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A presentation for the image of Burau(4) $\otimes$ $\mathbf{Z}_2$

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1 Introduction

Let B_n denote the n-strand braid group. We recall that this admits a representation

$$\beta_n : B_n \to GL_{n-1}(\mathbf{Z}[t, t^{-1}])$$

the (*reduced*) *Burau representation* [1]. Although open for a long time, it is now known that this representation is not faithful for $n \geqq 6$, (See [8, 5]) and it is an old result that it is faithful for $n = 3$. (See [7]) Despite these counterexamples, there is no understanding of the nature of the image groups in the nonfaithful cases nor any kind of intrinsic characterisation of braids which lie in the kernel. The two cases $n = 4, 5$ remain open. Resolution of the case $n = 4$ is an important open problem, firstly for the implications for the automorphism group of a free group of rank 2 and secondly as a test case for the faithfulness of the Jones representation, [4]. In the case $n = 4$ the only summand which could be faithful is the Burau summand. There is a map $\alpha : GL_{n-1}(\mathbf{Z}[t, t^{-1}]) \to GL_{n-1}(\mathbf{Z}_2[t, t^{-1}])$ given by reducing coefficients modulo two and thus a simplified representation $\beta_n \otimes \mathbf{Z}_2$. Using the ideas contained in [8] or [5], it is not difficult to show that this representation continues to be faithful in the case $n = 3$, and it was observed in [5] that it is not faithful for $n = 5$. The main result of this paper is that we shall give a complete description of the image group in the case $n = 4$; this appears to be the first explicit description of the image group for any infinite linear representation of a braid group with $n \geqq 4$.

An especially intriguing aspect is the picture which emerges of the complex which carries the group which contains something rather analogous to an "geometrically infinite" end in the language of hyperbolic geometry [9]. It is

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the nature of this end which seems to be what foils all attempts to prove any kinds of faithfulness results by multiplying matrices.

We now give an outline of the proof, deferring careful definitions. The representation above can be considered to have its image in the general linear group of the quotient field; this is a field with a discrete rank one valuation, and standard methods [2] give an action on a Euclidean building which we denote Δ. By restriction, we obtain a representation

$$\rho : B_4 \to \mathrm{Aut}(\Delta)$$

It was first observed by Squier [10], that suitably interpreted, the Burau representation can be regarded as sesquilinear for a certain form J. This turns out to have the powerful consequence in this context that we can identify precisely the stabiliser of the trivial lattice and consequently that all vertex stabilisers for this action are finite. It's worth noting that the restriction to a field of characteristic two has not yet been used, any field of nonzero characteristic suffices up to this point. If however, we now assume that the field is $\mathbf{Z}_2(t)$ we can use this information to compute exactly the stabiliser of every vertex in the building and from this it turns out that we can construct explicitly a certain complex B and show:

Theorem 1.1 *$\Delta/\mathrm{im}(\rho) \cong B$ and B is contractible.*

In particular, the building is contractible, so that in the language of complexes of groups (see Appendix) we will have:

Theorem 1.2 $\mathrm{im}(\rho) \cong (B, G_\sigma, \psi_a, g_{a,b})$

We will see that the subgroup of $\mathrm{im}(\rho)$ stabilizing a vertex is finite, although arbitrarily large, and so it follows from arguments using group cohomology that:

Corollary 1.3 *The reduced homology $\hat{H}_*(\mathrm{im}(\rho); \mathbb{Q}) = 0$.*

Many approaches to the faithfulness question of the Burau representation involve a reduction to consideration of certain subgroup; these approaches usually involve showing that a certain pair of matrices generate a free group. An especially interesting feature of our method is the nature of the complex B, which highlights perhaps why such methods always seem to fail. The complex is the union of two pieces, one of which is a tube which metrically has bounded diameter out towards the end and as such seems to resemble a geometrically infinite end in the language of hyperbolic geometry. Most of the methods currently available for showing that matrix groups are free, amount to finding a quasi-isometry of the graph of an abstract free group with the group graph of the matrix group and this is exactly what is forbidden in a geometrically infinite end. A sharper version of this statement is that as a result of its Euclidean structure, the building carries a canonical metric which makes it into a $CAT(0)$ space, and we have shown that the convex hull of the orbit of any point is the whole building. Although the clearest picture is provided by the complex itself, we can also use the information to give a presentation:

Theorem 1.4 *The image group* $\beta_4 \otimes \mathbf{Z}_2(B_4)$ *is presented as*:
Generators: x, y
Relations:
1. $x^4 = z$
2. $y^3 = z$
3. $[x^2, yxy] = 1$
4. $[x, (yxy)^i]^4 = 1$ *for all* $i \geqq 0$
5. *The group generated by* $\langle (yxy)^i x (yxy)^{-i} \mid i \geqq 0 \rangle$ *is nilpotent of class* 3

Here z *denotes a generator of the centre of* B_4.

An interesting feature here is that all the nonfaithfulness is determined by the vertex stabilisers.

Moreover, it follows from our method of proof that:

Theorem 1.5 *The subgroup of* $GL_3(\mathbf{Z}_2[t, t^{-1}])$ *consisting of isometries of the form* J *is precisely the image of* $\beta_4 \otimes \mathbf{Z}_2$.

Of course, it is immediate from 1.4 that the representation $\beta_4 \otimes \mathbf{Z}_2$ is not faithful; this was already known to the authors previous to this work. Although this aspect is not the thrust of this paper, we point out some corollaries. The first is that there are knots whose Jones polynomials are identically 1 when reduced modulo 2. Vaughan Jones was kind enough to furnish us with some 13-crossing examples in the knot tables. However, as observed in [4], the existence of such knots does not suffice to prove that $\beta_4 \otimes \mathbf{Z}_2$ is non-faithful. Application of a condition which is equivalent yields:

Corollary 1.6 *There is a four-braid whose Jones polynomial is the same as that of the four component unlink when coefficients are reduced modulo* 2.

The smallest such braid has the order of 160 crossings.

2 Preliminaries

We have an inclusion $\mathbf{Z}_2[t, t^{-1}] \to \mathbf{Z}_2(t)$ and since the target field admits a discrete rank 1 valuation, this gives an action of the group B_n on the Euclidean building $\Delta(n-1)$ associated to the group $SL_{n-1}(\mathbf{Z}_2(t))$.

We recall how this building and action are defined, restricting our attention to the case $n = 4$, since this is the only case in which we shall subsequently be interested. This will serve the additional purpose of establishing notation. Denoting the nonzero elements of $\mathbf{Z}_2(t)$ by $\mathbf{Z}_2(t)^*$, let $v : \mathbf{Z}_2(t)^* \to \mathbf{Z}$ be a discrete rank one valuation on $\mathbf{Z}_2(t)$ given by $t^n \to -n$. Standard properties imply that

$$\mathcal{O} = \{x \in \mathbf{Z}_2(t) \mid v(x) \geqq 0\}$$

is a subring of $\mathbf{Z}_2(t)$, the *valuation ring* associated to v. This is a local ring and the unique maximal ideal is easily seen to be $\mathcal{M} = \{x \in \mathbf{Z}_2(t) \mid v(x) > 0\}$, a principal ideal. Choose some generator π for this ideal. This element is called a *uniformizing parameter* and by construction we have that $v(\pi) = 1$. Since $\mathscr{M}$ is maximal, the quotient $k = \mathbf{Z}_2(t)/\mathscr{M}$ is a field, the *residue class field*. One sees easily that in this case, the residue class field is $\mathbf{Z}_2$.

Now let V be the vector space $\mathbf{Z}_2(t)^3$. By a *lattice* in V we shall mean an $\mathscr{O}$-submodule, L, of the form $L = \mathscr{O}x_1 \oplus \mathscr{O}x_2 \oplus \mathscr{O}x_3$ where $\{x_1, x_2, x_3\}$ is some basis for V. Thus the columns of a non-singular 3×3 matrix with entries in $\mathbf{Z}_2(t)$ define a lattice. The standard lattice is the one corresponding to the identity matrix. We define two lattices L and L' to be equivalent, if for some $\lambda \in \mathbf{Z}_2(t)^*$ we have $L = \lambda L'$. We denote equivalence class by $[L]$. The building Δ is defined as a flag complex in the following way. The points are equivalence classes of lattices, and $[L_0], \dots, [L_k]$ span a k-simplex (in our situation $k = 0, 1, 2$ are the only possibilities) if and only if one can find representatives so that $\pi L_0 \subset L_1 \subset \cdots \subset L_k \subset L_0$. All two-simplices are of the form $\{[x_1, x_2, x_3], [x_1, x_2, \pi x_3], [x_1, \pi x_2, \pi x_3]\}$; this is usually referred to as a *chamber* and denoted by C. The set of chambers defined by all lattices of the form $[\pi^a x_1, \pi^b x_2, \pi^c x_3]$ where $a, b, c \in \mathbf{Z}$ is called the *apartment* associated to the basis $\{x_1, x_2, x_3\}$. We shall denote such an apartment by $\Sigma[x_1, x_2, x_3]$ or just by Σ if the context is clear. Clearly the group $SL_3(\mathbf{Z}_2(t))$ acts on lattices and one sees easily that incidence is preserved, so that the group acts simplicially on Δ. It is shown in [2] that this building is a so-called *Euclidean building*, in particular, it is contractible and can be equipped with a metric which makes it into a $CAT(0)$ space and for which $SL_3(\mathbf{Z}_2(t))$ acts as a group of isometries. The metric is such that each 2 dimensional simplex is isometric to a unit Euclidean equilateral triangle. Every triangle lies in infinitely many *apartments* each of which is isometric to the Euclidean plane.

To each lattice is associated a *type* (in our case $\{0, 1, 2\}$), defined as follows: If we consider the action of the full group $GL_3(\mathbf{Z}_2(t))$ this acts transitively on lattices, so given a lattice L, we choose some $g \in GL_3(\mathbf{Z}_2(t))$ throwing the standard lattice to L. It is easily seen that if we reduce $v(\det(g))$ modulo 3, this is well defined on the class of the lattice L; by definition this is the type of $[L]$. Both $GL_3(\mathbf{Z}_2(t))$ and $SL_3(\mathbf{Z}_2(t))$ act on the building, the main difference being that the group $SL_3(\mathbf{Z}_2(t))$ is type-preserving. The stabilizer of the standard lattice is $GL_3(\mathscr{O})$. A fact we shall make use of several times is:

Proposition 2.1 *$GL_3(\mathbf{Z}_2(t))$ acts without edge inversions on Δ.*

Proof. We note that an edge cannot be identified with itself with orientation reversed; for if we consider the vertex which lies in the triangle containing the edge but not on the edge, this type is preserved by any such map, hence all types are preserved and the edge could not have been reversed. □

The link of the standard lattice is the flag geometry for $V = \mathbf{Z}_2^3$: it is a bipartite graph whose 14 vertices are the 7 one-dimensional and 7 two-dimensional subspaces of V. This graph is shown in Fig. 0 (with certain labels).

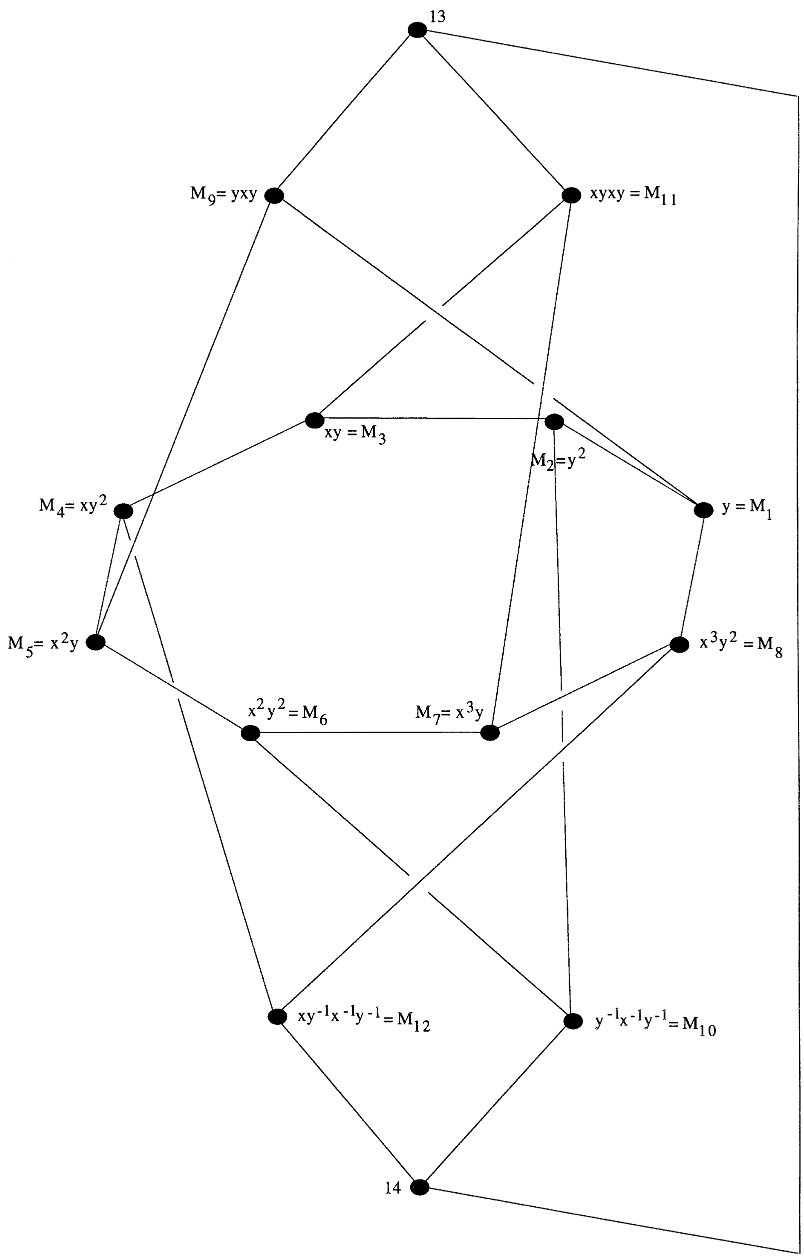

13
M9= yxy
xyxy = M11
xy = M3
M2=y2
M4= xy2
y = M1
M5= x2y
x3y2 = M8
x2y2 = M6
M7= x3y
xy-1x-1y-1 = M12
y-1x-1y-1 = M10
14

Fig. 0

In our case, the braid group maps

$$B_4 \to GL_3(\mathbf{Z}_2(t)) \to \mathrm{Aut}(\Delta)$$

We shall denote this composition by ρ; since central matrices acts trivially on the building, this factors through $B_4 \to B_4/Z$ where Z denotes the centre of B_4. We shall use:

Lemma 2.2 *Denoting the centre of* B_4 *by* Z, *we have*:

$$B_4/Z \cong \langle x, y \mid x^4 = y^3 = 1 \ [x^2, yxy] = 1 \rangle$$

where $x = \sigma_1\sigma_2\sigma_3$ *and* $y = \sigma_1\sigma_2\sigma_3\sigma_1$.

This is somewhat non-standard, albeit elementary, so rather than include it at this point in the exposition, we relegate it to an appendix. The matrices for these generators are:

$$\beta_4(x) = \begin{pmatrix} 0 & 0 & t \\ -t & 0 & t \\ 0 & -t & t \end{pmatrix} \qquad \beta_4(y) = \begin{pmatrix} 0 & 0 & -t \\ -t^2 & t & -t \\ 0 & t & -t \end{pmatrix}$$

3 Constructing the complex

In this section, we construct the complex B which will be the basis for the complex of groups description given in Sect. 4. We begin with an informal description of the whole construction before dealing with the details.

The complex B will consist of two parts. The first, denoted X, will be obtained by quotienting out the cone on the link of the identity lattice by $\mathrm{im}(\rho)$. The complex X has a single free edge which is a circle. The second part of the complex is topologically a half open annulus $S^1 \times [0, \infty)$ and it is glued onto this circle by a homeomorphism along its boundary component. Both X and this tube are constructed by making all obvious identifications forced by the image group and then we prove that no further identifications are possible. The main theorem of this section is the proof that $\Delta/\mathrm{im}(\rho) \cong B$. As part of this process, we are able to identify all stabilisers exactly and thus form the complex of groups.

Before embarking on the construction of X, we prove a simple lemma which plays a central role in all that follows. This lemma depends on the fact that the Burau representation can be considered as unitary for a certain form (the intersection pairing on the infinite cyclic covering of the punctured disc). This is due in essence to Squier. Since our notation is somewhat different from his, we establish this first.

Clearly we have an involution of the ring $\mathbf{Z}[t, t^{-1}]$ defined by mapping $* : t \to t^{-1}$ and this gives an involution on $GL_k(\mathbf{Z}[t, t^{-1}])$, also denoted $*$, defined by applying $*$ to all the entries of the matrix in question and transposing. Then Squier shows:

Lemma 3.1 [10] *The four strand Burau representation is sesquilinear, i.e.* $A^* \cdot J \cdot A = J$ *for* $A \in \mathrm{im}(\beta_4)$ *for the form*

$$J = \begin{pmatrix} -(s+s^{-1}) & s^{-1} & 0 \\ s & -(s+s^{-1}) & s^{-1} \\ 0 & s & -(s+s^{-1}) \end{pmatrix}$$

where $s^2 = t$.

We remark that Squier's form appears a little different in [10], but this is because in that paper, a preliminary conjugacy is applied to the Burau representation. The form of Lemma 3.1 applies to the most usual description of the Burau representation.

Lemma 3.2 *Suppose that A lies in* $\mathrm{im}(\rho)$ *and stabilises the identity vertex of* Δ. *Then there is* λ *in* $\mathbf{Z}_2(t)$ *such that the entries of* $\lambda \cdot A$ *are all constants.*

Proof. If an element A of $GL_3(\mathbf{Z}_2[t, t^{-1}])$ stabilises any vertex of Δ, then it must be type preserving and so it has determinant $\pm t^{3n}$ for some n. In particular it can be adjusted by homothety (i.e. replaced by $\pm t^{-n}A$) so that it lies in $SL_3(\mathbf{Z}_2[t, t^{-1}])$. Since by hypothesis, A stabilises the identity vertex of Δ, it follows that A lies inside $SL_3(\mathcal{O})$, that is to say that all its entries value $\geqq 0$. Notice that if in addition, the original element A lies in the image of the Burau representation, then its entries are all Laurent polynomials and it follows that the entries of the homothety adjusted matrix continue to be Laurent polynomials.

The fact that A lies in the image of the Burau representation means that we also have $A^* \cdot J \cdot A = J$ and so $A^* \cdot (J/s) \cdot A = J/s$. One now checks that J/s is $\mathcal{O}$-invertible and therefore we see from the description given in Lemma 3.1 that $J/s \in GL_3(\mathcal{O})$. (We have temporarily extended the valuation so that it is defined on $\mathbf{Z}[s, s^{-1}]$ by setting $v(s) = -1/2$). But this implies that $A^* \in SL_3(\mathcal{O})$. However, consideration of the action of the involution shows that the only matrices with Laurent polynomial entries which have both A and A^* lying in $SL_3(\mathcal{O})$ are the constant matrices. □

Since Lemma 3.2 shows that any matrix in $Q \in \mathrm{stab}(I) \cap \mathrm{im}(\rho)$ has no t dependence, it is in particular unchanged by the composition map

$$\xi = p_2 \circ \beta_4 \otimes \mathbf{Z}_2 : B_4 \to GL_3(\mathbf{Z}_2[t, t^{-1}]) \to GL_3(\mathbf{Z}_2)$$

where the map p_2 is given by the specialisation $t = 1$. This is a representation of Σ_4, the symmetric group on four letters, so that there are twenty four possibilities for the matrix Q. On the other hand, any such matrix lies in the image of $\beta_4 \otimes \mathbf{Z}_2$ (i.e. before setting $t = 1$) so that Q must satisfy $Q^* \cdot J \cdot Q = J$ where J continues to involve s. Using this observation a computation shows:

Lemma 3.3 *The only matrices in* $\mathrm{im}(\rho)$ *which stabilise the identity vertex are powers of the element* $\rho(x)$.

Since much reference will be made to the vertex stabilisers which can arise in $\mathrm{im}(\rho)$, we adopt the convention that unless the context makes it clear that this is not the case, $\mathrm{stab}(q)$ will mean $\mathrm{stab}(q) \cap \mathrm{im}(\rho)$. In passing we note that Lemma 3.3 has the following consequence:

Corollary 3.4 *The group $\rho(B_4)$ acts on Δ with finite vertex stabilisers.*

Proof. Pick any vertex $v \in \Delta$ and consider $\mathrm{stab}(v)$. This acts on $\mathrm{Link}(v)$ which contains finitely many points, so a subgroup H of finite index in $\mathrm{stab}(v)$ acts trivially on this link, so that $H \leqq \mathrm{stab}(w)$ for any w adjacent to v. There is a finite distance from v to I so repeating this process we see there is a subgroup of finite index in $\mathrm{stab}(v)$ which stabilises I, hence by Lemma 3.3 is a finite group. □

Remark 3.5 In fact, characteristic 2 does not play a role in the argument so far: If one puts $t = 1$ in the Burau representation, a computation reveals that for any prime it is only the element x which preserves the form J. Since vertex links are finite for any prime, it follows that 3.4 continues to hold for the action on any building $\Delta(p)$ coming from reduction modulo a prime.

3.1 Constructing X

With the notation established above one finds that the orbit under $\mathrm{im}(\rho)$ of the identity lattice I contains (at least) twelve points at distance 1 and so lying as vertices in $\mathrm{Link}(I)$. Since this latter complex is the flag manifold coming from the subspaces of $W \cong \mathbf{Z}_2^3$ (See [2]) it contains 14 vertices. To establish notation, we label twelve of these vertices by elements in the group:

$$\{y, y^2, xy, xy^2, x^2y, x^2y^2, x^3y, x^3y^2, yxy, xyxy, (yxy)^{-1}, x(yxy)^{-1}\}.$$

The two remaining vertices we denote simply by 13 and 14; we show below that in fact these are not in the orbit of I. See Fig. 0. We include representative lattices in the Appendix.

Our first task is to construct the complex X. To this end, we consider $\mathrm{star}(I)$ and some identifications which are forced by the group. For example, since x stabilises I, it acts on $\mathrm{Link}(I)$ and we compute from this description that it acts as the permutation which is the product of two four-cycles and two transpositions:

$$(y, xy, x^2y, x^3y)(y^2, xy^2, x^2y^2, x^3y^2)(yxy, xyxy)((yxy)^{-1}, x(yxy)^{-1})$$

Referring to Fig. 0, we see that since x exchanges yxy with $xyxy$ and 13 is the unique vertex distance 1 from these two points it follows that x fixes 13 and similarly 14. In fact x as an element of $GL_3(\mathbf{Z}_2)$ is a rotation of order 4 and thus preserves a unique line and plane which are 13 and 14 respectively.

The element y does not stabilise I, but it does give rise to further identifications, for example the triangle with vertices I, y, y^2 is mapped to itself by 3-cycle, and the edge (y^2, xy) is mapped to the edge (I, yxy).

The last and most interesting identification arises as follows. The vertex yxy is distance 1 from 13. Thus if we apply $(yxy)^{-1}$ to the edge $(yxy, 13)$ we obtain an edge $(I, (yxy)^{-1}(13))$, so that $(yxy)^{-1}(13) \in \mathrm{Link}(I)$. The element yxy acts on the building as an element of infinite order, so that Lemma 3.4 prohibits 13 being fixed and one finds that $yxy(14) = 13$. Taking account of all such identifications, we obtain the complex X shown in Fig. 1(a). This consists of four triangles glued with the identifications shown in that figure, where the triangle which is labelled 1 comes from the self identification of the triangle I, y, y^2 discussed above.

Of course *a fortiori* when the whole group acts there could be further identifications. We claim (See Theorem 3.7) that in fact this does not happen. The proof of this requires:

Theorem 3.6 *The vertices* 13 *and* 14 *do not lie in the orbit of the identity lattice.*

Proof of 3.6 Since $yxy(14) = 13$, if one of the vertices lies in the orbit, then they both do. Suppose in search of a contradiction, that 13 lies in the orbit of the image. From this it follows that every element of the link of I lies in the orbit of I. Since the building is connected, it follows that every vertex in the building lies in the orbit of I.

We have identified $\mathrm{stab}(I)$ as $\langle x \rangle$ so that vertices 13 and 14 have the property that they are fixed by all of $\mathrm{stab}(I)$ and moreover, they are the only vertices in the link of I which are fixed by all of $\mathrm{stab}(I)$.

Accordingly, we define a vertex $p \in \mathrm{Link}(q)$ to be an *s-point* for q if every element in $\mathrm{stab}(q)$ also fixes p. The comments in the above paragraph show that every point in the building has exactly two s-points in its link and that if ϕ is an element in $\mathrm{im}(\rho)$ then ϕ carries the s-points of q to the s-points of $\phi(q)$.

We have already observed that $yxy(14) = 13$ and it follows that $xyxy(14) = x(13) = 13$, so that 13 is an s-point for at least three points, namely I, yxy and $xyxy$. Thus every vertex in Δ must be an s-point for at least three points. Now any of the twelve group vertices is of the form $g(I)$, so that the stabiliser of such a point is $g \cdot \mathrm{stab}(I) \cdot g^{-1}$. In particular, Lemma 3.3 makes it easy to check if the generator of $\mathrm{stab}(g)$ also stabilises I; namely we need only compute $g \cdot x \cdot g^{-1}$ and see if it is one of four constant matrices. A calculation reveals that this does not happen, so that I is an s-point for at most two vertices, namely 13 and 14, which is the required contradiction. □

It follows from the proof that Δ contains three kinds of vertex:

1. Vertices in the $\rho(B_4)$ orbit of I.
2. Vertices which do not lie in the $\rho(B_4)$ orbit of I but are distance 1 from this orbit.
3. Vertices which are not of either of the above types.

Clearly, this classification is braid group invariant, and it follows from Theorem 3.6 that Type 2 points exist. We refer to the points of Type 1 as *group points* and continue to refer to the points of Type 2 as s-points. Our analysis below will show that there are countably many orbits of points of the third type and we shall exhibit explicitly one vertex in each orbit.

Theorem 3.7 *The complex X has no further identifications in $\Delta/\mathrm{im}(\rho)$.*

Proof. First we observe that there can be no more vertex identifications. For there are only two vertices in X, one is a group point and one is an s-point. Thus it follows from the fact that the group acts simplicially and Theorem 3.6 that these points cannot be further identified.

We now deal with the possibility of further identification amongst the triangles. Again, the invariance of s-points immediately implies that both triangles 3 and 4 have the property that they cannot be identified with any other triangle, nor can they admit any further self-identification.

If there is some group element which carries triangle 1 to triangle 2, this in particular implies that there is an element g in the group which maps triangle 2 to itself as a three cycle, since such an element exists for triangle 1. We claim that this is impossible.

First note that there is a map $B_4/Z \to \mathbf{Z}_3$ given by mapping x to the identity. The image is generated by y and the kernel is those elements in B_4/Z which preserve lattice type. It follows that (inverting g if necessary) that we have $g = w \cdot y$ where w is an element which preserves lattice type. We refer the reader to Fig. 1(a). Our first claim is that the element $w \cdot y$ cannot now be an anti-clockwise rotation of triangle 2. For then the element $y \cdot w \cdot y[I] = y[y^2] = [I]$ stabilises the identity lattice and is in particular type preserving, which it cannot be. Consider $y^{-1} \cdot w \cdot y$; since $w \cdot y$ rotates clockwise, this is type-preserving and a similar calculation shows that this element stabilises the vertex y^2. Thus $y^{-1} \cdot w \cdot y \in \mathrm{stab}(y^2) = y^{-1}\mathrm{stab}(I)y$ and so $w \in \mathrm{stab}(I)$. Now Lemma 3.3 shows that g is of the form $\rho(x^k \cdot y)$ for $k = 0, \ldots, 3$ which is easily checked to be impossible.

We now deal with the case of further identifications amongst edges. We have already shown that such identifications cannot arise from extra triangle identifications, but we have to deal with the possibility of an element of the image which causes an unexpected identification of an edge of a triangle in $Lk(I)$ with a triangle in an adjacent link.

By virtue of its s-points, the edge $(13, 14)$ cannot be identified with any other edge except itself and since there are no edge inversions, this edge can have no further identifications.

If edge $(yxy, 13)$ is to be identified any other edge, it can only be the class of $(I, 13)$. However such a group element stabilises 13 since this is the only s point, and thus the element is type preserving; a contradiction since the other end of the edge forces yxy and I to be identified and these have different type.

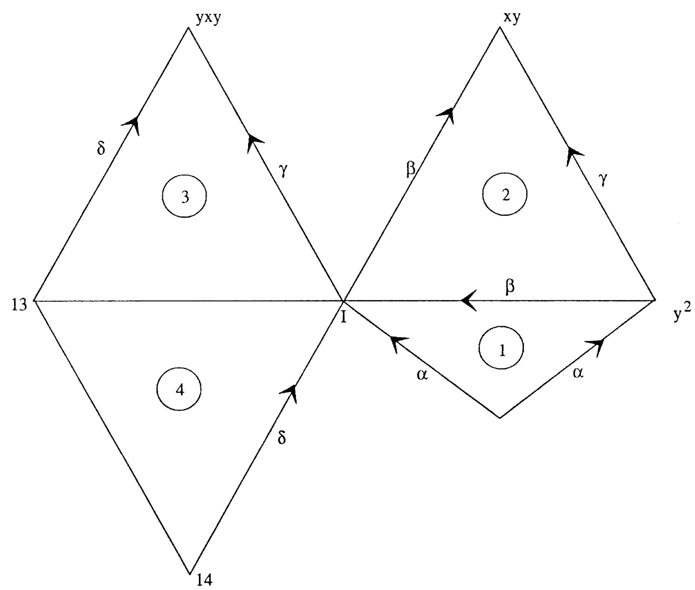


Fig. 1(a)

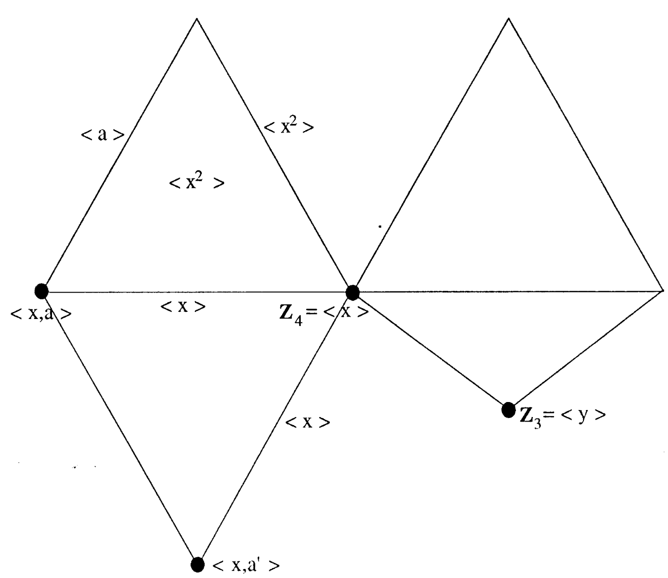


Fig. 1(b)

Finally, if edge (I, yxy) is identified with edge (I, y^2) this forces an identification of triangle 3 with either triangle 1 or 2 which we have already excluded. □

3.2. The tube

We now begin to construct the tube which forms the rest of the quotient complex. Again, for clarity, we summarize the procedure which builds up this tube, deferring some of the steps in its justification until later.

We note that the element x stabilises both 13 and 14. Moreover, we have already observed that $yxy(14) = 13$ so that the element $a = yxy \cdot x \cdot (yxy)^{-1}$ also stabilises 13.

One computes that the elements x and a both act on the link as elements of order 2 and that their product has order 4, so that the image of the map $i_0 : \mathrm{stab}(13) \to \mathrm{Aut}(\mathrm{Link}(13))$ contains a dihedral group $D_8 \cong \langle x, a \mid x^2 = a^2 = (a \cdot x)^4 = 1 \rangle$. We shall show below that this image is exactly this dihedral group.

We now introduce the following notation. The link of the identity can be described by fixing once and for all time, matrices $M_1, \ldots, M_{14}$ whose columns define the lattices at the relevant point in Δ. Such a choice is not canonical. All our calculations will be done with respect to the set described in the appendix to this paper. Then the link of 13 is graph isomorphic to the link of the identity and we choose the identification so that our numbering is given by premultiplication by the matrix M_{13} associated to lattice 13; so that for example, the vertex we shall identify with 1 in the link of 13 will be the equivalence class of the lattice $M_{13} \cdot M_1$. For convenience we shall denote this lattice class by 13.1 or for brevity 1^*. A somewhat more general example will define the notation completely; if we now look in the link of our new vertex 13.1 at the vertex labelled 7, this will be described by 13.1.7. Of course, links overlap so a vertex may have many guises; for example, $13.2 = [I]$ and $13.1 = [yxy]$.

We now compute that the permutation action of the generators is given as follows:

$$x = (1^*, 3^*)(4^*, 9^*)(8^*, 11^*)(12^*, 13^*)$$
$$a = (2^*, 9^*)(3^*, 13^*)(4^*, 14^*)(5^*, 10^*)$$

From this we compute the quotient complex $\mathrm{Link}(13)/D_8$ is the shaded hexagon in the chain of hexagons shown in Fig. 2. Notice that consideration of the generators shows that the vertices $6^* = 13.6$ and $7^* = 13.7$ are fixed for the entire action of $i_0(\mathrm{stab}(13))$ (this will be proved in 3.15). One finds that the element yxy acts as follows:

$$6^* \to 7^* \;\; 10^* \to 13 \;\; 13 \to 8^* \;\; 2^* \to 1^*$$

so that when we quotient out by the group $\mathrm{im}(\rho)$ we obtain an annulus, consisting of four triangles, two of which already appear in the complex X; we

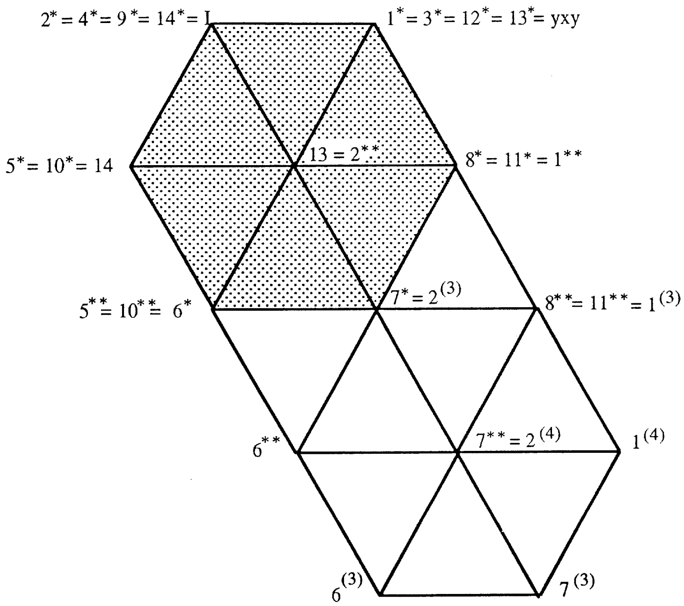


Fig. 2

identify these two triangles onto X. The new complex can be considered to have been constructed from X by adding an annulus consisting of two triangles. The annulus is added along the edge $(13, 14)$ and this leaves a new free edge coming from the edge $(6^*, 7^*)$.

Since x and a both stabilise 6^* and 7^*, we can compute the action of these elements on Link(6^*) and Link(7^*). One finds that a acts trivially on Link(7^*) and that x acts trivially on Link(6^*). However, x continues to be an element of order 2 when it acts on stab(7^*). Moreover, the above observations show that the element $a_1 = yxy \cdot a \cdot (yxy)^{-1}$ lies in stab(7^*). If, for brevity, we denote the vertex 13.7.k by k^{**}, then we may compute that the action of the elements x and a_1 is given by:

$$x = (1^{**}, 3^{**})(4^{**}, 9^{**})(8^{**}, 11^{**})(12^{**}, 13^{**})$$
$$a_1 = (2^{**}, 9^{**})(3^{**}, 13^{**})(4^{**}, 14^{**})(5^{**}, 10^{**})$$

It is part of the power of our notation that in these coordinates, the permutations induced on the link of 7^* are the same as those induced on 13.

Further, it turns out that the element yxy carries 6^{**} to 7^{**}. Thus we obtain another annulus; one boundary component of this annulus coming from the $(6^*, 7^*)$ edge of the hexagon, the other from the $(6^{**}, 7^{**})$ edge. It will turn out that this picture repeats itself and we add on a series of annuli coming from the links of the $13.7.7.\ldots.7 = 7^{(n)}$, giving rise to an infinite tube. The chain of hexagons so generated and a typical such hexagon are illustrated in Figs. 2 and 3 respectively.

We now give a more complete discussion of the construction of the tube. As above, we define a sequence of points $13.7.7.\ldots.7.k = k^{(n)}$.

Lemma 3.8 *For every n, $yxy(6^{(n)}) = 7^{(n)}$*

Proof. Unravelling the definitions, we see that the lemma requires that $y \cdot x \cdot y \cdot M_{13} \cdot M_7^{n-1} \cdot M_6 = M_{13} \cdot M_7^n \cdot \gamma$, where $\gamma \in SL_3(\mathcal{O})$. Equivalently, that $M_7^{-n} \cdot M_{13}^{-1} \cdot y \cdot x \cdot y \cdot M_{13} \cdot M_7^{n-1} \cdot M_6$ lies in $SL_3(\mathcal{O})$. This is a routine calculation. □

We now define a building map $\xi = M_{13} \cdot M_7 \cdot M_{13}^{-1}$; by construction, $\xi(k^{(n)}) = k^{(n+1)}$ for every $k \in \{1, \ldots, 14\}$. We will see later that these vertices are in distinct orbits under $\mathrm{im}(\rho)$ so that the element ξ does not lie in $\mathrm{im}(\rho)$. Using the map ξ we can identify the actions of elements in successive $\mathrm{stab}(7^{(n)})$. The calculations outlined on Link(13) give a pair of elements x and a lying in stab(13). For $k \geqq 0$, we define $a_k = (yxy)^{k+1} \cdot x \cdot (yxy)^{-k-1}$, with the convention that $a_0 = a$. We claim:

Lemma 3.9 *The following diagram commutes*:

$$\begin{array}{ccc} \mathrm{Link}(7^{(n)}) & \xrightarrow{\ \xi\ } & \mathrm{Link}(7^{(n+1)}) \\ \Big\downarrow x & & \Big\downarrow x \\ \mathrm{Link}(7^{(n)}) & \xrightarrow{\ \xi\ } & \mathrm{Link}(7^{(n+1)}) \end{array}$$

Lemma 3.10 *The following diagram commutes*:

$$\begin{array}{ccc} \mathrm{Link}(7^{(n)}) & \xrightarrow{\ \xi\ } & \mathrm{Link}(7^{(n+1)}) \\ \Big\downarrow a_n & & \Big\downarrow a_{n+1} \\ \mathrm{Link}(7^{(n)}) & \xrightarrow{\ \xi\ } & \mathrm{Link}(7^{(n+1)}) \end{array}$$

Lemma 3.11 *The map $a_n : \mathrm{Link}(7^{(m)}) \to \mathrm{Link}(7^{(m)})$ is the identity map for $m > n$.*

These are all routine calculations in linear algebra, made somewhat simpler by the fact that coefficients are in the field $\mathbf{Z}_2$. In particular we have as a consequence:

Corollary 3.12 *For every n, the image of the map $i_n : \mathrm{stab}(7^{(n)}) \to \mathrm{Aut}(\mathrm{Link}(7^{(n)}))$ contains a subgroup isomorphic to $D_8 \cong \langle x, a_n \mid x^2 = a_n^2 = (a_n \cdot x)^4 = 1 \rangle$.*

Given these ingredients we build up the second part of our quotient complex, namely an infinite tube. This is built up from the succession of annuli which come from quotienting out the hexagon $\mathrm{Link}(7^{(n)})/D_8$ by the map yxy.

Each such annulus consists of four triangles, two of which have occurred in the previous annulus, and the net effect is gluing on an annulus made of two triangles, one of whose boundary components comes from the edge $(6^{(n-1)}, 7^{(n-1)})$ and the other from the edge $(6^{(n)}, 7^{(n)})$. We denote this tube by T. We now form a complex B by glueing the tube T to the complex X along the circle coming from the edge $(13, 14)$. The chain of hexagons used in this construction was already depicted in Fig. 2; for a picture of a fundamental domain of the tube, the reader is referred ahead to Fig. 6. Our main theorem in this section is:

Theorem 3.13 *The quotient space* $\Delta/\mathrm{im}(\rho)$ *is isomorphic to* B.

The proof of this result takes several steps. The first is to show that Lemma 3.12 describes the whole image of the maps i_n. First we need the following (presumably well known) lemma:

Lemma 3.14 *Any subgroup* G *of* $GL_3(\mathbf{Z}_2)$ *of order* 24 *is either the stabiliser of a plane or the stabiliser of a line.*

If it is a line which is stabilized then there are exactly two orbits of lines, one containing one line and the other containing 6 *lines.*

If it is a plane which is stabilized then a similar statement holds.

Proof. The vector space $W \cong \mathbf{Z}_2^3$ on which $GL_3(\mathbf{Z}_2)$ acts has seven lines and seven planes; since the group acts transitively on lines, it follows that the stabiliser of any line has index 7 and therefore order 24. It follows that if G stabilises any line it would be a subgroup of, and hence equal to, some such stabiliser.

We may suppose then, that no orbit for G contains only one element. Thus the possibilities for the sizes of orbits for G are $\{2, 2, 3\}$ or $\{3, 4\}$.

We claim the first case is impossible; for by passing to a subgroup of index at most four, we obtain a subgroup of G which acts trivially on four distinct vectors. However any four nonzero vectors in W contain a basis and this is a contradiction.

Thus G must have orbit type $\{3, 4\}$. The action of G on the second orbit gives a representation of G to the symmetric group on 4 letters which must be faithful; since exactly as above the orbit contains a basis, so that any element of G acting trivially fixes a basis and is therefore trivial. Since both groups have order 24, G acts as the symmetric group on 4 letters on these four lines. Since some triple of these lines contains a basis, it follows that any such triple does. Thus after a conjugacy, G is the group of all permutations of the vectors $\{e_1, e_2, e_3, e_1 + e_2 + e_3\}$, where $\{e_1, e_2, e_3\}$ is a basis for V. One now sees easily that G stabilises the plane coming from the orbit of size 3; namely $\langle e_2 + e_3, e_1 + e_3\rangle$ and acts transitively on all other planes. □

Theorem 3.15 *For every* n, *the image of the map* $i_n : \mathrm{stab}(7^{(n)}) \to \mathrm{Aut}(\mathrm{Link}(7^{(n)}))$ *is precisely the group* D_8. *The quotient of* $\mathrm{Link}(7^{(n)})$ *is a hexagon, see Fig.* 3. *The points* $6^{(n+1)}, 7^{(n+1)}$ *are stabilized by this action, and are the only points in* $\mathrm{Link}(7^{(n)})$ *with this property.*

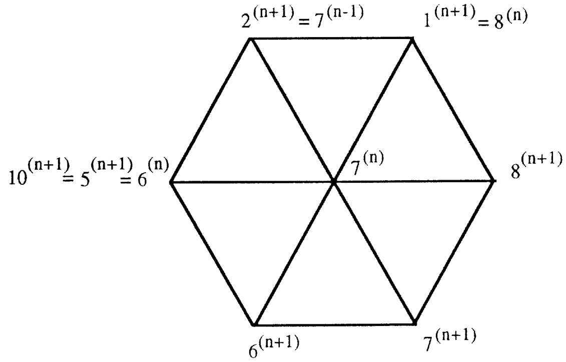


Fig. 3

Proof. First note that $GL_3(\mathbf{Z}_2)$ has order 7.24 and D_8 has order 8 thus we must show that the image is not the entire group and does not have order 24 or 56.

Now $GL_3(\mathbf{Z}_2)$ contains no subgroup of index 3; for the action of the group on the left cosets of such a subgroup gives a nontrivial representation $GL_3(\mathbf{Z}_2) \to \Sigma_3$ hence a normal subgroup of index at most 6; contradicting the simplicity of $GL_3(\mathbf{Z}_2)$.

Thus the only possibilities for groups strictly containing D_8 are subgroups of order 24 and the whole group. We shall prove the theorem by showing that a subgroup of order 24 cannot stabilise a tube vertex.

Since $yxy(6^{(n)}) = (7^{(n)})$, if either of these stabilisers has order 24, they both do. By the lemma, it follows that $i_n(\text{stab}(7^{(n)}))$ is either the stabiliser of a plane or of a line in V. Without loss we suppose the latter and suppose that $5^{(n+1)}$ corresponds to a line (the argument is the same if $5^{(n+1)}$ corresponds to a line.)

We refer the reader to Figs. 2 and 3, recalling that $6^{(n)} = 5^{(n+1)}$. Since the orbit of the vertex $5^{(n+1)}$ under the group $i_n(\text{stab}(7^{(n)}))$ already contains $10^{(n+1)}$, neither of these is stabilised by $i_n(\text{stab}(7^{(n)}))$. Thus $5^{(n+1)}$ is in an orbit of 6 lines so we can find $g \in i_n(\text{stab}(7^{(n)}))$ throwing this vertex to the vertex $1^{(n+1)} = 8^{(n)}$. It follows that the element $(yxy)^{-1}g(yxy)^{-1}$ throws the element $7^{(n)}$ to the element $7^{(n-1)}$, implying that $i_{n-1}(\text{stab}(7^{(n-1)}))$ is also a group of order at least 24.

Repeating this argument, we see that it suffices to show that $i_0(\text{stab}(13))$ cannot contain a subgroup of order 24. However, this also is impossible. For this means that either we can find an element of the image throwing $5^*(= 14)$ to yxy; impossible as the former is an s point and the latter is not. Or we can find an element throwing 8^* to I and this is impossible as $8^* = yxy(13)$ is the group image of an s point, hence an s point. □

This information will also be used in giving a complete iterative description of the groups $\text{stab}(7^{(n)})$ which we shall need when computing the complex of groups. (See Theorem 4.6 below.)

Our next series of results will be directed towards showing that the tube we have just constructed contains no further identifications.

Lemma 3.16 *There are no elements in the group mapping a triangle in the tube to itself.*

Proof. There are two cases, see Fig. 4. In the first case we see that $yxy\alpha(7^{(n)}) = yxy(6^{(n)}) = 7^{(n)}$, so $yxy\alpha \in \mathrm{stab}(7^{(n)})$. However, $yxy\alpha(6^{(n+1)}) = 8^{(n+1)}$ but by Theorem 3.15 the image of $\mathrm{stab}(7^{(n)}) \to \mathrm{Aut}(\mathrm{Link}(7^{(n)}))$ fixes $6^{(n+1)}$, a contradiction.

In the second case, we argue similarly, observing that $yxy\eta^{-1} \in \mathrm{stab}(7^{(n+1)})$ but throws $7^{(n)}$ to $8^{(n+2)}$ and an analogous contradiction. □

Lemma 3.17 *The point* 6^* *is neither a group point nor in the orbit of an* s*-point.*

Proof. The point 6^* cannot be a group point as group points have stabiliser $\mathbf{Z}_4$ and the stabiliser of 6^* maps onto D_8.

If it were an s-point, there would be an element η in the group such that $\eta(6^*) = 13$. We claim this is impossible. By Lemma 3.16, η cannot map the triangle $(13, 6^*, 7^*)$ to itself. Thus, there are two types of cases, depending on whether $\eta(13)$ is mapped to the orbit of 14 or to the orbit of 7^* (it cannot be a group point). These are exemplified in Fig. 5. In the first case, we have one of $yxy\eta$ or $(yxy)^{-1}\eta$ lies in $\mathrm{stab}(13)$ and yet moves 6^* which contradicts Theorem 3.15. In the second case, we may choose an element $\theta \in \mathrm{stab}(13)$ which moves $\eta(7^*)$ and deduce the element $\eta^{-1}\theta\eta \in \mathrm{stab}(13)$ moves the point 7^* contradicting Theorem 3.15. □

We observe that the second part of this argument actually shows a little more:

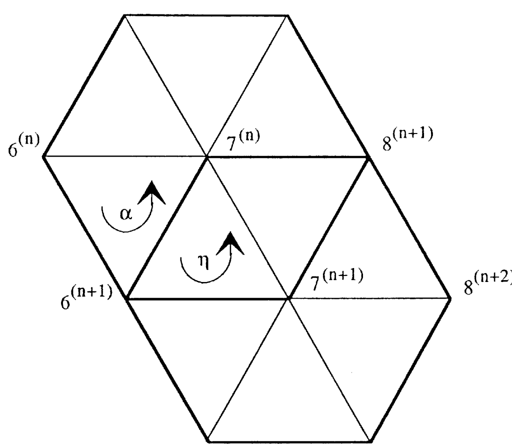


Fig. 4

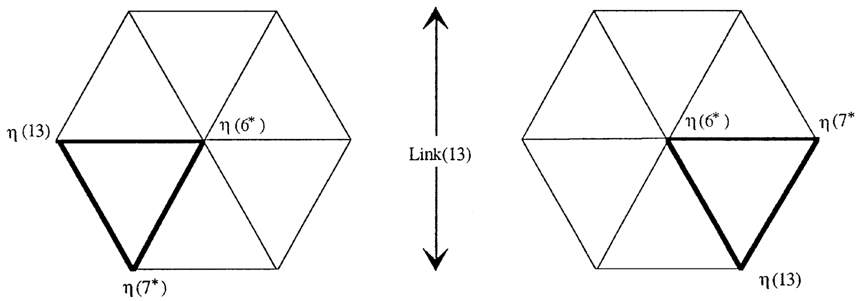


Fig. 5

Corollary 3.18 *For $n \geqq 2$, the point $6^{(n)}$ does not lie in the orbit of the point $6^{(n-1)}$.*

Remark 3.19 Notice that the Lemma implies that the orbit of every point in Link(13) is now accounted for as either a group point, the orbit of an s-point, or the orbit of 6^*.

Lemma 3.20 *For $n \geqq 2$, a point $6^{(n)}$ does not lie in the orbit of any $6^{(m)}$ for any $m < n$.*

Proof. We will argue by induction on n. Firstly observe that 6^{**} cannot be in the orbit of an s-point. For if it were, since the edge $(6^*, 7^*)$ lies in Link(6^{**}), it would follow that the group element mapping 6^{**} to 13 would map that edge into Link(13). We deduce (see the remark above) that one (and hence both) of 6^* or 7^* would be a group point or an s-point, contradicting Lemma 3.17 unless this edge is actually stabilised. However, since edge inversions are impossible, this would mean that the group element actually lay in stab(7^*) and we know that there are no such elements mapping 6^{**} to 13.

The proof that 6^{**} cannot lie in the same orbit as 6^* is the case $n = 2$ of Corollary 3.18. This completes the first step of the induction.

Now fix some $k > 2$ which is chosen as small as possible so that we have a counterexample to the Lemma. Suppose that there is a group element carrying $6^{(k)}$ to $6^{(m)}$ where $m \leqq k - 2$. We again consider the edge $(6^{(k-1)}, 7^{(k-1)})$; this is carried into the link of $6^{(m)}$. Since every point in this link is in the orbit of $6^{(m-1)}$, $6^{(m)}$ or $6^{(m+1)}$, this is already a contradiction to the minimality of k unless $m = k - 2$ or $m = k - 1$.

If $m = k - 2$, we argue as in the paragraph above: The minimality of k implies the edge must be stabilised, and the absence of inversions means we would find a group element in stab$(7^{(k-1)})$ mapping $6^{(k)}$ to $6^{(k-2)}$, a contradiction.

If $m = k - 1$, this is Corollary 3.18. □

Proof of Theorem 3.13 The picture of the tube is as shown in Fig. 6. We have already shown in Theorem 3.7 that the complex X is embedded.

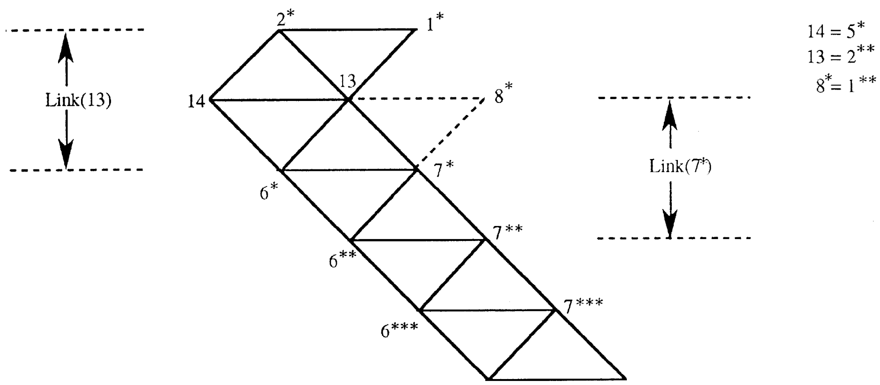


Fig. 6

We now claim that we may build an equivariant map $q : \Delta \to B$.

For we clearly have a map from the orbit of I under $\mathrm{im}(\rho)$ to X, and hence from orbits of the triangles numbered 1 and 2. We then use the orbits of the s-points to map in the triangles 3 and 4, this is well-defined because by Theorem 3.6, s-points are not group points. Thus we have defined q on a subcomplex $C_0 \subset \Delta$ which contains the orbit of the link of I. Theorem 3.7 shows that we have accounted for all possible identifications, so that q maps C_0 onto the complex X.

We now map in the tube, triangle by triangle. We refer to Fig. 7.

The next triangles of the construction are attached to C_0 along the edge coming from $(13, 14)$. Recall that $14 = 5^*$ so there are two triangles to be attached at this point, as the orbit of 5^* in the action of stab(13) on its link contains the two points 5^* and 10^*. So we append triangles $(13, 5^*, 6^*)$ and $(13, 10^*, 6^*)$ to C_0 and all triangles in the $\mathrm{im}(\rho)$ orbit of these. We have shown that 6^* is neither a group point nor in the orbit of an s-point, so that there is

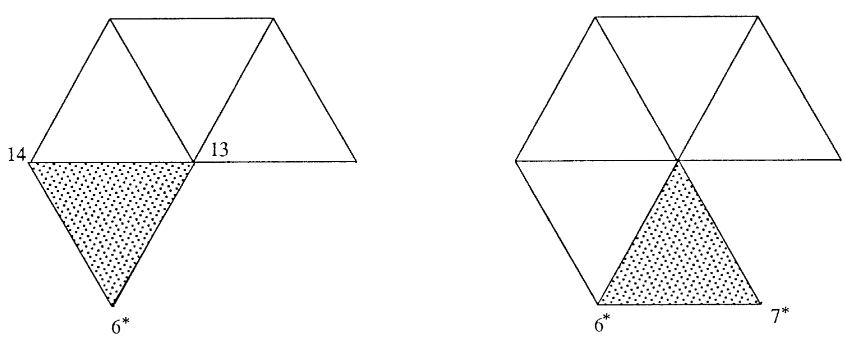


Fig. 7

no obstruction to extending the map from the orbit of 6^* to the corresponding vertex in B and whence from the orbit of these two triangles in Δ.

This leaves only one triangle $(13, 6^*, 7^*)$ in Link(13) unaccounted for and we attach the orbit of this triangle. This defines the map q on an equivariant complex $C_1 \subset \Delta$ which includes the links of I and of 13.

We can now continue this process. At each stage we seek to extend by triangles the complex C_k on which the map is defined. It follows exactly the same construction of the above paragraphs. By Lemma 3.20, the vertices of the form $6^{(k+1)}$ on which we wish to define the map does not appear in the complex C_k so that there is no obstruction to extending q over subsequent triangles and extending by equivariance.

Notice that at every stage of this process, if the map is defined on a simplex, then it is defined on all simplices its star. In particular, since the building is connected the map q is eventually defined on every simplex. The result now follows. □

This result identifies the quotient complex completely. The building is contractible, so if we denote the stabiliser of a simplex $\sigma \subset \Delta$ by G_σ, it follows from Haefliger's results on complexes of groups (see Theorem 5.1) that we have:

Theorem 3.21 $\mathrm{im}(\rho) \cong (B, G_\sigma, \psi_a, g_{a,b})$

Thus to compute the group of $\mathrm{im}(\rho)$ we need to compute the underlying group of this complex of groups. The next section is devoted to this calculation.

4. The complex of groups

In this section we compute the underlying group of B. In order to carefully define the conventions with which we will work, we have included an appendix which contains a brief summary (which follows Haefliger [3]) of complexes of groups.

4.1 The underlying group of X

The complex X consists of 2 vertices, 6 edges and 4 triangles. We begin by taking the subcomplex X_0 consisting of the closure of the first two triangles. The underlying group of this subcomplex of groups is B_4/Z:

Lemma 4.1 *The underlying group H of* $(X_0, G_\sigma, \phi_a, g_{a,b})$ *has the presentation*

$$\langle x, y \,|\, x^4 = y^3 = 1 \;\; [x^2, yxy] = 1 \rangle$$

where x and y are generators of the two vertex groups.

Proof. The procedure comes from 5.1.3, 5.1.4. For each cell σ in X_0 we choose a cell $\tilde{\sigma}$ with $p\tilde{\sigma} = \sigma$. These choices all lie in Link(I) and are shown as bold edges in Fig. 8(a). There are 2 vertices v_1, v_2, 3 edges $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and 2 triangles τ_1, τ_2. These choices determine the choice of lifts of cells in the 1-skeleton in the barycentric subdivision of X_0. A vertex or edge of the barycentric subdivision lies in the interior of a unique cell σ of X_0 and we choose the lift of the vertex or edge to lie in the chosen lift of σ.

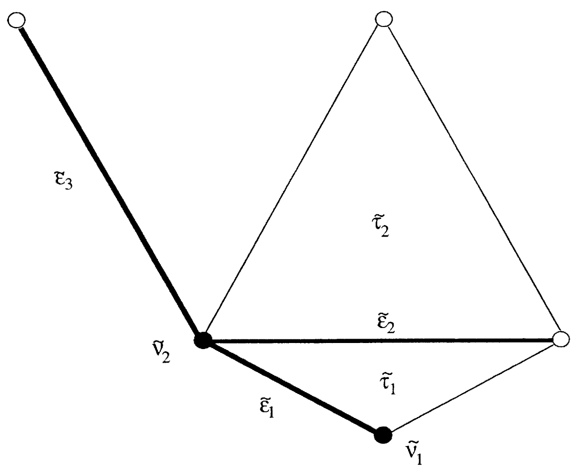


Fig. 8(a)

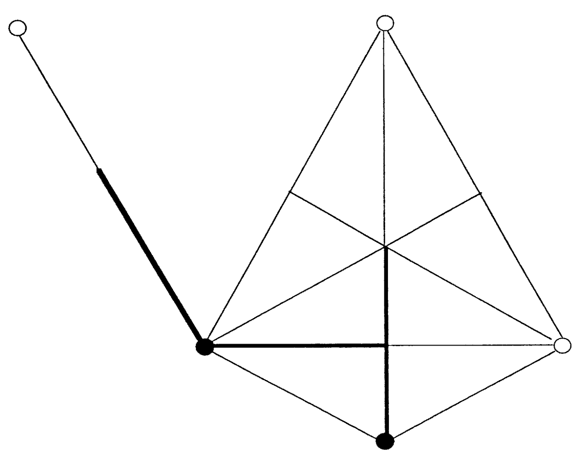


Fig. 8(b)

Next we compute the head elements h_a for each edge a of $X^{-(1)}$ and these are shown next to the edge in the barycentric subdivision they correspond to in Fig. 9(a). The chosen simplices are again indicated by bold lines and vertices in this figure.

From this we compute for each triangle with sides a, b, ab the monodromy element $g_{a,b}$. One finds that these elements are nontrivial in only two cases; these are the circled elements shown in Fig. 9(a).

Now we choose the maximal tree in $X^{-(1)}$ which is the projection of the corresponding tree upstairs shown as heavy lines in Fig. 8(b). Then Fig. 9(b) shows which elements of H each edge represents.

We will now repeatedly exploit the triangle relations in the underlying group of the form:

$$(ab)^+ = b^+ a^+ g_{a,b} \,.$$

For example, the shaded triangle in Fig. 9(b) has two sides in the maximal tree hence they are both trivial in H and since for this triangle $g_{a,b} = 1$ it follows that the third edge (labelled E_1 in the figure) is also trivial in H. Since edge E_1 is identified to the other edge labelled E_1 in 9(b), it follows that $E_2 = y^{-1}$ in H.

In a similar way one now easily fills in the remaining information on Fig. 9(b) using the triangle relations, the information in Fig. 8(b), and the identifications of the edges. In particular we see that the underlying group is generated by x and y.

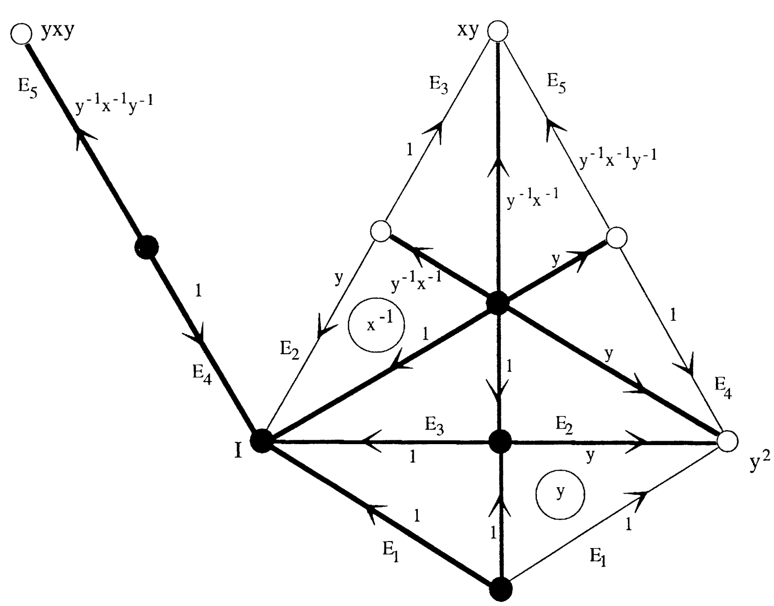


Fig. 9(a)

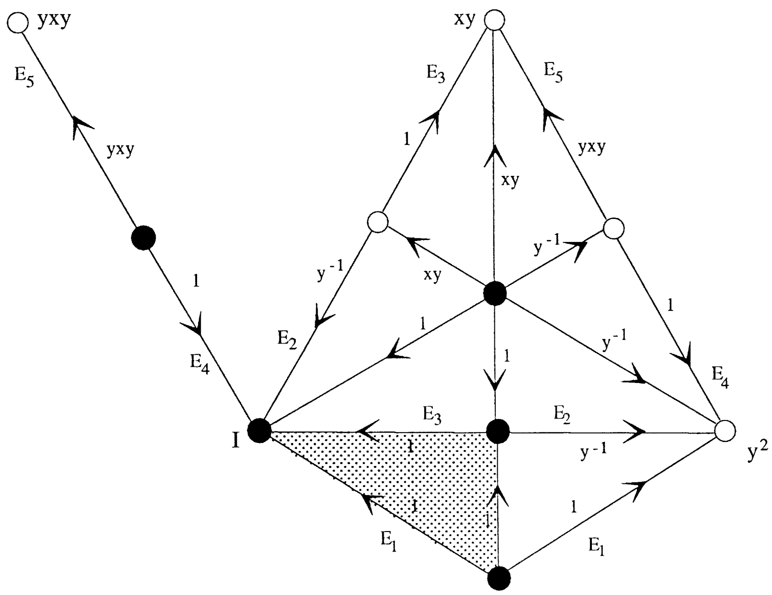


Fig. 9(b)

The only relations we have not yet incorporated are the inclusions of cell groups:

$$\psi_a(g) = a^- g a^+ .$$

These relations are non-trivial only when a cell and a face of the cell both have non-trivial groups assigned. This happens only along E_4 and E_5 (to see this recall that the stabilizer of the identity is $\langle x \rangle$). Figure 10 shows the two edges E_4, E_5 for which we must add relations. There is only one monomorphism $\mathbf{Z}_2 \to \mathbf{Z}_4$ and so ϕ_{E_4}, ϕ_{E_5} are equal. Since $E_4 = 1$ in H the relations from E_4 identify $\mathbf{Z}_2$ as a subgroup of $\mathbf{Z}_4$. The remaining edge $E_5 = yxy$ in H thus gives the relations

$$w = (yxy)^{-1} w (yxy)$$

for every w in $\mathbf{Z}_2$. But this is only non-trivial for $w = x^2$ which then says

$$x^2 = (yxy)^{-1} x^2 (yxy) . \qquad \square$$

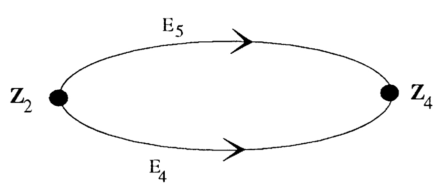


Fig. 10

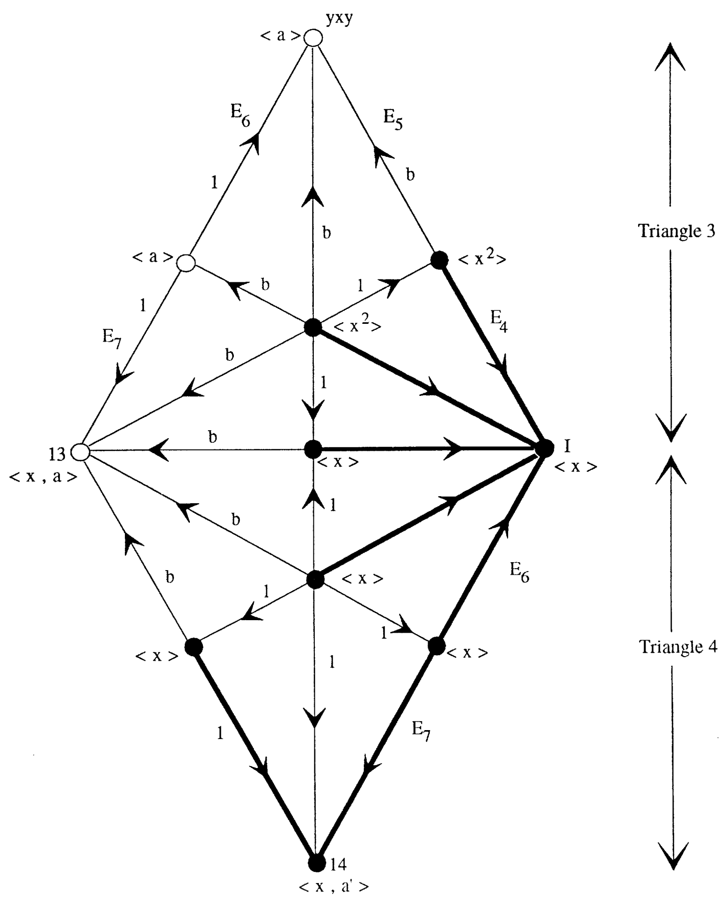


Fig. 11

Next we consider triangles 3 and 4. These form an annulus A in the quotient. The computation of the head elements is left to the reader in this case; as a result, one checks easily that the monodromy elements $g_{a,b}$ in this annulus are all trivial.

The groups assigned to the cells of A are as shown in Fig. 11. In A the edge $(I, 14)$ is identified to the edge $(yxy, 13)$. This figure also depicts the maximal tree for A by heavy edges. Using this choice of tree and the triangle relations, we have labelled the edges with the group elements of H that they represent – this is the edge labelling in Fig. 11.

An edge labelled 1 connecting two vertices creates relations which identify the group assigned to the tail of the edge with the obvious subgroup of the group at the head of the edge. Edges not labelled by 1 are labelled b and it remains to examine the relations which result. Those edges labelled b which start at a vertex carrying the group $\langle x^2 \rangle$, for example E_5, identifies x^2 with bx^2b^{-1}. This leaves a number of edges labelled b starting at the group $\langle x \rangle$ and ending on $\langle x, a' \rangle$. These give relations which identify $b^{-1}xb$ with a.

Thus the underlying group of A has a presentation with the generators and relations for Stab(13) together with an element b and extra relations saying b commutes with x^2 and conjugates x into a. The presentation of Stab(13) is given in Theorem 4.7 when $n = 0$. When the annulus is glued onto triangles 1 and 2, b is identified with yxy; we refer to Figs. 9(b) and 11. Thus we have:

Lemma 4.2 *The underlying group of X is*

$$\langle x, y \,|\, x^4 = y^3 = 1 \;\; [x^2, yxy] = 1 \;\; [x, yxy]^4 = 1 \rangle \,.$$

4.2 The group of the tube

We will compute the group associated to the complex of groups for a single annulus made from a pair of triangles. This will serve the dual purpose of computing the contributions from Link(13) as well as the contributions from the successive annuli which build up the tube.

We refer the reader to Fig. 12(a) which shows the barycentric subdivision of the first two triangles in the sequence embedded in the building Δ; these lie in Link(13). The distinguished choices of edges and vertices are indicated as usual by the heavy lines. Each edge is oriented according to the conventions described above and labelled with its head element h_a. One then sees easily:

Lemma 4.3 *For every composable pair of edges a and b, we have* $g_{a,b} = 1$.

This means that the presentation for the group in this case is especially simple as the relations of Type (d) (see Sect. 5.1.4) are just those of the usual edge group of algebraic topology. We choose a maximal tree, consistent with

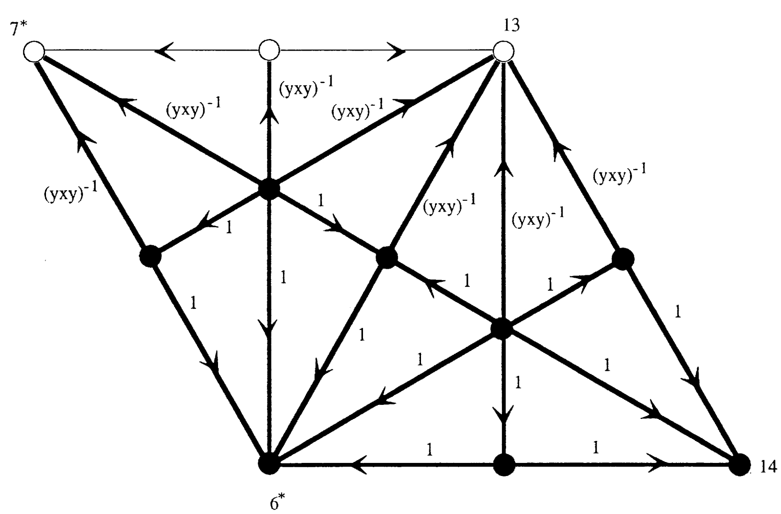


Fig. 12(a)

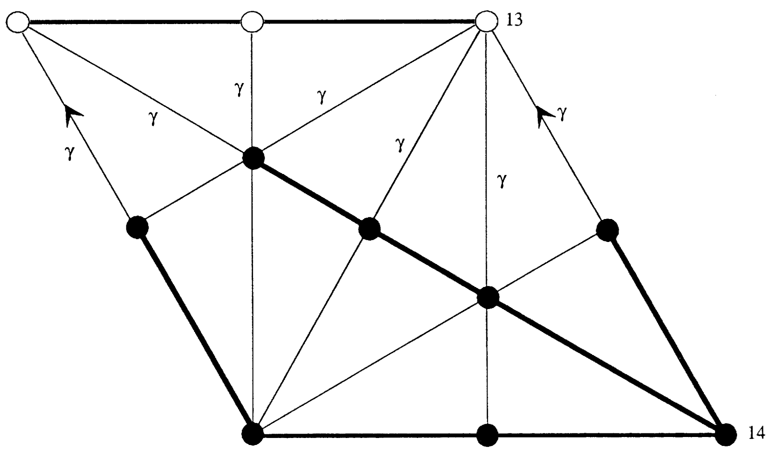


Fig. 12(b)

and extending the one in Fig. 11, indicated by bold lines as in Fig. 12(b) and the labelling of this figure is by the resulting group elements, where unlabelled edges correspond to the identity group element. We see that we obtain one generator from the edges and all other groups come from vertex, edge and 2-simplex stabilisers of Δ. Almost all the relations of Sect. 5.1.4 Type (c) now say that the maps ψ_a are inclusions. Some do not; for example, the inclusion of the stabiliser of the edge $a = (13, 14)$ into stab(14) yields that for $g \in$ stab((13, 14)) that $\gamma^- g \gamma^+ = \psi(g) = (yxy)^{-1} g(yxy)$. However the identification of the edge (13, 14) in Figs. 11 and 12(b) show that $\gamma = b = yxy$. Thus we have shown:

Lemma 4.4 *If A denotes the annular graph of groups shown in Fig.* 12(a), *then*

$$\pi_1(A_1) \cong \langle yxy, \mathrm{stab}(7^*)\rangle$$

The picture now repeats itself exactly with the groups stab($7^{(n)}$) as we add successive annuli. In every case these stabilisers are subgroups of the group generated by the elements x and yxy, so that we have shown:

Theorem 4.5

$$\pi_1(A_\infty) \cong \langle x, yxy\rangle$$

4.3 A presentation for im(ρ)

We now compute explicitly all the data necessary to find a presentation for the fundamental group of the complex of groups associated to the action of im(ρ) on Δ. To this end we need:

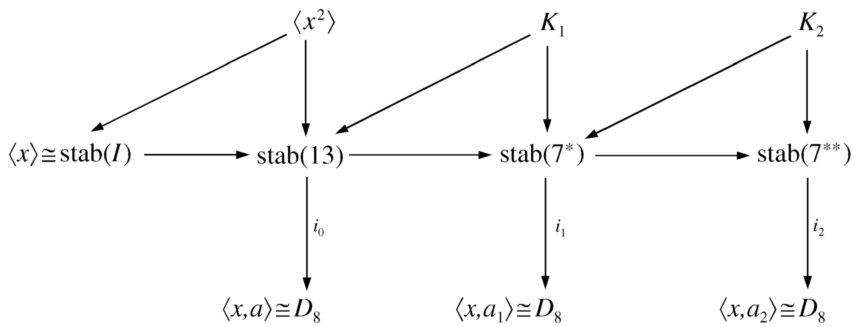
Theorem 4.6 *The groups* stab($7^{(n)}$) *are all finite groups of order* $16 \cdot 4^n$ *which are generated by* $x, a_0, \ldots, a_n$.

Proof. We have a map $i_0 : \mathrm{stab}(13) \to D_8 \leqq \mathrm{Aut}(\mathrm{Link}((13)))$. Anything in the kernel of this map must in particular stabilise I, so that by Lemma 3.3, $\ker(i_0) \leqq \langle x\rangle$, whence $\ker(i_0) = \langle x^2\rangle$. Thus there is an exact sequence

$$1 \to \langle x^2\rangle \to \mathrm{stab}(13) \to D_8 \to 1$$

where the dihedral group is generated by x and a. It follows that stab(13) has order 16 and is generated by x and a.

This is the basis for an inductive argument: If we consider $i_n : \mathrm{stab}(7^{(n)}) \to D_8$, we will again have $\ker(i_n) \leqq \mathrm{stab}(7^{(n-1)})$ so that $\mathrm{stab}(7^{(n)})$ is described as an extension $K_n \to \mathrm{stab}(7^{(n)}) \to D_8$. In sum we have a diagram:



By Lemma 3.11, the elements $a_0, \dots, a_{n-1}$ all lie in $\mathrm{stab}(7^{(n)})$, so our inductive hypothesis shows that $\mathrm{stab}(7^{(n)})$ is generated by $x, a, a_1, \dots, a_n$.

Referring back to Figs. 2 and 3, we recall that x acts as a transposition on the vertices $8^{(n-1)}$ and $11^{(n-1)}$, while the element a_{n-1} moves them not at all, so that K_n is the kernel of the map $\mathrm{stab}(7^{(n-1)}) \to \mathbf{Z}_2$ given by mapping every a_i to 0 and x to 1. Thus $\mathrm{stab}(7^{(n)})$ has order $8 \cdot 16 \cdot 4^{n-1}/2 = 16 \cdot 4^n$ as required. □

Theorem 4.7 *The group* $\mathrm{stab}(7^{(n)})$ *has presentation*:
Generators: $x, a_0, a_1, \dots, a_n$
Relations:
1. $x^4 = 1$
2. $x^2 = a_0^2 = \cdots = a_n^2$
3. $(xa_i)^4 = 1$ *for all* $i \geqq 0$
4. $[a_i, a_j] = [a_{i+k}, a_{j+k}]$ *for all* i, j, k
5. $[x, a_i] = [a_{k-1}, a_{i+k}]$ *for all* i
6. *The group is nilpotent of class* 3.

Proof. Each of the groups $\mathrm{stab}(7^{(n)})$ is a subgroup of $\langle x, yxy\rangle$. Setting $b = yxy$, one discovers after defining a change of basis matrix by:

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

a calculation reveals that:

$$P^{-1}\beta_4 \otimes \mathbf{Z}_2(b)P = \begin{pmatrix} t & 1+t & t \\ 0 & 1 & t \\ 0 & 0 & t \end{pmatrix} \quad P^{-1}\beta_4 \otimes \mathbf{Z}_2(x)P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

This observation makes it easy to check that the group stab($7^{(n)}$) satisfies all the stated relations and moreover that the commutator subgroup is the direct sum of $n+1$ copies of $\mathbf{Z}_2$. Define an abstract group $T(n)$ with generators and relations as in the statement, so that we have a surjective homomorphism $\pi : T(n) \to$ stab($7^{(n)}$). Because relation 6 implies that $T(n)$ is nilpotent of class 3, whence relations 4 and 5 taken with Theorem 5.4 of [6] implies that the commutator subgroup of $T(n)$ is generated by commutators $[x, a_j]$. Further, relation 2 gives $[x, a_j] = (xa_j)^2$, so that by relation 3 each of these commutators has order two. It follows that the commutator subgroup of $T(n)$ can be no larger than the direct product of $n+1$ copies of $\mathbf{Z}_2$. Restriction of the map π to commutator subgroups implies that the commutator subgroup of $T(n)$ is actually isomorphic to $n+1$ copies of $\mathbf{Z}_2$.

Now computing the abelianisation of $T(n)$, we see that it is a direct sum of groups generated by $x, x^{-1}a_0, \ldots, x^{-1}a_n$, where x has order 4 and each of the remaining generators has order 2. Thus the abelianisation is a group of order $4 \cdot 2^{n+1}$. It follows that $T(n)$ is a group of order $2^{n+1} \cdot 4 \cdot 2^{n+1} = 16 \cdot 4^n$ which is the order of stab($7^{(n)}$). Whence π is an isomorphism, completing the proof. □

Theorem 4.8 *The group* im(ρ) *is presented as*:
Generators: x, y
Relations:
1. $x^4 = 1$
2. $y^3 = 1$
3. $[x^2, yxy] = 1$
4. $[x, (yxy)^i]^4 = 1$ *for all* $i \geqq 0$
5. *The group generated by* $\langle (yxy)^i x (yxy)^{-i} | i \geqq 0 \rangle$ *is nilpotent of class* 3

Proof. The calculations above make it clear that all relations must lie in the group generated by x and yxy. Since yxy acts on the building as an element of infinite order, it follows that if one writes the relation in this group in terms of these two elements, it must have zero exponent in yxy. The reason is that any element u of exponent zero can be written as a product of conjugates of the form $(yxy)^r x (yxy)^{-r}$. It follows that u lies in stab($7^{(n)}$), for some sufficiently large n and in particular it fixes a vertex.

Now yxy fixes no vertex, so that writing a purported relation as some power of the element yxy multiplying an element of exponent zero shows that the yxy exponent is forced to be zero.

Thus any relation can be written as a word in the group $\langle (yxy)^i x (yxy)^{-i} | i \geqq 0 \rangle$ and hence lies in stab($7^{(n)}$) for some n. Now one sees that the relations of Theorem 4.7 are exactly the relations above restricted to this subgroup. □

Corollary 4.9 *The image group $\beta_4 \otimes \mathbf{Z}_2(B_4)$ is presented as*:
Generators: x, y
Relations:
1. $x^4 = z$
2. $y^3 = z$
3. $[x^2, yxy] = 1$
4. $[x, (yxy)^i]^4 = 1$ *for all* $i \geqq 0$
5. *The group generated by* $\langle (yxy)^i x (yxy)^{-i} | i \geqq 0 \rangle$ *is nilpotent of class* 3

Here z denotes a generator of the centre of B_4.

Remarks. An interesting feature of this presentation is that all the extraneous relations are contained in vertex stabilisers.

It is also interesting to observe that the use of the form J bypasses the need for calculations involving the matrices x and y; all the proof really uses is the fact that these matrices are isometries of the form J. To underline this we observe that our methods show:

Theorem 4.10 *The subgroup of $GL_3(\mathbf{Z}_2[t, t^{-1}])$ consisting of isometries of the form J is precisely the image of $\beta_4 \otimes \mathbf{Z}_2$.*

Proof. In the key Lemma 3.2, we only made use of the fact that the J isometry had Laurent polynomial coefficients to prove that all its entries were constants. The paragraph which followed this then used this fact to deduce that only four possibilities arose by checking the twenty four elements in the image of the symmetric group after specialising $t = 1$. However one easily checks that in fact one does not obtain any new elements in the entire group $GL_3(\mathbf{Z}_2)$. To sum up, we have shown:

Lemma 4.11 *The only matrices in* $\mathrm{Isom}(J) \leqq GL_3(\mathbf{Z}_2[t, t^{-1}])$ *which stabilise the identity vertex are powers of the element $\rho(x)$.*

We can now follow the computation for the vertex stabilisers through from the beginning noting that all that was ever used was the precise description of $\mathrm{stab}(I)$ and the fact we were dealing with $\mathrm{Isom}(J)$. In particular, all vertex stabilisers have the same size in $\mathrm{Isom}(J)$ as they do in $\mathrm{im}(\rho)$ and these sizes are all distinct so that $\mathrm{Isom}(J)$ does not cause any more vertex identifications, hence $\Delta/\mathrm{im}(\rho) \cong \Delta/\mathrm{Isom}(J)$ and since $\mathrm{im}(\rho) \leqq \mathrm{Isom}(J)$ the groups coincide. □

5 Appendix

5.1 Complexes of groups

We recall some of the theory of complexes of groups as developed by Haefliger.

5.1.1 Simplicial cell complexes. Let X be a CW complex and σ an n-cell of X, then an *ordering* of σ is a continuous map of the standard n-simplex Δ^n

onto σ which is a homeomorphism of the interior of Δ onto the interior of σ. Two orderings of σ are *consistent* if they differ by an isometry of Δ. A *simplicial cell complex* is a CW complex X together with a maximal set of $(n+1)!$ consistent orderings on each n-cell of X such that the restriction of an ordering of an n-cell σ to a face τ of σ is an ordering of τ.

A *simplicial map* between simplicial cell complexes is a continuous map f whose restriction to each n-cell σ is a homeomorphism onto an n-cell τ and such that for each ordering ϕ of σ the composition $f \circ \phi$ is an ordering of τ.

An *inversion* is a simplicial map f which maps at least one n-cell σ to itself with the property that the restriction $f|\sigma$ is not the identity. We will consider a group G acting by simplicial maps on a simplicial cell complex X having the property that no element of G is an inversion. In this case we say that *G acts without inversions.*

The *Barycentric subdivision* of a simplicial cell complex X is a simplicial cell complex X^1. It is formed by taking the images under the orderings of the barycentric subdivisions of the simplexes Δ^n. Since orderings are consistent, there is a well defined barycenter in each cell σ of X. Thus the images under the orderings of the simplices of the barycentric subdivision of the standard simplexes Δ^n provides the structure of X^1. The vertices of X^1 correspond to the cells of X. An edge a of the barycentric subdivision of X^1 corresponds to a face $\tau < \sigma$ of a cell σ of X and is to be thought of as a directed edge in σ between the barycenters of τ and σ directed from the larger cell σ to the smaller face τ. The *initial vertex* of a is $i(a) = \sigma$ and the *terminal vertex* of e is $t(a) = \tau$. The edge with this orientation is denoted a^+ and with the opposite orientation by a^-.

Two edges a, b in X^1 are *composable* if there is some cell σ of X they both lie in and if $t(a) = i(b)$. In this case there is an edge $c = ab$ in σ with $i(c) = i(a)$ and $t(c) = t(b)$ and there is a 2-simplex in X^1 containing the 3 edges a, b, c.

5.1.2 Complexes of Groups. A Complex of Groups is the data $(X, G_\sigma, \psi_a, g_{a,b})$ where X is a simplicial cell complex and for each cell σ of X there is a group G_σ such that

- for each edge a of X^1 there is a monomorphism

$$\psi_a : G_{i(a)} \to G_{t(a)}$$

- If a, b are composable edges of X^1 there is a given *monodromy* element $g_{a,b} \in G_{t(b)}$ such that

$$g_{a,b}\psi_{ab}(g_{a,b})^{-1} = \psi_a\psi_b$$

- the *cocycle condition* (which is vacuous in the case of interest that $\dim(X) < 3$) whenever a, b, c are composable edges then

$$\psi_a(g_{b,c})g_{a,bc} = g_{a,b}g_{ab,c} .$$

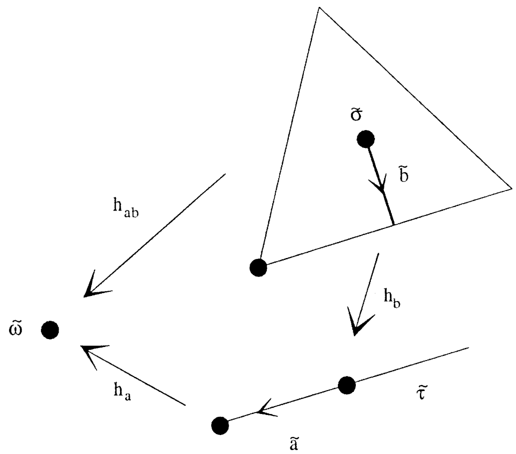


Fig. 13(a)

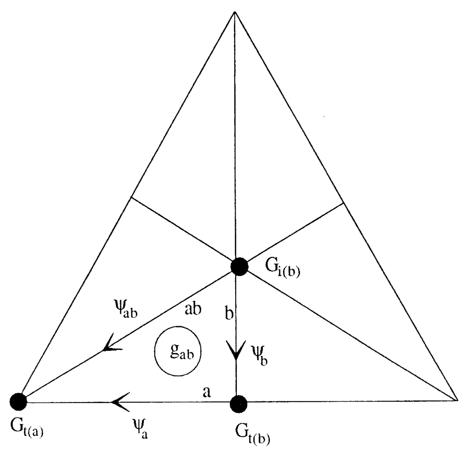


Fig. 13(b)

For a 2-dimensional complex of groups, the data needed is indicated in the Fig. 13(b). This shows the barycentric subdivision of a single triangle. There is a group assigned to each vertex and edge of the triangle, and one for the triangle. There are inclusions of the triangle group into each edge group and into each vertex group. There are inclusions of the edge groups into adjacent vertex groups. Finally there are 6 monodromy elements $g_{a,b}$ corresponding to the 6 choices of starting at the barycenter of the triangle and going via the barycenter of an edge to a vertex adjacent to that edge.

5.1.3 The Group complex associated to a group action. Suppose that a group G acts simplicially without inversions on a simplicial cell complex $\tilde{X}$ then there is an associated complex of groups which is unique up to isomorphism.

This is constructed as follows. First the quotient space $X = \tilde{X}/G$ inherits a simplicial cell structure from X due to the hypothesis of no inversions. Let

$$p : \tilde{X} \to X$$

be the quotient map. Next for each cell σ of X choose a cell $\tilde{\sigma}$ lying over σ and define

$$G_\sigma = \mathrm{Stab}(\tilde{\sigma}) .$$

For each edge a of X^1 with $i(a) = \sigma$ and $t(a) = \tau$ let $\tilde{\sigma}$ and $\tilde{\tau}$ be the choices made above. Now it is likely that $\tilde{\sigma}$ does not contain $\tilde{\tau}$ therefore we choose a different $\tilde{\sigma}_1$ lying over σ and containing $\tilde{\tau}$ and choose $g \in G$ an element such that $g\tilde{\sigma} = \tilde{\sigma}_1$. Now define

$$\psi_a = \mathrm{incl} \circ c_g : G_{i(a)} \to G_{t(a)}$$

where $c_g(x) = g^{-1}xg$ is conjugation.

Finally the monodromy elements $g_{a,b}$ arise as follows. If a, b are composable edges with $i(a) = \sigma$, and $t(a) = i(b) = \tau$ and $t(\tau) = \omega$ let $\tilde{\sigma}, \tilde{\tau}, \tilde{\omega}$ be the choices made above of simplices in $\tilde{X}$ over the simplices σ, τ, ω in X. In defining $\psi_a, \psi_b, \psi_{ab}$ we chose cells $\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\tau}_1$ in $\tilde{X}$ and *head* elements $h_a, h_b, h_{ab} \in G$ with $\tau < h_a\tilde{\sigma} = \tilde{\sigma}_1$ and $\omega < h_{ab}\tilde{\sigma} = \tilde{\sigma}_2$ and $\omega < \tilde{\tau}_1$. Then define the *monodromy* elements by:

$$g_{a,b} = h_a h_b h_{ab}^{-1} .$$

See Fig. 13(a).

5.1.4 The Underlying Group of a Complex of Groups. Given a complex of groups $(X, G_\sigma, \psi_a, g_{a,b})$ we define FG to be the group:
Generators
(a) elements of G_σ for the cells σ of X.
(b) directed edges of X^1.

Relations
(a) relations of G_σ
(b) when a^+, a^- are opposite orientations on the same edge

$$a^+ = (a^-)^{-1} .$$

(c) For $g \in G_{i(a)}$ then

$$\psi_a(g) = a^- g a^+$$

(d) If a, b are composable then

$$(ab)^+ = b^+ a^+ g_{a,b}$$

Given two vertices σ, τ of X^1 a $G(X)$ path from σ to τ is a sequence g_0, $e_1, g_1, \ldots, e_n, g_n$ where $e_1, e_2, \ldots, e_n$ is an edge path in X^1 (thus $t(e_k) = i(e_{k+1})$)

with $i(e_1) = \sigma$ and $t(e_n) = \tau$, and where g_k is an element of $G_{t(e_k)}$. The *Underlying group* based at x of the complex of groups is the set of elements in FG representable as $G(X)$ paths starting and ending at the basepoint, a vertex σ of X^1. We need the following basic fact from [3], p. 516, Theorem (4.1):

Theorem 5.1 *Suppose that a group G acts simplicially without inversions on a simplicial complex X. Let $(X, G_\sigma, \psi_a, g_{a,b})$ be the complex of groups constructed from this data, and let H be the underlying group of this complex. Then there is a natural isomorphism $H \cong G$.*

It is also shown by Haefliger that one may obtain a presentation of the underlying group by collapsing a maximal tree and then considering all edge paths. This is the presentation we use, the extra relations are:

(e) Choose a maximal tree T in the barycentric subdivision $X^{(1)}$ and for each edge $a \in T$ set

$$a = 1\,.$$

5.2 Presentation of B_4/Z

Theorem 5.2 *The* 4-*string braid group admits a presentation*

$$\langle x, y \,|\, x^4 = y^3 \ \ [x^2, yxy] = 1\rangle$$

where $x = \sigma_1\sigma_2\sigma_3$ and $y = \sigma_1\sigma_2\sigma_3\sigma_1$. Then $\sigma_1 = x^{-1}y$, $\sigma_2 = yx^{-1}$, $\sigma_3 = xy^{-2}x^2$. Furthermore the center is generated by x^4.

Proof. The 4-string braid group has a presentation:

$$B_4 = \langle \sigma_1, \sigma_2, \sigma_3 \,|\, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \quad \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3 \quad \sigma_1\sigma_3 = \sigma_3\sigma_1\rangle$$

We obtain another presentation using as generators $x = \sigma_1\sigma_2\sigma_3$, $y = \sigma_1\sigma_2\sigma_3\sigma_1$. To see these are generators we express $\sigma_1, \sigma_2, \sigma_3$ in terms of them.

$$\begin{aligned} y &= \sigma_1\sigma_2\sigma_3\sigma_1 \\ &= \sigma_1\sigma_2\sigma_1\sigma_3 \\ &= \sigma_2\sigma_1\sigma_2\sigma_3 \\ &= \sigma_2 x \end{aligned}$$

Thus

$$\sigma_1 = x^{-1}y \qquad \sigma_2 = yx^{-1} \qquad \sigma_3 = (\sigma_1\sigma_2)^{-1}x = xy^{-2}x^2\,.$$

The center of B_4 is infinite cyclic generated by x^4 and we compute that $x^4 = y^3$ as follows:

$$\begin{aligned}
y^3 &= (\sigma_1\sigma_2\sigma_3\sigma_1)(\sigma_1\sigma_2\sigma_3\sigma_1)(\sigma_1\sigma_2\sigma_3\sigma_1)\\
&= \sigma_1\sigma_2(\sigma_3\sigma_1)\sigma_1\sigma_2(\sigma_3\sigma_1)\sigma_1\sigma_2\sigma_3\sigma_1\\
&= \sigma_1\sigma_2(\sigma_1\sigma_3)\sigma_1\sigma_2(\sigma_1\sigma_3)\sigma_1\sigma_2\sigma_3\sigma_1\\
&= (\sigma_1\sigma_2\sigma_1)\sigma_3(\sigma_1\sigma_2\sigma_1)\sigma_3\sigma_1\sigma_2\sigma_3\sigma_1\\
&= (\sigma_2\sigma_1\sigma_2)\sigma_3(\sigma_2\sigma_1\sigma_2)\sigma_3\sigma_1\sigma_2\sigma_3\sigma_1\\
&= \sigma_2\sigma_1(\sigma_2\sigma_3\sigma_2)\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3\sigma_1\\
&= \sigma_2\sigma_1(\sigma_3\sigma_2\sigma_3)\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3\sigma_1\\
&= \sigma_2(\sigma_1\sigma_3)\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3\sigma_1\\
&= \sigma_2(\sigma_3\sigma_1)\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3\sigma_1\\
&= \sigma_2\sigma_3(\sigma_1\sigma_2\sigma_3)(\sigma_1\sigma_2\sigma_3)(\sigma_1\sigma_2\sigma_3)\sigma_1\\
&= \sigma_2\sigma_3(\sigma_1\sigma_2\sigma_3)^4(\sigma_2\sigma_3)^{-1}\\
&= \sigma_2\sigma_3x^4(\sigma_2\sigma_3)^{-1}\\
&= x^4\,.
\end{aligned}$$

Next we compute the relations between x and y. The relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ gives

$$\begin{aligned}
(x^{-1}y)(yx^{-1})(x^{-1}y) &= (yx^{-1}(x^{-1}y)(yx^{-1})\\
x^{-1}(yy)x^{-2}y &= yx^{-2}(yy)x^{-1}\\
x^{-1}(x^4y^{-1})x^{-2}y &= yx^{-2}(x^4y^{-1})x^{-1}\\
x^3y^{-1}x^{-2}y &= yx^2y^{-1}x^{-1}\\
(x^3y^{-1}x^{-2})y &= yx^2(y^{-1}x^{-1})\\
y(xy) &= (x^2yx^{-3})yx^2\\
(x^4)yxy &= (x^4)x^2yx^{-3}yx^2\\
x^4yxy &= x^2y(x^4)x^{-3}yx^2\\
x^4yxy &= x^2y(x)yx^2\\
x^2(yxy) &= (yxy)x^2
\end{aligned}$$

Next we compute the relation $\sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3$ which gives

$$\begin{aligned}
(yx^{-1})(xy^{-2}x^2)(yx^{-1}) &= (xy^{-2}x^2)(yx^{-1})(xy^{-2}x^2)\\
y^{-1}x^2yx^{-1} &= xy^{-2}x^2y^{-1}x^2\\
(y^{-1})x^2yx^{-1} &= xy^{-2}x^2(y^{-1}x^2)\\
x^2yx^{-1}x^{-2}y &= yxy^{-2}x^2\\
(x^4)x^2yx^{-3}y &= (x^4)yxy^{-2}x^2\\
x^2yxy &= yx(y^{-2}x^4)x^2\\
x^2yxy &= yx(y)x^2\\
x^2(yxy) &= (yxy)x^2
\end{aligned}$$

Finally we compute $\sigma_1\sigma_3 = \sigma_3\sigma_1$ which gives:

$$\begin{aligned}
(x^{-1}y)(xy^{-2}x^2) &= (xy^{-2}x^2)(x^{-1}y)\\
x^{-1}yxy^{-2}x^2 &= xy^{-2}xy\\
x^{-1}yx(y^{-2})x^2 &= x(y^{-2})xy\\
x^{-1}yx(x^{-4}y)x^2 &= x(x^{-4}y)xy\\
x^{-1}yx(y)x^2 &= x(y)xy\\
yxyx^2 &= x^2yxy\\
(yxy)x^2 &= x^2(yxy)
\end{aligned}$$

□

Whence we obtain:

Corollary 5.3 *Denoting the centre by Z, a presentation of B_4/Z is given by*:

$$\langle x, y \,|\, x^4 = y^3 = 1 \;\; [x^2, yxy] = 1\rangle$$

where $x = \sigma_1\sigma_2\sigma_3$ *and* $y = \sigma_1\sigma_2\sigma_3\sigma_1$.

5.3 Lattice representatives

For the convenience of the reader who wishes to duplicate some of the calculations, we indicate the simple representatives of the fourteen vertices in Link(I) which we have used. The first eight points are the two orbits on which x acts as four cycle:

$$M_1 = [y] = \begin{pmatrix} 0 & 0 & 1\\ t & 1 & 1\\ 0 & 1 & 1\end{pmatrix} \qquad M_2 = [y^2] = \begin{pmatrix} 0 & 1/t & 0\\ 1 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}$$

$$M_3 = [xy] = \begin{pmatrix} 0 & 1 & 1\\ 0 & 1 & 0\\ t & 0 & 0\end{pmatrix} \qquad M_4 = [xy^2] = \begin{pmatrix} 1 & 0 & 0\\ 1 & 1/t & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$M_5 = [x^2y] = \begin{pmatrix} t & 0 & 0\\ t & 1 & 1\\ t & 1 & 0\end{pmatrix} \qquad M_6 = [x^2y^2] = \begin{pmatrix} 0 & 0 & 1\\ 1 & 0 & 0\\ 1 & 1/t & 0\end{pmatrix}$$

$$M_7 = [x^3y] = \begin{pmatrix} t & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \qquad M_8 = [x^3y^2] = \begin{pmatrix} 1 & 1/t & 1\\ 1 & 1/t & 0\\ 0 & 1/t & 0\end{pmatrix}$$

There are then two orbits on which x acts as a transposition:

$$M_9 = [yxy] = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1/t & 1 \\ 1 & 1/t & 0 \end{pmatrix} \qquad M_{11} = [xyxy] = \begin{pmatrix} 1 & 1/t & 0 \\ 0 & 1/t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_{10} = [(yxy)^{-1}] = \begin{pmatrix} 0 & 0 & 1 \\ t & 0 & 0 \\ t & 1 & 0 \end{pmatrix} \qquad M_{12} = [x(yxy)^{-1}] = \begin{pmatrix} t & 1 & 0 \\ t & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and finally the two lattices which are not group points; these are fixed by x:

$$M_{13} = [13] = \begin{pmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ t & 0 & 1 \end{pmatrix} \qquad M_{14} = [14] = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1/t \end{pmatrix}$$

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