated by all the entries of the matrices g_i, their complex conjugates and 1. Then R is a finitely generated integral domain with 1 so that for any non-zero element there is a maximal ideal which does not contain that element. Note that both Γ and $\tau(\Gamma)$ embed in $\mathrm{PSL}(2, R)$.

If $\gamma \in \Gamma \setminus H$, then $\gamma = P(g)$ where at least one element from each set $\{\tau(g_{ij}) - g_{ij}\}$,$\{\tau(g_{ij}) + g_{ij}\}$, $i, j = 1, 2$, is non-zero. Call these x, y and choose a maximal ideal $\mathcal{M}$ such that $xy \notin \mathcal{M}$. Let

$$\rho : \mathrm{PSL}(2, R) \to \mathrm{PSL}(2, R/\mathcal{M}) \times \mathrm{PSL}(2, R/\mathcal{M})$$

be the homomorphism defined by $\rho(\gamma) = (\pi(\gamma), \pi(\tau(\gamma)))$ where π is induced by the natural projection $R \to R/\mathcal{M}$. The image group is finite since $R/\mathcal{M}$ is a finite field. By construction, the image of γ is a pair of distinct elements in $\mathrm{PSL}(2, R)/\mathcal{M})$, while the image of H lies in the diagonal. This proves the lemma.

Proof of Theorem 3.2.2: Let $i : F \to M$ be a totally geodesic immersion of a closed surface. Let Γ be the covering group of M in $\mathrm{PSL}(2, \mathbf{C})$ and let $H = i_*(\pi_1(F))$. Then H is Fuchsian and preserves some circle or straight line $\mathcal{C}$ in $\mathbf{C} \cup \infty$. Notice that $\mathrm{stab}(\mathcal{C}, \Gamma)$ contains H and is a discrete group acting on the hyperbolic plane spanned by $\mathcal{C}$; therefore we have a finite sheeted covering $\mathbf{H}^2/H \to \mathbf{H}^2/\mathrm{stab}(\mathcal{C}, \Gamma)$ which implies that $\mathrm{stab}(\mathcal{C}, \Gamma)$ is the fundamental group of a closed orientable surface.

Now by Lemma 3.2.3 the group $\mathrm{stab}(\mathcal{C}, \Gamma)$ is separable in Γ and there is an embedded closed surface in the covering $\mathbf{H}^2/\mathrm{stab}(\mathcal{C}, \Gamma)$. As discussed at the start of §4, separability now implies that there is a finite sheeted covering M_K in which this surface embeds. By untwisting this surface if it happens to be nonorientable, it follows that this manifold or its double covering contains the required embedded orientable totally geodesic surface.

In fact one can go further and produce infinite virtual Betti number in this case:

Theorem 3.2.4. *If a closed hyperbolic* 3*-manifold* M *contains a totally geodesic closed surface group then it has infinite virtual Betti number.*

Proof: As we showed above, the manifold virtually contains an embedded totally geodesic surface, F. Suppose that this surface is separating, then it expresses the fundamental group of the manifold as a free product with amalgamation $\pi_1(M) \cong \pi_1(L) *_{\pi_1(F)} \pi_1(R)$ where L and R are the two sides. By untwisting if necessary, we may suppose that the indices $[\pi_1(L) : \pi_1(F)]$ and $[\pi_1(R) : \pi_1(F)]$ are both infinite. Now separability guarantees that we can find a homomorphism of $\phi : \pi_1(M) \to A$ onto a finite group A so that the indices $[\phi(\pi_1(L)) : \phi(\pi_1(F))]$ and $[\phi(\pi_1(R)) : \phi(\pi_1(F))]$ are both strictly larger than two. By restricting this homomorphism to both sides, we assemble a new map

$$\pi_1(M) \longrightarrow \phi(\pi_1(L)) *_{\phi(\pi_1(F))} \phi(\pi_1(R))$$

onto a free product with amalgamation of finite groups. Now as in [26], the target group here is virtually a free group of rank two or greater so that $\pi_1(M)$ has infinite virtual Betti number as required.

Remark: Theorems 3.2.2 and 3.2.4 and Lemma 3.2.3 were stated only for closed hyperbolic 3-manifolds. However, these results also hold, and are proved the same way for non-compact finite volume hyperbolic 3-manifolds.

3.3

In light of Theorem 3.2.4 and Conjectures 0, 1 and ∞ from §2, a natural question is whether every finite volume hyperbolic 3-manifold contains an immersion of a closed totally geodesic surface. The answer to this is no, but to discuss this in more detail we will require some arithmetic properties of hyperbolic 3-manifold groups, and we shall develop some of this below (see [24] for more on this topic).

Let Γ be a subgroup of $\mathrm{PSL}(2, \mathbf{C})$ that does not contain a soluble subgroup of finite index. The *trace-field* of Γ, denoted $\mathbf{Q}(\mathrm{tr}\Gamma)$ is the field:

$$\mathbf{Q}(\mathrm{tr}\gamma : \gamma \in \Gamma).$$

Note that, for any $\gamma \in \mathrm{PSL}(2, \mathbf{C})$, the traces of any lifts to $\mathrm{SL}(2, \mathbf{C})$ will only differ by $\pm$, and so the trace-field is well-defined. The trace-field is a conjugacy invariant of Γ. The connection to finite volume 3-manifolds is made interesting by the following result that is basically a consequence of Mostow Rigidity in dimension 3.

Theorem 3.3.1. *Let $\Gamma < \mathrm{PSL}(2, \mathbf{C})$ be a discrete group of finite covolume, then $\mathbf{Q}(\mathrm{tr}\Gamma)$ is a finite extension of $\mathbf{Q}$.*

It follows that if $M = \mathbf{H}^3/\Gamma$ is a hyperbolic 3-manifold of finite volume, then $\mathbf{Q}(\mathrm{tr}\Gamma)$ is a topological invariant of M.

One deficiency of the trace-field is that it is not a commensurability invariant, however, this is easily remedied.

Definition 3.3.2. *Let Γ be a finitely generated group. Define $\Gamma^{(2)} = < \gamma^2 \mid \gamma \in \Gamma >$.*

Then it is easy to check that $\Gamma^{(2)}$ is a finite index normal subgroup of Γ whose quotient is an elementary abelian 2-group.

Theorem 3.3.3. *Let Γ be a finitely generated non-elementary subgroup of $\mathrm{SL}(2, \mathbf{C})$. The field $\mathbf{Q}(\mathrm{tr}\Gamma^{(2)})$ is an invariant of the commensurability class of Γ.*

Notation: The field $\mathbf{Q}(\mathrm{tr}\Gamma^{(2)})$ is denoted $k\Gamma$ and is called the *invariant trace-field* of Γ.

Another algebraic object that plays a role in the theory of hyperbolic 3-manifolds is a quaternion algebra over the invariant trace-field.

Suppose Γ is a subgroup of $\mathrm{PSL}(2, \mathbf{C})$ that does not contain a soluble subgroup of finite index (in fact we should work in $\mathrm{SL}(2, \mathbf{C})$). Here we associate to Γ a quaternion algebra over $\mathbf{Q}(\mathrm{tr}\Gamma)$. Let

$$A_0\Gamma = \{\Sigma a_i \gamma_i \mid a_i \in \mathbf{Q}(\mathrm{tr}\Gamma), \gamma_i \in \Gamma\},$$

where only finitely many of the a_i are non-zero.

Theorem 3.3.4. *$A_0\Gamma$ is a quaternion algebra over $\mathbf{Q}(\mathrm{tr}\Gamma)$.*

In the case of $A_0\Gamma^{(2)}$, we denote this by $A\Gamma$ and call this the *invariant quaternion algebra* of Γ.

Example: Suppose that $\mathbf{H}^3/\Gamma$ is non-compact but finite volume (eg the figure-eight knot complement). Then $A\Gamma \cong M(2, k\Gamma)$. The only thing to note is that the manifold being finite volume and non-compact implies that Γ contains parabolic elements. These give rise to zero divisors in $A\Gamma$ and hence the invariant quaternion algebra is not a division algebra. The isomorphism follows.

Notation: A quaternion algebra B over a field k (of characteristic $\neq 2$) can be described as follows. Let $a, b \in k^*$ and let B be the 4-dimensional vector space over k with basis $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$. Multiplication is defined on B by requiring that 1 is a multiplicative identity element, that

$$\mathbf{i^2} = a.1, \quad \mathbf{j^2} = b.1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k} \tag{3.1}$$

and extending the multiplication linearly so that B is an associative algebra over k. This algebra is denoted by the Hilbert symbol $\left(\frac{a,b}{k}\right)$.

3.4

We now produce manifolds without totally geodesics surfaces. This will follow from our next theorem which requires one more definition.

Let B be a quaternion algebra over a number field k. We say that B is *ramified* at an embedding $\sigma : k \to \mathbf{C}$ if $\sigma(k) \subset \mathbf{R}$ and the quaternion algebra

$$B^\sigma \otimes_{\sigma(k)} \mathbf{R} \cong \mathbf{D}$$

where D is the Hamiltonian quaternions, and B^σ is the quaternion algebra over $\sigma(k)$ obtained by applying σ.

If $\mathcal{P}$ is a prime ideal of R_k (the ring of integers of k) we say that a quaternion algebra B/k is *ramified* at $\mathcal{P}$ if $B \otimes_k k_{\mathcal{P}}$ is a division algebra over the local field $k_{\mathcal{P}}$ (ie the completion of k at $\mathcal{P}$).

Theorem 3.4.1. *Let Γ be a Kleinian group of finite co-volume which satisfies the following conditions:*

- *$k\Gamma$ contains no proper subfield other than $\mathbf{Q}$.*
- *$A\Gamma$ is ramified at at least one embedding of $k\Gamma$.*

Then Γ contains no purely hyperbolic elements.

Before discussing the proof, we recall that a purely hyperbolic element is a hyperbolic element γ with $\mathrm{tr}^2(\gamma) > 4$. Since a Fuchsian group is conjugate to a subgroup of $\mathrm{PSL}(2, \mathbf{R})$, the "generic" element is purely hyperbolic. More precisely, if F is a cocompact Fuchsian group then all elements of infinite order are purely hyperbolic. Thus Theorem 3.4.1 provides a method of checking the lack of existence (in a very strong way) of totally geodesic surface subgroups.

Proof: Note that Γ contains a hyperbolic element if and only if $\Gamma^{(2)}$ contains a hyperbolic element. Let us suppose that $\gamma \in \Gamma^{(2)}$ is hyperbolic, and let $t = \mathrm{tr}(\gamma)$. By assumption $t \in k\Gamma \cap \mathbf{R} = \mathbf{Q}$, and $|t| > 2$.

Now $A\Gamma$ is ramified at an embedding σ of $k\Gamma$, which is necessarily real. Let $\sigma : k\Gamma \to \mathbf{R}$ be the Galois embedding of $k\Gamma$, and $\psi : A\Gamma \to \mathbf{D}$ extending σ. Thus

$$\psi(\Gamma^{(2)}) \subset \psi(A\Gamma^1) \subset \mathbf{D}^1.$$

But then, since $t \in \mathbf{Q}$,

$$t = \sigma(t) = \psi(\gamma + \bar{\gamma}) = \psi(\gamma) + \overline{\psi(\gamma)} = \mathrm{tr}\psi(\gamma).$$

Since tr $\mathbf{D}^1 \subset [-2, 2]$ we obtain a contradiction. □

Now although it may appear the conditions of Theorem 3.4.1 are hard to check, many examples of manifolds are known to satisfy these conditions. This is the class of *arithmetic hyperbolic 3-manifolds* (see [24] for details).

These are defined as follows. Let k be a number field with exactly one pair of complex conjugate embeddings, let R_k denote the ring of integers of k, and let B be a quaternion algebra over k which is ramified at all the real embeddings. Let ρ be a k-embedding of B into $M_2(\mathbf{C})$ and let $\mathcal{O}$ be an R_k-order of B. Then a subgroup Γ of $\mathrm{PSL}(2, \mathbf{C})$ is an *arithmetic Kleinian group* if it is commensurable with some such $\mathrm{P}\rho(\mathcal{O}^1)$.

Examples of arithmetic Kleinian groups that satisfy the hypothesis of Theorem 3.4.1 are then easily constructed. One just takes for example a cubic number field and B a quaternion algebra with the properties given in the definition. A particular example of such a group/manifold is the Weeks manifold, the hyperbolic 3-manifold with the smallest known volume at this point. The field $k = \mathbf{Q}(\theta)$ where θ is a complex root of $x^3 - x + 1 = 0$, and B is a quaternion algebra over k ramified at the real embedding of k and at a prime ideal of norm 5 in k.

3.5

An important subclass of arithmetic Kleinian groups are the *Bianchi groups*. These are the generalization to dimension 3 of the modular group and are defined as $\mathrm{PSL}(2, O_d)$ where O_d is the ring of integers in $\mathbf{Q}(\sqrt{-d})$. These groups all contain a copy of $\mathrm{PSL}(2, \mathbf{Z})$ and so all contain a non-elementary Fuchsian group. However, they also contain lots of cocompact Fuchsian subgroups. These are constructed as follows.

Lemma 3.5.1. *Let F be a Fuchsian subgroup of the Bianchi group $\mathrm{PSL}(2, O_d)$ which contains two non-commuting hyperbolic elements. Then F preserves a circle or straight-line in $\mathbf{C} \cup \infty$*

$$a|z|^2 + Bz + \overline{Bz} + c = 0,$$

where $a, c \in \mathbf{Z}$ and $B \in O_d$.

Proof: Since F is a Fuchsian subgroup, it does preserve a circle or straight-line $\mathcal{C}$ in $\mathbf{C} \cup \infty$. Assume this has equation $a|z|^2 + Bz + \overline{Bz} + c = 0$ with a and c real numbers and B complex. By conjugating in $\mathrm{PSL}(2, O_d)$ we may assume that $a \neq 0$. Hence on further dividing, we can assume that $a = 1$.

F contains a pair of non-commuting hyperbolic elements, and these have distinct fixed points which lie on $\mathcal{C}$ (recall §2.2). If one such element g is represented by $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ then its fixed points are $\frac{\alpha - \delta \pm \lambda}{2\gamma}$ where $\lambda^2 = (\alpha + \delta)^2 - 4 > 0$. An easy calculation shows that the perpendicular bisector of the line (in $\mathbf{C}$) joining these fixed points has the equation

$$\gamma z + \overline{\gamma z} = \gamma\mu + \overline{\gamma\mu},$$

where $\mu = \frac{\alpha-\delta}{2\gamma}$. Since the centre of $\mathcal{C}$ is the intersection of two such lines we deduce, since all these coefficients are in $\mathbf{Q}(\sqrt{-d})$, that $B \in \mathbf{Q}(\sqrt{-d})$. Rewriting the equation of the circle $\mathcal{C}$ as

$$|z+\overline{B}|^2 = |B|^2 - c,$$

we find that, since the fixed points of the hyperbolic element g lie on $\mathcal{C}$, we can solve for $c \in \mathbf{Q}$. Clearing denominators completes the proof of the lemma. □

Using arithmetic methods, it can be shown that, if $\mathcal{C}$ is a circle or straight line as above, and $\mathbf{H}_{\mathcal{C}}$ the hyperbolic plane in $\mathbf{H}^3$ erected on $\mathcal{C}$, then $\mathrm{stab}(\mathcal{C}, \mathrm{PSL}(2, O_d))$ if a Fuchsian group acting with finite co-area, and indeed one can arrange that the group acts cocompactly.

Example: In the case of $d = 1$, the Bianchi group is the Picard group $\mathrm{PSL}(2, \mathbf{Z}[i])$. If we let $p \in \mathbf{Z}$ be a prime congruent to 3 mod 4, then the circle

$$\mathcal{C}_p = \{z \in \mathbf{C} : |z|^2 = p\}$$

gives rise to a subgroup $\mathrm{stab}(\mathcal{C}_p, \mathrm{PSL}(2, \mathbf{Z}[i])$ that is cocompact.

3.6

The discussion in §4.3–4.5, exploits the description of $\mathrm{Isom}^+(\mathbf{H}^3)$ as $\mathrm{PSL}(2, \mathbf{C})$. In higher dimensions, since $\mathrm{Isom}(\mathbf{H}^n) = \mathrm{O}_0(n,1)$, arithmetic methods of constructing lattices exploit the theory of quadratic forms.

We now discuss this construction. This goes back to Borel and Harish-Chandra, [8] and [7]) and will require some standard facts about quadratic forms and orthogonal groups of such forms; [17] is a standard reference.

3.7

If f is a quadratic form in $n+1$ variables with coefficients in k and associated symmetric matrix F, let

$$\mathrm{O}(f) = \{X \in \mathrm{GL}(n+1, \mathbf{C}) \mid X^t F X = F\}$$

be the *Orthogonal group* of f, and

$$\mathrm{SO}(f) = \mathrm{O}(f) \cap \mathrm{SL}(n+1, \mathbf{C}),$$

the *Special Orthogonal group* of f. These are algebraic groups defined over k.

Definition 3.7.1. *Two n-dimensional quadratic forms f and q defined over a field k (with associated symmetric matrices F and Q) are equivalent over k if there exists $P \in \mathrm{GL}(n, K)$ with $P^t F P = Q$.*

If $k \subset \mathbf{R}$ is a number field, and R_k its ring of integers, then $\mathrm{SO}(f; R_k)$ is an arithmetic subgroup of $\mathrm{SO}(f; \mathbf{R})$, [8] or [7]. The following is well-known and proved in [2] for example.

Lemma 3.7.2. *Let $k \subset \mathbf{R}$ be a number field and R_k its ring of integers. Let f and q be n-dimensional quadratic forms with coefficients in R_K which are equivalent over k.*

- $\mathrm{SO}(f; \mathbf{R})$ *is conjugate to* $\mathrm{SO}(q; \mathbf{R})$ *and* $\mathrm{SO}(f; k)$ *is conjugate to* $\mathrm{SO}(q; k)$.
- $\mathrm{SO}(f; R_k)$ *is conjugate to a subgroup of* $\mathrm{SO}(q; k)$ *commensurable with* $\mathrm{SO}(q; R_k)$. $\square$

There is a converse to the second part of Lemma 3.7.2 which we record here (see [31] for example). Note that if $f' = \lambda f$, for $\lambda \in k$ (non-zero), then $\mathrm{SO}(f'; k) = \mathrm{SO}(f; k)$. With notation as above,

Lemma 3.7.3. *Suppose $\mathrm{SO}(f; R_k)$ and $\mathrm{SO}(q; R_k)$ are commensurable. Then f is equivalent to λq for some non-zero $\lambda \in K$.* $\square$

Assume that $k \subset \mathbf{R}$ is totally real, and let f be a form in $n+1$-variables with coefficients in k, and be equivalent over $\mathbf{R}$ to the form f_n. Furthermore, if $\sigma : k \to \mathbf{R}$ is a field embedding, then the form f^σ obtained by applying σ to f is defined over the real number field $\sigma(k)$. We insist that for embeddings $\sigma \neq id$, f^σ is equivalent over $\mathbf{R}$ to the form in $(n+1)$-dimensions, of signature $(n+1, 0)$. Since f is equivalent over $\mathbf{R}$ to f_n, it from follows Lemma 3.7.2 that $\mathrm{O}(f; \mathbf{R})$ is conjugate, by a matrix P say in $\mathrm{GL}(n+1, \mathbf{R})$ to $\mathrm{O}(f_n; \mathbf{R})$. From [8] (or [7]) $P\mathrm{SO}_0(f; R_k)P^{-1}$ defines an arithmetic subgroup in

$\mathrm{Isom}^+(\mathbf{H}^n)$, and so necessarily of finite co-volume. In what follows we will abuse notation, and suppress the conjugating matrix, and simply identify $\mathrm{SO}_0(f;R_k)$ as an arithmetic subgroup of $\mathrm{Isom}^+(\mathbf{H}^n)$.

The group $\mathrm{SO}_0(f;R_k)$ is cocompact if and only if the form f does not represent 0 non-trivially with values in k, see [8]. Whenever $n \geq 4$, the arithmetic groups constructed above are non-cocompact if and only if the form has rational coefficients, since it is well known every indefinite quadratic form over $\mathbf{Q}$ in at least 5 variables represents 0 non-trivially, see [17].

The following theorem summarizes a case that what we shall make use of (see [31] Chapter 6).

Theorem 3.7.4. *If Γ is a non-cocompact arithmetic subgroup of $\mathrm{SO}_0(f_n;\mathbf{R})$ then Γ is commensurable (up to conjugacy) with a group $\mathrm{SO}_0(f;\mathbf{Z})$ where f is a diagonal quadratic form with rational coefficients and signature $(n,1)$.* $\square$

3.8

We will now discuss the proof of the following theorem. This was proved in [2] for geometrically finite subgroups. Since appearing, the solution to the Tameness conjecture ([1] and [12]) has allowed for the extension to LERF.

Theorem 3.8.1. *The groups $\mathrm{PSL}(2,O_d)$ are LERF.*

Proof: We first remark that if H is a finitely generated subgroup of $\mathrm{PSL}(2,O_d)$ it contains a torsion-free subgroup of finite index, and so if F is geometrically infinite then it will be separable by Theorem 3.0.4 and the fact that a finite index supergroup is separable. Thus it remains to show that $\mathrm{PSL}(2,O_d)$ is H-separable for H a geometrically finite subgroup.

The key geometric idea is contained in the following lemma.

Lemma 3.8.2. *For every d there is a finite index subgroup Δ_d of $\mathrm{PSL}(2,O_d)$ such that Δ is contained in the group generated by reflections in a finite volume all right polyhedron P in $\mathbf{H}^6$.*

Given this Theorem 3.8.1 follows immediately from Theorem 3.1.1.

Sketch of the Proof of Lemma 3.8.2

This makes use of the description of arithmetic groups arising from quadratic forms. Theorem 3.7.4 affords a description of the non-cocompact lattices, and it can be shown that if p_d is the quadratic form $< d, 1, 1, -1 >$, then p_d is isotropic and so $\mathrm{SO}_0(p_d; \mathbf{Z})$ is a non-cocompact arithmetic group. Indeed it represents the commensurability class (as discussed above) of the image of $\mathrm{PSL}(2, O_d)$ in $\mathrm{SO}_0(3, 1)$.

We now discuss the construction of the all right polyhedron, a related arithmetic group and the group Δ_d.

An all right ideal polyhedron in hyperbolic 6-space:

In $\mathbf{H}^6$ there is a simplex Σ with one ideal vertex given by the following Coxeter diagram (see [25] p. 301).

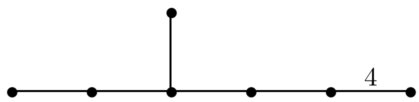

Figure 1

Notice that deleting the right most vertex of this Coxeter symbol gives an irreducible diagram for a finite Coxeter group, namely E_6. This group has order $2^7.3^4.5$.

The connection to arithmetic groups is given in the following lemma.

Lemma 3.8.3. *(i)* $G^+(\Sigma) = \mathrm{SO}_0(f_6; \mathbf{Z})$.

(ii) There is an all right polyhedron Q built from $2^7.3^4.5$ copies of Σ. In particular the reflection group $G(Q)$ is commensurable with $\mathrm{SO}_0(f_6; \mathbf{Z})$.

Proof: The first part is due to Vinberg [30], and also discussed in [25] p. 301. For the second part, as noted above, if one deletes the face F of the hyperbolic simplex corresponding to the right hand vertex to the given Coxeter diagram, the remaining reflection planes pass through a single (finite) vertex and these reflections generate the finite Coxeter group E_6. Take all the translates of the simplex by this group; this yields a polyhedron whose faces all correspond to copies of F. Two such copies meet at an angle which is twice the angle of the reflection plane of the hyperbolic simplex which lies between them. One sees from the Coxeter diagram that the plane F makes angles $\pi/2$ and $\pi/4$ with the other faces of the hyperbolic simplex, so the resulting polyhedron is all right as required. □

We can now construct Δ_d.

Lemma 3.8.4. *Let f be the quadratic form $< 1, 1, 1, 1, 1, 1, -1 >$. Then for all d, $\mathrm{SO}(f; \mathbf{Z})$ contains a group Δ_d which is conjugate to a subgroup of finite index in the Bianchi group $\mathrm{PSL}(2, O_d)$.*

The proof requires an additional lemma. Assume that j is a diagonal quaternary quadratic form with integer coefficients of signature $(3, 1)$; so that j is equivalent over $\mathbf{R}$ to the form $< 1, 1, 1, -1 >$. Let $a \in \mathbf{Z}$ be a square-free positive integer and consider the seven dimensional form $j_a =< a, a, a > \oplus j$, where $\oplus$ denotes orthogonal sum. Being more precise, if we consider the 7-dimensional $\mathbf{Q}$-vector space V equipped with the form j_a there is a natural 4-dimensional subspace V_0 for which the restriction of the form is j. Using this it easily follows that,

Lemma 3.8.5. *In the notation above, the group $\mathrm{SO}(j; \mathbf{Z})$ is a subgroup of $\mathrm{SO}(j_a; \mathbf{Z})$.* □

Proof of Lemma 3.8.4

Let p_d be as above. The key claim is that $q_d =< d, d, d > \oplus p_d$ is equivalent over $\mathbf{Q}$ to the form f.

Assuming this claim for the moment, by Lemma 3.7.2 we deduce that there exists $R_d \in \mathrm{GL}(7, \mathbf{Q})$ such that $R_d \mathrm{SO}(q_d; \mathbf{Z}) R_d^{-1}$ and $\mathrm{SO}(f; \mathbf{Z})$ are commensurable. This together with Lemma 3.8.5 gives the required group G_d.

To prove the claim, since every positive integer can be written as the sum of four squares, write $d = w^2 + x^2 + y^2 + z^2$. Let A_d be the 7×7 matrix

$$\begin{pmatrix} w & x & y & z & 0 & 0 & 0 \\ -x & w & -z & y & 0 & 0 & 0 \\ -y & z & w & -x & 0 & 0 & 0 \\ -z & -y & x & w & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Note A_d has determinant d^2, so is in $\mathrm{GL}(7, \mathbf{Q})$. Let F the diagonal matrix associated to the form f and Q_d be the 7×7 diagonal matrix of the form $< d, d, d, d, 1, 1, -1 >$ (i.e. of $< d, d, d > \oplus p_d$). Then a direct check shows that $A_d F A_d^t = Q_d$ as is required. $\square$

Remark. The polyhedron Q is finite covolume since there is only one infinite vertex: deleting the plane corresponding to the left hand vertex of the Coxeter group is the only way of obtaining an infinite group and this group is a 5 dimensional Euclidean Coxeter group. By inspecting other Coxeter diagrams it can be shown that there are ideal all right polyhedra in $\mathbf{H}^k$ at least for $2 \leq k \leq 8$. This was exploited in [22] to prove that many other discrete subgroups of $\mathrm{Isom}(\mathbf{H}^n)$ are separable on geometrically finite subgroups for $2 \leq n \leq 8$.

Chapter 4

New directions.

While examination of subgroup separability in the geometric context has generated a good deal of interesting mathematics, it is unclear the extent to which it will resolve the central conjectures which it was introduced to resolve. For example, although it seems somewhat unlikely, one might be able to show that hyperbolic 3-manifold groups were subgroup separable and this on its own actually answers *none* of the questions in which we were originally interested. In this section we shall formulate some more recent work which appears to be more germane. We begin with a simple definition that underpins much of what follows.

Definition 4.0.6. *Let G be a group and H a subgroup. Then a homomorphism $\theta : H \longrightarrow A$* **extends over the finite index subgroup** *$V \leq G$ if*

- *$H \leq V$*
- *There is a homomorphism $\Theta : V \longrightarrow A$ which is the homomorphism θ when restricted to H.*

One of the motivations in our setting for making this definition is the following straightforward theorem:

Theorem 4.0.7. *Suppose that G is LERF, and H a finitely generated subgroup. Suppose $\theta : H \rightarrow A$ is a homomorphism onto a finite group.*

Then there is subgroup of finite index V of G containing H and a homomorphism $\Theta : V \to A$ with $\Theta|_H = \theta$.

Proof. We can assume that H has infinite index in G. Let $K = \ker \theta$, a finite index subgroup of H, and hence finitely generated. Since G is LERF, there is a finite index subgroup $K' < G$ such that $K' \cap H = K$. Define $\Delta = \bigcap\{hK'h^{-1} : h \in H\}$. Note that since K is normal in H, $K < \Delta$ and moreover, since K has finite index in H, the above intersection consists of only a finite number of conjugates of K'. Hence Δ has finite index in K' (and also G). Let V denote the group generated by H and Δ. It is easy to check that, by construction, Δ is a normal subgroup of V, so that $V = H.\Delta$. The canonical projection $\Theta : V \to V/\Delta$ defines the required extension.

In other words, if the ambient group is LERF, then every homomorphism of a finitely generated subgroup onto a finite group can be extended to some subgroup of finite index in G. We shall say that the homomorphism θ *virtually extends* and that G has the *local extension property* for homomorphisms onto finite groups. It turns out that these extension properties give elegant expressions of several well established notions, for example residual finiteness:

Theorem 4.0.8. *G is residually finite if and only if G has the property that for each of its cyclic groups there is a virtual extension of at least one of the maps onto a nontrivial (cyclic) group.*

Proof. If G has the stated extension property, then given a nonidentity $g \in G$ we can extend some homomorphism of $\langle g \rangle \longrightarrow \mathbf{Z}/k$ to $V \longrightarrow \mathbf{Z}/k$, where V has finite index in G. Then the kernel of this map has finite index in G and excludes g so that G is residually finite. Conversely, suppose that G is residually finite, and let some nonidentity $g \in G$ be given. Choose some normal subgroup of finite index in G, N say, which excludes g.

Then the map

$$N.\langle g \rangle \longrightarrow N.\langle g \rangle/N \cong \langle g \rangle/\langle g^k \rangle$$

for some k. This extends the map $\langle g \rangle \to \mathbf{Z}/k$, this map being nontrivial by choice of N.

Another nice application of Theorem 4.0.7 is the following.

Theorem 4.0.9. $\mathrm{SL}(n,\mathbf{Z})$ *is not subgroup separable for all* $n \geq 3$.

Proof: Assume to the contrary that $\mathrm{SL}(n,\mathbf{Z})$ is subgroup separable. Then since $\mathrm{SL}(n,\mathbf{Z})$ contains free subgroups of all possible ranks, given an finite group G, Theorem 4.0.7 constructs a finite index subgroup Γ_G of $\mathrm{SL}(n,\mathbf{Z})$ and an onto homomorphism $\theta : \Gamma_G \to G$. However, $\mathrm{SL}(n,\mathbf{Z})$ for $n \geq 3$ has the Congruence Subgroup Property; ie every subgroup Γ of finite index in $\mathrm{SL}(n,\mathbf{Z})$ contains the kernel of some reduction homomorphism;

$$\mathrm{SL}(n,\mathbf{Z}) \to \mathrm{SL}(n,\mathbf{Z}/m\mathbf{Z}),$$

for $m \in \mathbf{Z}$, $m \geq 2$. Thus $\ker\theta$ contains a group of the form $\Gamma(m)$ for some $m \geq 2$. It follows that G is a quotient of a subgroup of $\mathrm{SL}(n,\mathbf{Z}/m\mathbf{Z})$. However, as we discuss below, this is impossible for certain groups G. For example we can choose $G = A_\ell$ for ℓ very large, and we obtain a contradiction.

To establish the existence of the groups G we use the following lemma (see for example [20] Window 2).

Lemma 4.0.10. *Let* p *be a prime, let* $\mathrm{SL}(n,p)$ *denote the finite group* $\mathrm{SL}(n,\mathbf{Z}/p\mathbf{Z})$. *For fixed* n, *if* A_ℓ *is a quotient of a subgroup of* $\mathrm{SL}(n,p)$, *then* $n \geq \frac{2\ell-6}{3}$.

Thus for large enough ℓ the alternating group A_ℓ is not a quotient of a subgroup of $\mathrm{SL}(n,p)$. This lemma completes the proof since by the structure theory of the finite groups $\mathrm{SL}(n,\mathbf{Z}/m\mathbf{Z})$, given the prime factorization $m = p_1^{m_1} p_2^{m_2} \dots p_r^{m_r}$ it can be shown (essentially the Chinese Remainder Theorem) that:

$$\mathrm{SL}(n,\mathbf{Z}/m\mathbf{Z}) \cong \prod \mathrm{SL}(n,\mathbf{Z}/p_i^{m_i}\mathbf{Z}).$$

Furthermore, the homomorphism $\mathrm{SL}(n,\mathbf{Z}/p_i^{m_i}\mathbf{Z}) \to \mathrm{SL}(n,p)$ has kernel a finite p-group. Putting these statements together it follows that if $G = A_\ell$ as above, so that in particular A_ℓ is simple, elementary finite group theory shows that A_ℓ is necessarily a quotient of a subgroup of $\mathrm{SL}(n,p)$ which is false by Lemma 4.0.10. □

4.1

An important generalization of the extension property is the following:

Definition 4.1.1. *Let G be a group and H a subgroup. Then G* virtually retracts *to H if there is a finite index subgroup V of G with*

- $H \leq V$
- *There is a homomorphism $\theta : V \longrightarrow H$ which is the identity when restricted to H.*

The finite index subgroup V will be called a *retractor*. One should regard a virtual retraction as an extension of the identity homomorphism $H \longrightarrow H$, to some finite index subgroup V of G and clearly, given any such retraction, we can extend any homomorphism $H \rightarrow A$ over the finite index subgroup V. While retractions are presumably somewhat rare, one can still ask for their existence in more restricted circumstances, for example, we might require that H be a finite subgroup, or an infinite cyclic subgroup or a geometrically finite (or quasi-convex) subgroup in some more geometric setting. The most important incarnation of this comes from the following conjecture:

Conjecture 4.1.2. *Suppose that G is the fundamental group of a closed hyperbolic* 3*-manifold. Then G virtually retracts to any of its cyclic subgroups*

A group that satisfies this conjecture is said to *virtually retract* over its cyclic subgroups.

We note that although this question is significantly stronger than the traditional virtual Betti number conjecture, phrased in these terms it places the question closer in spirit to questions about extensions of cyclic groups as in the notion of residual finiteness. We also note that it neither implies, nor is implied by LERF. For example if M is a closed 3-manifold that is modeled on the SOL geometry, then $\pi_1(M)$ is LERF, but it does not virtually retract over all of its cyclic subgroups. To see this, since M admits a SOL geometry, it has a finite sheeted cover M_1 that is a torus bundle over the circle (recall §2.4, where in this case the surface is a torus), and it is easy to see that $\pi_1(M_1)$ does not virtually retract onto infinite cyclic groups in the fiber group.

On the other hand, an examination of the proof of the generalized Scott's theorem given §4.1 shows:

Theorem 4.1.3. *Suppose that G is a finitely generated group which virtually embeds into an all right hyperbolic Coxeter subgroup of Isom*$(\mathbf{H}^n)$.

Then G virtually retracts to its geometrically finite subgroups.

As with the subgroup separability property, the property of virtually retracting over $\mathbf{Z}$ is well behaved for subgroups. That is to say, if G virtually retracts over $\mathbf{Z}$ and $K \leq G$, then K virtually retracts over $\mathbf{Z}$. This is easily seen: If $\mathbf{Z} < K$ is given and V a subgroup of finite index in G which is a retractor for this $\mathbf{Z}$, then $V \cap K$ is a retractor for $\mathbf{Z}$ in K.

For finite supergroups there is also a similar result:

Theorem 4.1.4. *Suppose that G is a group and K a subgroup of G of finite index.*

Then if K virtually retracts over $\mathbf{Z}$*, so does G.*

Proof: We begin by noting that from the argument above, we may assume that K is normal in G. Given a $\langle g \rangle \cong \mathbf{Z}$ in G, choose a retraction $r : A \longrightarrow \langle g \rangle \cap K$, where A has finite index in K. By the normality of K, $\langle g \rangle$ acts by conjugacy on A while stabilizing $\langle g \rangle \cap K$, so that we may intersect all these conjugates and restrict the original retraction, and we may suppose that A is normalised by $\langle g \rangle$.

Choose a faithful linear representation $\rho : \langle g \rangle \cap K \to \mathrm{GL}(V)$, for some finite dimensional vector space V. By composition with the retraction, we get a (non-faithful) linear representation $A \to \mathrm{GL}(V)$. Note that $A.\langle g \rangle / A \cong \langle g \rangle / (\langle g \rangle \cap A)$ is finite, so that we may induce upwards to get a representation $\rho^* : A.\langle g \rangle \to \mathrm{GL}(V')$.

Since A is normal in $A.\langle g \rangle$, the restriction of the induced representation ρ^* down to A gives $[A.\langle g \rangle : A]$ copies of ρ which is therefore a faithful representation of $\langle g \rangle \cap K$. It follows that when ρ^* is restricted to $\langle g \rangle$, it is faithful on a subgroup of finite index, namely $\langle g \rangle \cap K$. This forces $\ker\rho^*$ to be finite and hence trivial, since $\mathbf{Z}$ is torsion free.

We have already mentioned the connection with the classical virtual Betti number problem. In fact we have more:

Theorem 4.1.5. *Suppose that M is a hyperbolic n-manifold for which $\pi_1(M)$ virtually retracts over its cyclic subgroups.*
Then M has infinite virtual Betti number.

Proof: Suppose that the first Betti number of M is k and let γ be an element lying in the kernel of the map $\pi_1(M) \to H_1(M)$. Let $q : \widetilde{M} \to M$ be a finite sheeted covering in which the lift of (some power of) γ becomes an element of infinite order in $H_1(\widetilde{M})$.

By considering the transfer map, we see that with rational coefficients $H_1(\widetilde{M}) \cong H_1(M) \oplus \ker(q_*)$ and the element γ^r lies in $\ker(q_*)$. It follows that $H_1(\widetilde{M})$ has rank at least $k+1$. □

Remark: Thus virtually retracting over cyclic subgroups proves infinite Betti number in a way which seems more natural than the traditional approach of finding a surjection to a nonabelian free group. It is also rather easy to show that this condition is somewhat more robust than subgroup separability. For example, if A and B virtually retract over their cyclic subgroups, so does $A \times B$.

4.2

We now discuss two classes of group where virtually retracting over cyclic subgroups can be established without using the full power of LERF.

Case 1: The Bianchi groups: Although the Bianchi groups are known to be LERF by Theorem 3.8.1, and the method of proof shows that the Bianchi groups will virtually retract to all geometrically finite subgroups, we can give a proof of virtual retraction to infinite cyclic subgroups. Because our main interests are in the topology of 3-manifolds, we will work with torsion-free subgroups of finite index in the Bianchi groups to avoid some technicalities.

Theorem 4.2.1. *Let $\Gamma < \mathrm{PSL}(2, O_d)$ be a torsion-free subgroup of finite index and $\gamma \in \Gamma$ be a non-trivial element. Then Γ virtually retracts onto $< \gamma >$.*

Proof: We will assume that γ is hyperbolic (the case of parabolic is similar). Let A_γ denote the axis of γ (recall §2.2). The theorem will follow from the next claim.

Claim: *There exists a hyperbolic plane $\mathcal{H} \subset \mathbf{H}^3$ such that:*

(i) $A_\gamma \cap \mathcal{H}$ in one point.

(ii) $\Gamma(\mathcal{H}) = \mathrm{stab}(\mathcal{H};\Gamma)$ acts with finite covolume on $\mathcal{H}$.

Assuming the claim we proceed to complete the proof. By Lemma 3.2.3 (in the finite volume setting), $\Gamma(\mathcal{H})$ is separable in Γ. Furthermore by passage to a subgroup of index 2 (if needed) we can assume that there is a $\Gamma_1 < \Gamma$ of finite index, and

$$\Sigma(\mathcal{H}) = \mathcal{H}/(\Gamma_1 \cap \Gamma(\mathcal{H})) \hookrightarrow \mathbf{H}^3/\Gamma_1$$

embeds as a non-separating orientable surface (recall Theorem 3.2.2). By assumption $A_\gamma \cap \mathcal{H}$ and so this implies that the projection of A_γ to $M_1 = \mathbf{H}^3/\Gamma_1$ meets $\Sigma(\mathcal{H})$. Now the geometric version of separability can be used to find a further finite sheeted covering for which intersection pairing with $\Sigma(H)$ defines a retraction on some power of γ. The proof is completed by the next lemma.

Lemma 4.2.2. *Suppose that G is a group which virtually retracts over $\langle\gamma^r\rangle$. Then G virtually retracts over $\langle\gamma\rangle$.*

Proof. Suppose that $\pi : K \to \mathbf{Z}$ is a retraction over $\langle\gamma^r\rangle$, where $\langle\gamma^r\rangle < K$ and K has finite index in G.

The element γ acts by conjugation on K, stabilising $\langle\gamma^r\rangle$, so replacing K by all its γ conjugates, we may assume that K is normalised by γ. Set $K_+ =< K, \gamma >= K \cdot \langle\gamma\rangle$.

Then K_+ has finite index in G. Moreover,

$$K_+/K \cong \langle\gamma\rangle/\langle\gamma^r\rangle, \text{ so that } [K_+ : K] < \infty.$$

Choose some one dimensional faithful representation $\rho : \mathbf{Z} \to \mathbf{C}$ and induce the composition $\rho\circ\pi$ up to K_+ to obtain a representation $\rho_+ : K_+ \to V$ for some complex vector space V.

Since K is normal in K_+, the restriction of ρ_+ down to K gives a direct sum of $[K_+ : K]$ copies of the original representation, in particular, ρ_+ is faithful on $\langle\gamma^r\rangle$ and hence faithful on $\langle\gamma\rangle$. Restricting ρ_+ to $\rho_+^{-1}(\rho_+(\langle\gamma\rangle)$ gives the required retraction. $\square$

The proof of the claim is completed as follows. Firstly, part (ii) of the claim follows from the discussion in §4.5. To prove part (i) of the claim, it is easy to see using the density of $\mathbf{Q}(\sqrt{-d})$ in $\mathbf{C}$ that we can construct a circle $\mathcal{C}$ that encloses one of the fixed points β of γ and excludes the other and is centered at $z_0 = u_0/v_0$ ($u_0, v_0 \in O_d$) with radius a small rational number q. Such a circle has an equation of the form $|z - \frac{u_0}{v_0}|^2 = q^2$. Expanding, clearing denominators and rearranging puts this equation in the form of Lemma 3.5.1. $\square$

Case 2: Coxeter groups: Scott's theorem and the proof of Theorem 3.8.1 highlights the importance of groups generated by reflections in the faces of all right polyhedra. We now discuss Coxeter groups more generally in the context of virtual retractions to infinite cyclic groups.

We first recall some basic statements about Coxeter groups, see [16] for details.

Suppose that W is a group and S is a set of generators all of order 2. Then (W, S) is a *Coxeter system* if W admits a presentation:

$$< S \quad | \quad (s \cdot t)^{m(s,t)} = 1 >$$

where $m(s,t)$ is the order of $s \cdot t$ and there is one relation for each pair s, t with $m(s,t) < \infty$.

We refer to W as a *Coxeter group.* The *Coxeter diagram* of this presentation consists of a vertex for each element of S together with an edge connecting distinct vertices s, t whenever $m(s,t) \neq 2$ and the edge is labelled by $m(s,t)$. It is also standard practice in the case when $m(s,t) = 3$ to leave the edge unlabelled, and we follow that convention here. Since the generators have order 2, this means that if two vertices are not connected by an edge then the generators corresponding to the vertices commute. Thus, if the diagram is not connected, the Coxeter group is the direct sum of the subgroups given by the connected components. A Coxeter group (W, S) is called *reducible* if its diagram is not connected. Otherwise the Coxeter group is irreducible, in our context we may as well restrict to irreducible Coxeter groups. We shall sketch a proof of (see below for definitions):

Theorem 4.2.3. *Suppose that W is a Coxeter group with all its two generator special subgroups finite. Let $\gamma \in W$ be an element acting hyperbolically on the Coxeter complex. Then W virtually retracts over $< \gamma >$.*

This implies:

Corollary 4.2.4. *A Coxeter group is either virtually abelian or has infinite virtual Betti number.*

Proof: If a Coxeter group isn't virtually abelian, we can add relations of the form $(s.t)^k = 1$ to find an infinite non-virtually abelian Coxeter group with all two generator special subgroups finite.

We now form the Davis version of the Coxeter complex. We briefly recall the construction. Firstly by a *special subgroup* of W we mean a subgroup $< S' >$ of W where $S' \subset S$. The finite special subgroups form a poset under inclusion and the Davis complex Σ consists of left cosets of all finite special subgroups where inclusion of faces is defined by reverse inclusion of cosets. In particular, if $n = |S|$, the $(n-1)$-simplices correspond to the elements of W and the dual 1-skeleton of Σ is a modified Cayley graph of W with generating set S. (The modification consists of identifying the edge labelled s with the edge labelled s^{-1} for each generator $s \in S$.) The action of W on the left cosets by left multiplication induces a simplicial action of W on Σ. A top dimensional simplex, C, of Σ is called a *chamber.* Observe that the only element of W which maps some chamber to itself is the identity (see [10], Chapter III, §4 Lemma 6). In his thesis, it was shown by Moussong that the cells of this complex can be metrized as Euclidean polyhedra so that in the induced piecewise Euclidean metric, Σ is a $CAT(0)$ space.

Following [10], (Chapter III, §4), we see that given any pair of adjacent chambers C and C', there is a unique automorphism s of the Coxeter complex of order 2 which exchanges C and C' while fixing $C \cap C'$, and this gives rise to a *wall* (denoted by $\mathcal{H}_s$) in the Coxeter complex, namely $\mathcal{H}_s = Fix(s)$. Conversely, any reflection in W gives rise to a unique wall. Note that walls separate the Coxeter complex (See [10], for example, Chapter III §3 Corollary 3) and are totally geodesic in the $CAT(0)$ metric, since they are the fixed set of an orientation reversing isometry.

Fix the following notation: let $g \in W$, then $C_W(g)$ denotes the centralizer of g in W.

Lemma 4.2.5. $\mathrm{stab}(\mathcal{H}_s) = C_W(s)$.

Proof. Let γ be an element of $\mathrm{stab}(\mathcal{H}_s)$. Then $\gamma(C \cap C')$ is some codimension-1 face in $\mathcal{H}_s$ and is therefore fixed by s. It follows that s and $\gamma^{-1} \cdot s \cdot \gamma$ are both automorphisms of order two fixing $C \cap C'$ pointwise and exchanging γC with $\gamma C'$. Thus $\gamma^{-1} \cdot s \cdot \gamma \cdot s^{-1}$ maps C to itself and therefore is the identity element of W. (See [10], Chapter III, §4 Lemma 6)

Conversely, if $\gamma \in C_W(s)$, then it follows that $s(\gamma\mathcal{H}_s) = \gamma\mathcal{H}_s$. As s fixes a unique wall, we deduce that $\gamma\mathcal{H}_s = \mathcal{H}_s$, and $\gamma \in \mathrm{stab}(\mathcal{H}_s)$ as was required. $\square$

The key use of this lemma is the following result of [19]. For the convenience of the reader we include a proof:

Theorem 4.2.6. *([19]) Let $\alpha : G \to G$ be an automorphism of a residually finite group G. Then $Fix(\alpha)$ is separable in G.*

Proof. Choose an element γ not lying in $Fix(\alpha)$. This means that the element $\gamma^{-1} \cdot \alpha(\gamma)$ is not the identity element, so that there is a homomorphism $\phi : G \to F$ onto a finite group F, so that $\phi(\gamma^{-1} \cdot \alpha(\gamma))$ is not the identity element. Define a homomorphism

$$\Phi : G \longrightarrow F \times F$$

by $\Phi(g) = (\phi(g), \phi(\alpha(g)))$. Note that Φ maps $Fix(\alpha)$ into the diagonal subgroup of $F \times F$, however by construction, $\Phi(\gamma) = (\phi(\gamma), \phi(\alpha(\gamma)))$ does not lie in the diagonal subgroup, so that $\Phi^{-1}\{\ (f,f)\ \mid\ f \in F\}$ is the required subgroup of finite index. $\square$

It is a theorem of Tits that Coxeter groups are linear, it follows that they are residually finite and we deduce

Corollary 4.2.7. *In the notation above* $\mathrm{stab}\mathcal{H}_s$ *is a separable subgroup of W.*

Fix some element $g \in W$ which which acts hyperbolically (See [9] p. 229); in particular, there is a geodesic line γ in Σ along which g

acts as translation. As in [9] Theorem 6.8, the translation distance along γ is the minimal distance points in Σ are moved. The key claim is now:

Lemma 4.2.8. *Then there is a wall X_s so that one end of γ lies on one side of X_s, the other end lies on the other.*

Proof: Choose some axis γ for the element g; this must meet some wall X_s transversely. The wall is totally geodesic so γ cannot meet it more than once; hence the ends of γ lie on either side of X_s. $\square$

Of course, this implies in particular that $X_s \cap \gamma$ is nonempty. We can use this result to prove the main result.

Consider the subgroup W_+, of index 2 in W which is the kernel of the map $W \to \mathbf{Z}_2$ given by sending each generator in S to $1 \in \mathbf{Z}/2$; the action of W^+ on walls is now orientation preserving. Moreover, W is linear so there is a torsion free subgroup of finite index inside W^+ which we denote by W_T. Let $\mathrm{stab}(X_s)$ be the stabilizer of X_s inside the group W_T.

Choose some point p_- so far towards the $-\infty$ end of γ, that the $-\infty$ end of γ never returns to X_s. Now choose some very large power $g^t \in W_T$ so that $g^t(p_-) = p_+$ lies a similarly long way towards the $+\infty$ end of γ. In particular, p_- and p_+ are on either side of X_s. Consider $\pi : X \longrightarrow X/\mathrm{stab}(X_s)$; this contains the compact subcomplex formed by $X_s/\mathrm{stab}(X_s)$ together with the image of the subarc of γ between p_- and p_+. Denote this subcomplex by C.

Since $\mathrm{stab}(X_s)$ acts by isometries which do not exchange the sides, the points $\pi(p_-)$ and $\pi(p_+)$ lie on opposite sides of $X_s/\mathrm{stab}(X_s)$. Moreover, the ends of the projection of the geodesic γ never return to $X_s/\mathrm{stab}(X_s)$ past the points $\pi(p_-)$ and $\pi(p_+)$ In particular the arc meets $X_s/\mathrm{stab}(X_s)$ an odd number of times.

The subgroup $\mathrm{stab}(X_s)$ is separable inside W_T so by REF, there is a subgroup $\mathrm{stab}(X_s) \leq K$ of finite index in W_T, so that the projection of C in the covering $X/\mathrm{stab}(X_s) \longrightarrow X/K$ is an *embedding* of C.

Since C is embedded in X/K, and the ends of γ in the covering $X/\mathrm{stab}(X_s)$ never return to $X_s/\mathrm{stab}(X_s)$ past the chosen points, so that the lift of $g \subset X/K$ running through the arc portion of C must meet the surface portion $X_s/\mathrm{stab}(X_s)$ in an odd number of points. It follows that taking intersection number with $X_s/\mathrm{stab}(X_s)$ gives the relevant element of $H^1(X/K;\mathbf{Z})$ which retracts some power of g. The theorem follows from Theorem 4.2.2.

Bibliography

[1] I. Agol, *Tameness of hyperbolic 3-manifolds*, preprint.

[2] I. Agol, D. D. Long and A. W. Reid, *The Bianchi groups are separable on geometrically finite subgroups*, Ann. of Math. **153** (2001), 599–621.

[3] G. Baumslag and D. Solitar, *Some two-generator one-relator non-Hopfian groups*, Bull. A. M. S. **68** (1962), 199-201.

[4] R. Benedetti and C. Petronio, *Lectures on Hyperbolic Geometry*, Universitext, Springer-Verlag (1992).

[5] A. Blass and P. M. Neumann, *An application of universal algebra in group theory*, Michigan J. Math **21** (1974), 167–169.

[6] F. Bonahon, *Bouts des variétés hyperboliques de dimension 3*, Annals of Math. **124** (1986), 71–158.

[7] A. Borel, *Compact Clifford-Klein forms of symmetric spaces*, Topology **2** (1963), 111 – 122.

[8] A. Borel and Harish-Chandra, *Arithmetic subgroups of algebraic groups*, Annals of Math. **75** (1962), 485–535.

[9] M. Bridson & A. Haefliger *Metric spaces of nonpositive curvature*, Grundlehren der Mathematischen Wissenschaften **319** Springer-Verlag, (1999).

[10] K. Brown *Buildings*, Springer-Verlag (1989).

[11] R. G. Burns, D. Karrass and D. Solitar, *A note on subgroups with separable finitely generated subgroups*, Bull. Australian Math. Soc. **36** (1987), 153–160.

[12] D. Calegari and D. Gabai, *Shrinkwrapping and the taming of hyperbolic 3-manifolds*, preprint.

[13] D. Cooper and D. D. Long, *Some surface subgroups survive surgery*, Geometry and Topology **5** (2001), 347–367.

[14] M. Hall Jr., *Coset representations in free groups*, Trans. A. M. S. **67** (1949), 421–432.

[15] J. Hempel. *Residual finiteness for* 3*-Manifolds.* In Combinatorial Group Theory and Topology. Ann. of Math. Studies, **111** 379 –396, P. U. P (1987).

[16] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Studies in Advanced Mathematics, **29**, C.U.P. (1990).

[17] T. Y. Lam, *The Algebraic Theory of Quadratic Forms*, Benjamin (1973).

[18] D. D. Long, *Immersions and embeddings of totally geodesic surfaces*, Bull. London Math. Soc. **19** (1987), 481–484.

[19] D.D. Long and G. Niblo, *Subgroup separability and* 3*-manifold groups*, Math. Zeit. **207** (1991), 209 – 215.

[20] A. Lubotzky and D. Segal, *Subgroup Growth*, Progress in Math. **212**, Birkhauser (2003).

[21] C. F. Miller III, *On group-theoretic decision problems*, Annals of Math. Study **68** P.U.P (1971).

[22] D. D. Long and A. W. Reid, *On subgroup separability in hyperbolic Coxeter groups*, Geom. Dedicata **87** (2001), 245–260.

[23] T. Li, *Immersed essential surfaces in hyperbolic 3-manifolds*, Comm. Anal. and Geom. **10** (2002), 275–290.

[24] C. Maclachlan and A. W. Reid, *The Arithmetic of Hyperbolic 3-Manifolds*, Grad. Texts in Math. **219** Springer-Verlag (2003).

[25] J. G. Ratcliffe, *Foundations of Hyperbolic Manifolds*, G.T.M. **149**, Springer-Verlag, (1994).

[26] G. P. Scott and C. T. C. Wall, *Topological methods in group theory*, in *Homological Group Theory*, L.M.S. Lecture Notes **36**, 137–203, Cambridge University Press (1979).

[27] G. P. Scott, *Subgroups of surface groups are almost geometric*, J. London Math. Soc. **17** (1978), 555 - 565. See also *ibid Correction:* J. London Math. Soc. **32** (1985), 217–220.

[28] J-P, Serre, *Trees*, Springer-Verlag (1980).

[29] W.P. Thurston. *The Geometry and Topology of 3-manifolds.* Princeton University mimeographed notes. (1979)

[30] E. B. Vinberg, *Discrete groups in Lobachevskii space generated by reflections*, Mat. Sb. **72** (1967), 471–488.

[31] E. B. Vinberg and O. V. Shvartsman, *Discrete groups of motions of spaces of constant curvature*, Geometry II, Encyc. Math. Sci. **29**, 139–248, Springer-Verlag (1993).

[32] F. Waldhausen. *On irreducible three-manifolds which are sufficiently large.* Ann. of Math. **87** (1968), 56 - 68.