Pseudomodular surfaces

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1 Introduction.

A Fuchsian group is a discrete subgroup of $\mathrm{PSL}(2,\mathbf{R})$. As such it acts discontinuously on $\mathbf{H}^2$ (the upper half plane model of the hyperbolic plane) by fractional linear transformations. This action induces an action on the real line. It is well known that if an isometry of $\mathbf{H}^2$ fixes a point of the real line then the point is one of a pair, in the case that the isometry is hyperbolic or the isometry in question is parabolic and the point in question is unique. Points fixed by parabolic elements of a Fuchsian group Γ shall be referred to as the *cusps* of Γ. If $\Gamma < \mathrm{PSL}(2,k)$ and k is the smallest such field, then consideration of the equation which must be satisfied by a fixed point shows that a cusp must always lie inside $k \cup \{\infty\}$. A classical case where the cusp set is completely understood is the case when $\Gamma = \mathrm{PSL}(2,\mathbf{Z})$, and the cusp set coincides with $\mathbf{Q} \cup \{\infty\}$. More generally determining the cusp set has been hard to do, with only some moderate success—there is a large literature on this type of problem, see for example [10], [11], [15] and [16] to name a few.

Recall that Fuchsian or Kleinian groups Γ_1 and Γ_2 are *commensurable* if Γ_1 has a subgroup of finite index which is conjugate to a subgroup of finite index in Γ_2. This paper is motivated by the following question:

Question: Let $\mathbf{F}$ denote either $\mathbf{R}$ or $\mathbf{C}$. If Γ_1 and Γ_2 are finite covolume subgroups of $\mathrm{PSL}(2,\mathbf{F})$ with the same cusp set, are they commensurable?

Notice that in dimension 3 if $M = \mathbf{H}^3/\Gamma$ is a fibered manifold with a single cusp, with fiber group F, then using the fact that F is normal in Γ, it is easy to see that F and Γ have the same cusp set. Thus in this case finite volume cannot be dropped. In dimension 2, it is sufficient that one has finite co-volume and the second is finitely generated (see below).

Our main result is the somewhat surprising:

Theorem 1.1 *There is a finite coarea discrete group* $\Gamma \leq \mathrm{PSL}(2,\mathbf{Q})$ *not commensurable with the modular group whose cusp set is precisely* $\mathbf{Q} \cup \{\infty\}$.

In fact we will exhibit four mutually noncommensurable such groups; we do not know whether there are infinitely many such commensurability classes.

Note that the condition on the cusp set of Γ implies that its limit set will coincide with that of $\mathrm{PSL}(2,\mathbf{Z})$, namely the whole of $\mathbf{R} \cup \{\infty\}$ and hence one needs only stipulate that such a group is finitely generated to ensure that $\mathbf{H}^2/\Gamma$ has finite area.

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For our purposes, it is slightly more convenient to work with a subgroup $\Delta \leq \mathrm{PGL}(2, \mathbf{Q})$ for which $\mathbf{H}^2/\Delta$ is topologically a torus with a single cusp; we will have an identification of Γ with the subgroup of index four defined by the canonical mapping $ker\{\pi_1(\mathbf{H}^2/\Delta) \to H_1(\mathbf{H}^2/\Delta; \mathbf{Z}_2)\}$.

A complete hyperbolic surface $\mathbf{H}^2/\Delta$ arising from a discrete group $\Delta \leq \mathrm{PGL}(2, \mathbf{Q})$ which is not commensurable with $\mathrm{PSL}(2, \mathbf{Z})$ and has cusp set precisely $\mathbf{Q} \cup \{\infty\}$ we shall call a **pseudomodular surface** and Δ a **pseudomodular group**.

The existence of pseudomodular surfaces has various interesting corollaries. For example they provide a kind of nonstandard Euclidean algorithm for the rational numbers as well as a pseudo-Farey tessellation of the hyperbolic plane. We shall show elsewhere that they can be used to construct new generalizations of Dedekind sums.

The paper is organized as follows. In §2 we introduce a family of groups $\Delta(u^2, 2\tau)$ each of which defines a finite area complete torus for $0 < u^2 < \tau - 1$. Roughly speaking, u^2 is a degree of freedom coming from the placement of a single cusp and 2τ measures the translation length of the parabolic element stabilising infinity. If u^2 and τ are chosen to be rational, then the cusp set of $\Delta(u^2, 2\tau)$ will be contained in the rational numbers. We note in passing that it follows from results in [7] (See Chapter 9, Theorem 1.4.2) that the groups $\Delta(u^2, 2\tau)$ are all quasiconformal deformations of the punctured torus group inside the modular group.

In this paper we focus on what appear to be the two simplest cases, namely $2\tau = 4, 6$.

Our normalization is such that $\Delta(1, 6)$ is the modular torus arising as a subgroup of index 6 in $\mathrm{PSL}(2, \mathbf{Z})$. In fact we are able to analyse completely the values for which the groups $\Delta(u^2, 6)$ and $\Delta(u^2, 4)$ are arithmetic; there are only finitely many, we give the (small) list in Theorems 2.2 and 2.3.

The main theorem is proved in 2.1; there it is shown that for certain values of $(u^2, 2\tau)$, the group $\Delta(u^2, 2\tau)$ is a nonarithmetic group whose cusp set is precisely the rational numbers, as promised by Theorem 1.1. Specifically, we show that

Theorem 1.2 *The groups $\Delta(u^2, 2\tau)$ for $(u^2, 2\tau)$ in the set $\{(5/7, 6), (2/5, 4), (3/7, 4), (3/11, 4)\}$ are all pseudomodular and noncommensurable.*

We examine various aspects of the group $\Delta(5/7, 6)$, including giving an estimate for its Hurwitz constant and proving some comparison theorems for the size of a horoball at the rational p/q coming from the groups $\mathrm{PSL}(2, \mathbf{Z})$ and $\Delta(5/7, 6)$. In §3.3, we describe how pseudomodular groups can be used to give different versions of the Farey tessellation of $\mathbf{H}^2$.

In §4 we use these groups to prove some new results concerning the finitely generated intersection property, giving negative answers to questions 11 and 12 of the problem list [2]. We show:

Theorem 1.3 *Let V be the collection of all rational primes excluding 2 and 5. Then $\mathrm{SL}(2, \mathbf{Z}[S])$ does not have fgip for any finite set of $S \subset V$ of primes inverted.*

The method of proof for 1.1 is to construct a certain covering of the reals by "killer intervals"; these intervals are associated to a certain finite collection of group elements which yield a nonstandard version of the Euclidean algorithm. When such a covering exists the group is pseudomodular.

In fact, general considerations show that for any value of $(u^2, 2\tau)$ the killer intervals form an open dense subset of the reals and a group may fail to be pseudomodular because some rational lies outside this set. However, it seems worth noting that there are several values of u^2 for which there is very strong computer evidence that $\Delta(u^2, 2\tau)$ is pseudomodular but the computation seems to be a degree of difficulty harder than that for the values listed in the above theorem; such groups seem to suggest the possibility that there are groups where the open dense set contains the rational numbers, but fails to to be a covering. In §5 we include some calculations for the groups $\Delta(u^2, 2\tau)$ for small values of u^2 and $2\tau = 4, 6$ which illustrate these various behaviours.

In §6 we compile a list of open questions related to the groups $\Delta(u^2, 2\tau)$. Finally, §7 is an appendix containing the data for a calculation used in 2.1.

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2 The groups $\Delta(u^2, 2\tau)$.

In this section we exhibit a family of groups $\Delta(u^2, 2\tau)$, each of which is $\mathbf{H}^2/\Delta(u^2, 2\tau)$ a complete finite area once punctured torus. It is convenient to work in $\mathrm{GL}(2,\mathbf{R})$. If u^2 and τ are rational, then the group $\Delta(u^2, 2\tau)$ can be taken to lie inside $\mathrm{GL}(2,\mathbf{Q})$. Actually, we prefer to allow nonrational matrices of determinant 1, but each such will be of the form $\sqrt{r}M$ for $M \in \mathrm{GL}(2,\mathbf{Q})$ and $r \in \mathbf{Q}$ so that the passage to the canonical subgroup of index 4 has image in $\mathrm{SL}(2,\mathbf{Q})$.

Consider the fundamental domain depicted in Figure 1.1 below.

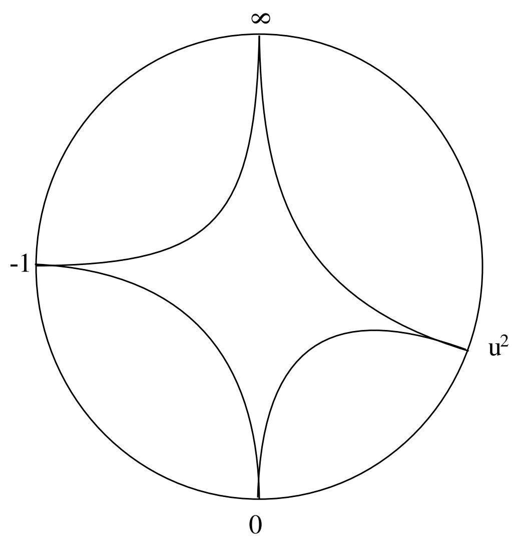


Figure 1.1

Choose isometries g_1 mapping the directed edge $\{-1, 0\}$ to the directed edge $\{\infty, u^2\}$ and g_2 mapping $\{\infty, -1\}$ to $\{u^2, 0\}$. Any such choice will give a discrete group, abstractly isomorphic to a free group of rank two as well as a hyperbolic metric on the quotient $\mathbf{H}^2/\Delta(u^2, 2\tau)$. In general this metric will not be complete; the completeness condition is arranged by making choices so that the commutator is a parabolic element. A convenient choice is

$$g_1.g_2^{-1}.g_1^{-1}.g_2 = \begin{pmatrix} -1 & -2\tau \\ 0 & -1 \end{pmatrix}$$

After a small computation, one finds that

$$g_1 = \begin{pmatrix} (-1+\tau)/\sqrt{-1+\tau-u^2} & u^2/\sqrt{-1+\tau-u^2} \\ \\ 1/\sqrt{-1+\tau-u^2} & 1/\sqrt{-1+\tau-u^2} \end{pmatrix}$$

and

$$g_2 = \begin{pmatrix} u/\sqrt{-1+\tau-u^2} & u/\sqrt{-1+\tau-u^2} \\ \\ 1/(u\sqrt{-1+\tau-u^2}) & (\tau-u^2)/\sqrt{-1+\tau-u^2} \end{pmatrix}$$

In this paper we will explore the cases that the parabolic is translation by 4 and 6; other choices are, of course, possible, but they seem computationally more difficult.

Understanding the behaviour of the cusp sets of these groups for various values of u^2 and τ seems to be a difficult problem. We begin with some trivial observations.

Lemma 2.1 *(i) The group $\Delta(u^2, 6)$ is equivalent to the group $\Delta(2 - u^2, 6)$ after remarking and conjugacy.*
(ii) The group $\Delta(u^2, 4)$ is equivalent to the group $\Delta(1 - u^2, 4)$ after remarking and conjugacy.

Proof. For $\Delta(u^2, 6)$, one checks that the traces of the triple $\{g_1, g_2, g_1.g_2\}$ at the value $2 - u^2$ are the same as those of $\{g_1.g_2^{-1}, g_2^{-1}, g_1.g_2^{-2}\}$ at the value u^2. Standard facts [6] now imply that the image groups are conjugate in $\mathrm{SL}(2, \mathbf{C})$ and hence in $\mathrm{GL}(2, \mathbf{Q})$, since rational points are preserved.

A similar computation (applied to the same triple) also proves the result in the other case. $\square$

This lemma implies that we need never consider u^2 outside the range $0 < u^2 \leq 1$ in the first case and $0 < u^2 \leq 1/2$ in the second.

Notice that it is elementary that for most values of u^2, the groups $\Delta(u^2, 2\tau)$ are not commensurable with $\mathrm{SL}(2, \mathbf{Z})$ since there will be a trace in the canonical kernel which is not a rational integer, for example in the group $\Delta(2/3, 6)$ the trace of $g_1^2 = 19/4$.

The condition that $\Delta(u^2, 2\tau)$ is commensurable with $\mathrm{PSL}(2, \mathbf{Z})$ is equivalent to the statement that $\Delta(u^2, 2\tau)$ is arithmetic; it turns out that we can analyse this situation completely for the groups in which we are interested:

Theorem 2.2 *When $0 < u^2 \leq 1$, the group $\Delta(u^2, 6)$ is arithmetic precisely for*

$$u^2 = 1, 1/5, 1/2$$

Proof. It is well-known [18] that to check that a non-cocompact Fuchsian group Γ of finite co-area is arithmetic it suffices to check that $\mathrm{tr}\gamma^2 \in \mathbf{Z}$ for all $\gamma \in \Gamma$. By [9] Corollary 3.2 it suffices to check $\mathrm{tr}\gamma^2$ on a generating set (assuming the elements are not of order 2). One checks easily that at these values the three traces $tr(g_1.g_1)$, $tr(g_2.g_2)$, $tr(g_1.g_2.g_1.g_2)$ are rational integers so that those groups are arithmetic.

For the converse, we note that if we write $tr(g_2.g_2) = \ell$ this yields the equation

$$0 = u^4(2 + \ell) - (4 + 2\ell)u^2 + 9$$

and considering this as an equation in u^2 it has discriminant $4(\ell + 2)(\ell - 7)$; so that for a rational solution in u^2, we need to find for which rational integers ℓ one has that $(\ell+2)(\ell-7)$ is a nonnegative square.

Clearly, there are solutions $\ell = 7, -2$; henceforth we suppose that $(\ell + 2)(\ell - 7)$ is a nonzero square.

If we further suppose that $(\ell + 2)$ and $(\ell - 7)$ are coprime to each other, then if $(\ell + 2)(\ell - 7)$ is a square, we have

$$\alpha^2 = \pm(\ell + 2)$$

and

$$\beta^2 = \pm(\ell - 7)$$

for (coprime) integers α and β. In the case that both signs are positive, this implies $\alpha^2 = 3^2 + \beta^2$ and Pythagorus' theorem implies that the only integer solutions are $\beta = 0$ and $\beta = 4$ corresponding to $\ell = 7, 23$ respectively. If both signs are negative, we get $\beta^2 = 3^2 + \alpha^2$, so that $\alpha = 0, 4$ and corresponding to $\ell = -2, -18$.

Finally, we suppose that $\alpha = (\ell+2)$ and $\beta = (\ell-7)$ are not coprime to each other. Consideration of $\alpha - \beta$ shows that this prime must be 3.

Writing $(\ell+2) = 3\ell_1$ and $(\ell-7) = 3\ell_2$, the hypothesis gives that $9\ell_1\ell_2$ and hence $\ell_1\ell_2$ is a square. Expressing this in terms of ℓ_2 we see that $\ell_2(\ell_2+3)$ is a square.

In the case that ℓ_2 and (ℓ_2+3) are coprime to each other, the same analysis as above shows that the only solutions are $\ell_2 = 1, -4$ which gives $\ell = 10, -5$. If not, the prime is still 3 and an entirely analogous argument shows that $\ell_2 = 0$ which has been ruled out.

Examination of the solutions for u^2 for these values of ℓ implies the theorem. $\square$

Remark. Professor Masser has pointed out that one can give an argument without appealing to Pythagorus' theorem: One may check that that if $l > 23$ then $(2\ell-6)^2 < 4(\ell+2)(\ell-7) < (2\ell-5)^2$ and so $4(\ell+2)(\ell-7)$ cannot be a square. Similarly for $\ell < -18$, then $(2\ell-4)^2 < 4(\ell+2)(\ell-7) < (2\ell-5)^2$. One examines the intermediate cases directly.

A similar analysis reveals

Theorem 2.3 *When $0 < u^2 \leq 1/2$, the group $\Delta(u^2, 4)$ is arithmetic precisely for*

$$u^2 = 1/5, 1/3, 1/2$$

As observed in the introduction, it is clear that for u^2 and 2τ rational, the cusp set of $\Delta(u^2, 2\tau)$ is a subset of the rationals. However one can show that for certain values, the cusp set must be a *proper* subset of $\mathbf{Q}$. A simple example is given by $u^2 = 2/3$ and $2\tau = 6$ - in this case we have

$$g_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{3}{2\sqrt{2}} & \frac{7}{2\sqrt{2}} \end{pmatrix}$$

and one finds that the fixed point equation for this matrix (see below) is $(2+Q)(-1+3Q) = 0$; that is to say that the rationals -2 and $1/3$ are fixed by g_2. It is a standard fact [4], that in a discrete group, a point fixed by a hyperbolic element of the group cannot also be fixed by a parabolic element of the group, so that neither -2 or $1/3$ can be cusps of $\Delta(2/3, 6)$.

Let Γ be a discrete subgroup of $\mathrm{PSL}(2, \mathbf{C})$, and $\gamma \in \Gamma$ a hyperbolic element, represented by the matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with $c \neq 0$. The fixed-point equation is given by $cz^2 + (d-a)z - b = 0$, and solving gives the two fixed points:

$$\frac{(a-d) \pm \sqrt{\mathrm{tr}^2\gamma - 4}}{2c}.$$

The case when $\sqrt{\mathrm{tr}^2\gamma - 4}$ lies in the trace-field of Γ seems intrinsically interesting in our (and other) contexts, as in the case of the example above: we refer to them as *special hyperbolics* or if the context is clear, just as *specials*. Similarly, rationals fixed by a special hyperbolic will be referred to as special (values).

In §5 we compile some values of u^2 for which $\Delta(u^2, 2\tau)$ is known to contain a special hyperbolic. In these cases one knows that the cusp set of the group is a proper subset of $\mathbf{Q}$.

Remarks. Define $\Delta^{(2)} = ker\{\Delta \to H_1(\Delta; \mathbf{Z}_2)\}$. Since the groups $\Delta^{(2)}(u^2, 2\tau) < \mathrm{PSL}(2, \mathbf{Q})$, if $\gamma \in \Delta(u^2, 2\tau)$ is a special hyperbolic its fixed points will lie in $\mathbf{Q}$. Since the cusp set of $\mathrm{PSL}(2, \mathbf{Z})$ (and hence any group commensurable with $\mathrm{PSL}(2, \mathbf{Z})$) is $\mathbf{Q} \cup \{\infty\}$, arithmetic Fuchsian groups contain no special hyperbolics. Similarly, pseudomodular groups also have no special hyperbolics.

2.1 Pseudomodular $\Delta(u^2, 2\tau)$.

Showing that a given value of $(u^2, 2\tau)$ does give a pseudomodular surface is a good deal more subtle and is accomplished by establishing a reduction procedure which will move any rational to ∞ (in essence a Euclidean algorithm).

Since each $\Delta(u^2, 2\tau)$ comes equipped with a version of the Euclidean algorithm coming from the fundamental domain, it is relatively simple (with the aid of a computer) to find candidate values of u^2 by checking whether all the fractions p/q in the interval $[0, 2\tau]$ (in fact $[0, \tau]$ suffices) with q fairly small can be moved to ∞ by the group $\Delta(u^2, 2\tau)$. The rigourous proof that such candidate examples are in fact pseudomodular is harder. We illustrate the reduction principle in this section by showing:

Theorem 2.4 *The group $\Delta(5/7, 6)$ is pseudomodular.*

Notice that for $\Delta(5/7, 6)$, $\mathrm{tr}(g_2^2) = \frac{117}{15}$ and so $\Delta(5/7, 6)$ cannot be commensurable with $\mathrm{SL}(2, \mathbf{Z})$.

The final step of this theorem involves some computation, but we provide in §7 all the information so that this step can be easily reconstructed.

The principle lying at the heart of our procedure is the following. To economise on notation we shall fix $(u^2, 2\tau) = (5/7, 6)$. We give two illustrative examples.

Example 1. One finds that

$$g_1.g_1.g_2^{-1} = \begin{pmatrix} \frac{47}{3\sqrt{5}} & \frac{-2\sqrt{5}}{3} \\ \frac{28}{3\sqrt{5}} & \frac{-\sqrt{5}}{3} \end{pmatrix}$$

throws $5/28$ to ∞. One computes that the action of this group element on a generic rational p/q is given by

$$g_1.g_1.g_2^{-1}(p/q) = \frac{47p - 10q}{28p - 5q}$$

It follows that the solutions to the inequality $|28p-5q| < |q|$ form an interval about $5/28$ of the form $(5/28-1/28, 5/28+1/28)$. We refer to this interval as the **killer interval** associated to the fraction $5/28$; any rational in this interval will have its denominator strictly decreased by application of the killer word $g_1.g_1.g_2^{-1}$. Notice that the killer interval is determined by the rational number since although there is an ambiguity in the group elements which throw $5/28$ to infinity, this ambiguity is completely accounted for by postmultiplication by an element of $stab(\infty)$ and this upper triangular subgroup does not alter denominators. We note that killer intervals are only defined for those rational number which are cusps.

Example 2. One finds that

$$g_1^{-1}g_2^{-1} = \begin{pmatrix} \frac{7\sqrt{7/5}}{3} & \frac{-4\sqrt{5/7}}{3} \\ \frac{-2\sqrt{35}}{3} & \frac{\sqrt{35}}{3} \end{pmatrix}$$

throws $1/2$ to ∞. In this case one computes:

$$g_1^{-1}g_2^{-1}(p/q) = \frac{-49p + 20q}{35(2p - q)}$$

and in this case the solutions of the inequality $|35(2p-q)| < |q|$ still give rise to a killer interval, but it is the much thinner interval $(\frac{1}{2} - \frac{1}{2\cdot 35}, \frac{1}{2} + \frac{1}{2\cdot 35})$. We refer to the integer 35 as the **contraction constant** for $1/2$.

In general, given a word of the group w throwing α/β to ∞ we obtain a killer interval by solving an inequality of the form $|c(\beta p-\alpha q)|<|q|$; this is the region where the word w is guaranteed to reduce denominators. We denote such a region by $(\alpha/\beta : c)$.

It is this second behaviour which does not occur for the modular group; we sketch a proof below that for the modular group the contraction factor for any rational number is 1.

Our reduction principle is easily summed up:

Theorem 2.5 *Suppose that $\Delta(u^2, 2\tau)$ is such that the interval $[0, 2\tau]$ can be covered by killer intervals.*

Then $\Delta(u^2, 2\tau)$ has cusp set all of $\mathbf{Q}\cup\{\infty\}$.

Proof. Given a rational number p/q it is equivalent by a translation in the group (an operation which does not alter denominators) to a rational number in the interval $[0, 2\tau]$, so we may as well assume $p/q \in [0, 2\tau]$. The existence of a killer interval about p/q implies that there is a $\gamma_1 \in \Delta(u^2, 2\tau)$ such that $\gamma_1(p/q)$ has denominator q' with $q' < q$. By subsequent reductions via killer intervals and elements γ_i we get to denominator 0, and hence a word $W(\gamma_1, \dots \gamma_n) \in \Delta(u^2, 2\tau)$ such that $W(\gamma_1, \dots \gamma_n)(p/q) = \infty$ as required.□

Remark. For any value of $(u^2, 2\tau)$ the union of the killer intervals is always a dense open subset of $[0, 2\tau]$; it is not actually necessary for the union of the killer intervals to cover $[0, 2\tau]$ in order for the group to be pseudomodular, only that the complement of this union contains no rational points.

We comment briefly on the situation when $(u^2, 2\tau) = (1, 6)$ thereby giving the modular torus $\mathbf{H}^2/\Delta(1)$ with $\Delta(1) < \mathrm{PSL}(2, \mathbf{Z})$. Of course it is well-known in this case that the cusp set is $\mathbf{Q}\cup\{\infty\}$.

Using the group $\Delta(1, 6)$, one easily computes that the integers between 0 and 5 all have contraction constant 1 and this set of intervals covers $[0, 6]$—the intervals have length 1 with integer endpoints. This then implies (after the fact) that all the contraction constants are 1.

Indeed the task of covering the interval with killer intervals is simplified because the groups $\Delta(u^2, 2\tau)$ are all hyperelliptic. It will follow from the lemma below, and (as remarked in the introduction), the fact that the cusp set is preserved by passage to a subgroup of finite index, that it suffices to cover the interval $[0, \tau]$.

The following lemma is well known:

Lemma 2.6 *There exists a $\mathbf{Z}_2$-supergroup $\Delta_0(u^2, 2\tau)$ of $\Delta(u^2, 2\tau)$ of signature $(0; 2, 2, 2; 1)$ where the parabolic element is translation by τ.*

Proof. A punctured torus equipped with any complete hyperbolic metric admits a hyperelliptic involution which acts as rotation by π around an axis through the puncture and meeting the surface in three points. The quotient orbifold is easily seen to be a 2-sphere with 3 cone points of order 2, hence the claim about the signature of the supergroup. Now a standard presentation for such a Fuchsian group is given by

$$< x, y, z, \gamma \mid x^2 = y^2 = z^2 = 1, xyz\gamma = 1 >,$$

where γ is a parabolic element. Notice that γ does not live in $\Delta(u^2, 2\tau)$ but its square does, and so under the normalization for $\Delta_0(u^2, 2\tau)$ we deduce that γ acts as translation by τ. □

In particular, the hyperelliptic involution to the torus induces an automorphism of the free group of rank two given by $g_i \to g_i^{-1}$ for $i = 1, 2$. It follows from the above argument that after applying this automorphism, a killer word for the interval $[0,\tau]$ becomes (possibly after translation by 2τ) a killer word for the translated interval in $[\tau, 2\tau]$. As simple examples of this, one checks easily that for $\Delta(5/7, 6)$, $g_2^{-1}.g_1$ throws 0 to ∞ while $g_2.g_1^{-1}$ throws 3 to for ∞.

Setting aside the arithmetic cases, for a given u^2 the contraction constant can be arbitrarily large, and can be large even for quite simple rationals, for example one finds that when $u^2 = 5/7$, the contraction factor for 2/9 is 109375. Given $(u^2, 2\tau)$, it appears to be quite difficult to predict what the contraction constant is for a given p/q, even assuming that $\Delta(u^2, 2\tau)$ is pseudomodular.

The proof of Theorem 2.4 in the case $(u^2, 2\tau) = (5/7, 6)$ is completed by exhibiting a collection of killer intervals covering $[0, 3]$; one such family is provided in the Appendix §7.

A complete list of noncommensurable nonarithmetic groups which have been proved to be pseudomodular in this fashion is contained in:

Theorem 2.7 *The groups* $\Delta(u^2, 2\tau)$ *for* $(u^2, 2\tau)$ *in the set* $\{(5/7, 6), (2/5, 4), (3/7, 4), (3/11, 4)\}$ *are all pseudomodular and noncommensurable.*

A covering by killer intervals in each of these cases is exhibited in §7. That the four groups listed here are noncommensurable can be proved by simple denominator considerations in all but one case; this case can be dealt with by computing the minimal element in the commensurability class and applying results of Margulis.

We note that in addition, there are other choices (listed in §7) for which there is extensive computer evidence that the group is pseudomodular, but an approach based on finding a finite set of killer intervals seems to fail.

2.2 Groups containing special hyperbolics.

An interesting (and apparently exceptional) case occurs for $\Delta(3/4, 4)$, where one can find the collection of killer intervals

$$\{(0:4), (1:4), (2:4), (3/4:1), (5/4:1), (3/8:1), (13/8:1), (1/4:3), (7/4:3)\}$$

We recall the notation $(p/q : c)$ indicates a killer interval centred at p/q and with contraction constant c, that is to say, the interval $(p/q - 1/(c \cdot q), p/q + 1/(c \cdot q))$.

This collection has the property that the *closures* of the intervals cover $[0, 2]$, but the interiors miss out the two points 1/2 and 3/2. This is a reflection of the fact that this group contains specials fixing these points. The covering by killer intervals allows one to prove:

Theorem 2.8 *(i) The group* $\Delta(3/4, 4)$ *contains exactly one conjugacy class of special hyperbolic. (ii) Every rational number is either a cusp of* $\Delta(3/4, 4)$ *or is equivalent to* $1/2$.

We conjecture this kind of phenomenom should happen much more generally.

This example also has the interesting feature:

Theorem 2.9 *The rational* p/q *is a special value for* $\Delta(3/4, 4)$ *if and only if* q *is even and not divisible by* 4.

Sketch Proof. Since the killer intervals cover all of $[0, 2]$, we can reduce the denominator of any rational until it is either 1 or 2 (as in 2.8) and an elementary case-by-case analysis with each generator shows that the property that p/q has q even and not divisible by 4 is preserved by the group. $\square$

3 The group $\Delta(5/7, 6)$.

3.1 The Hurwitz constant for the group $\Delta(5/7, 6)$.

In this section we examine some aspects of the pseudomodular group $\Delta(5/7, 6) = \Delta$, in particular we estimate its Hurwitz constant and compare its horoball pattern with that of the classical modular torus.

To estimate the Hurwitz constant we use the ideas of Vulakh, see [19]. We recall that the Hurwitz constant in our setting is defined as follows. The Hurwitz constant h is the largest constant so that for every irrational number α there are infinitely many group elements $\gamma \in \Delta$ with

$$|\alpha - \gamma(\infty)| < h^{-1}c(\gamma)^{-2}$$

In [19] it is shown that if F denotes a Ford domain for Δ, then one can estimate h. For convenience, we recall some of [19] in the context of F. Using the symmetry described in §2.1, the Ford domain F is symmetric about the line $x = 3$, and so to give the estimates it suffices to consider only the part of the Ford domain lying over the interval $[0, 3]$. This is depicted in Figure 3.1. We claim that the four isometric circles centred at $\{0, 5/7, 2, 3\}$ of radii respectively $\sqrt{\frac{5}{7}}, \frac{3\sqrt{5}}{7}, \frac{3}{\sqrt{7}}, \sqrt{\frac{5}{7}}$ define this part of the Ford domain of Δ.

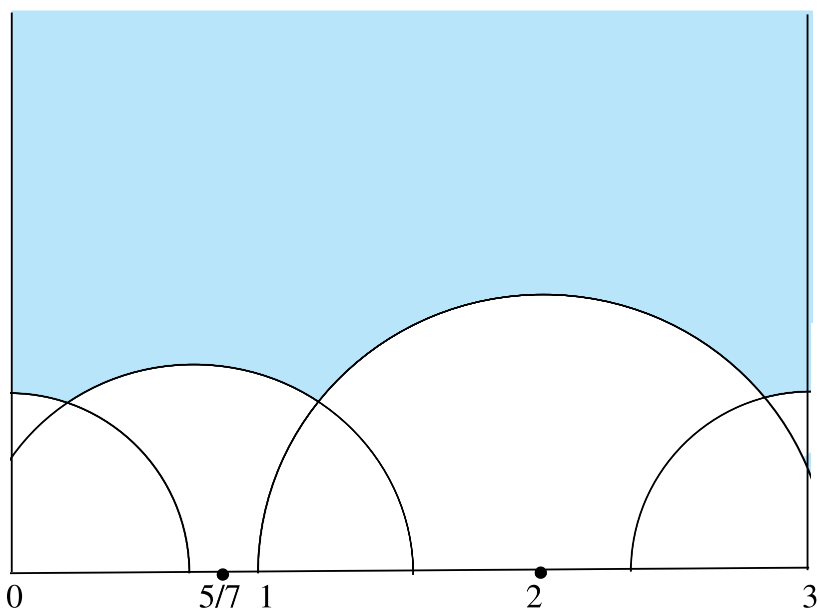


Figure 3.1

Referring to Figure 3.1, the result of [19] says that to obtain an estimate for h, it suffices to find the height of the lowest vertex of the Ford domain. If h_v denotes this height, then Theorem 1 of [19] gives $h \leq 1/2h_v$. An exercise in trigonometry gives the height of the lowest vertex in Figure 3.1 as $\frac{\sqrt{131}}{14}$. Thus the Hurwitz constant for $\Delta(5/7)$ satisfies $h \leq \frac{\sqrt{131}}{28}$.

For more on how the Hurwitz constant varies over the moduli space of 1-punctured tori, see [8].

3.2 The horoball patterns.

We begin by recalling some facts about parabolic elements in $\mathrm{PSL}(2,\mathbf{Q})$ which will be useful in what follows.

Lemma 3.1 *Let p,q be co-prime integers, and $\gamma \in \mathrm{PSL}(2,\mathbf{Q})$ a parabolic element fixing p/q. Then γ has the form*

$$\begin{pmatrix} 1-k^2pq & p^2k^2 \\ -q^2k^2 & 1+k^2pq \end{pmatrix}$$

for some constant k with $k^2 \in \mathbf{Q}$. Furthermore, there exists an integer $0 < n(\gamma)$ such that $\gamma^{n(\gamma)} \in \mathrm{PSL}(2,\mathbf{Z})$.

Proof. Every p/q is $\mathrm{PSL}(2,\mathbf{Q})$-equivalent to infinity. The form of an element mapping infinity to p/q is:

$$\begin{pmatrix} pk & 0 \\ qk & 1/pk \end{pmatrix},$$

with k as above. Conjugating $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ by the above element gives the form of γ shown.

Now if $m \in \mathbf{Z}$, then γ^m has the form:

$$\begin{pmatrix} 1-mk^2pq & mp^2k^2 \\ -mq^2k^2 & 1+mk^2pq \end{pmatrix}.$$

Choosing an appropriate $m = n(\gamma)$ to clear denominators of k^2 proves the second statement.$\square$

We begin by proving that there is no uniform bound on the $n(\gamma)$'s for parabolic elements in a pseudomodular group:

Theorem 3.2 *Suppose that Γ is pseudomodular. Let $N(p/q)$ be the smallest integer for which the generator of the parabolic subgroup of Γ stabilising p/q powers into the modular group.*

Then the set $\{N(p/q) \mid p/q \in \mathbf{Q}\}$ is unbounded.

Proof. Suppose to the contrary that this set were bounded, so that there would be a uniform power K which worked for every rational. In particular, choosing a generator c for the stabiliser of ∞ in Γ we see that the elements $\gamma(c^K)\gamma^{-1} \in \mathrm{SL}(2,\mathbf{Z})$ for every $\gamma \in \Gamma$. It follows that the subgroup N say, generated by these elements is a subgroup of $\mathrm{SL}(2,\mathbf{Z})$. Clearly, N is simply the normal closure of c^K in Γ, and in particular is a non-trivial normal subgroup of Γ. However it follows from results of Bass (cf. [3], Prop. 2.8) that the presence of a normal subgroup with all traces in $\mathbf{Z}$ implies the ambient group has traces which are all algebraic integers, a contradiction in the pseudomodular case. $\square$

We now examine the failure of the existence of any uniform power; it turns out that for the group $\Delta(5/7,6)$ one can give a fairly elegant answer. To this end we recall that one can define a *maximal horoball* in $\mathbf{H}^2/\Gamma$ and that the preimage of this horoball in $\mathbf{H}^2$ gives a complex of horoballs centred on the cusp set of Γ any pair of which are tangent or disjoint. For the modular torus one sees rather easily that this complex is connected, a reflection of the fact that the maximal horoball carries the entire fundamental group of the modular torus.

This can fail for other values of $(u^2, 2\tau)$. In particular, in the case $\Delta(5/7, 6)$, one sees that the maximal horoball centred at ∞ in this group is tangent only to maximal horoballs centred at 2 and -1 (and of course the translates of this pair by $stab(\infty)$.) It follows that the set of horoballs which are connected to ∞ is the orbit of the subgroup carried by this complex in $\mathbf{H}^2/\Delta$ which one sees is $< g_1, g_1 g_2^{-1} g_1^{-1} g_2 >=< g_1, g_2^{-1} g_1 g_2 >$. We denote this subgroup by Σ_∞. Components of the horoball complex correspond to cosets of Σ_∞ in $\Delta(5/7, 6)$. We claim:

Theorem 3.3 *The group Σ_∞ inside $\Delta(5/7, 6)$ contains a subgroup of finite index which lies inside* $\mathrm{SL}(2, \mathbf{Z})$*.*

In particular, $\{N(p/q) \mid p/q \in \Sigma_\infty(\infty)\}$ is bounded.

Proof. One checks that the traces of the elements $g_1.g_1$, $(g_2^{-1} g_1 g_2)^2$ and $(g_1 g_2^{-1} g_1^{-1} g_2)^2$ are all rational integers and it follows that the canonical subgroup of index four can be conjugated into $\mathrm{SL}(2, \mathbf{Z})$. This conjugation carries rationals to rationals so must lie inside $\mathrm{GL}(2, \mathbf{Q})$. By passing to a congruence subgroup deep enough to clear denominators in the conjugating matrix, it follows that Σ_∞ contains a subgroup of finite index which actually lies in $\mathrm{SL}(2, \mathbf{Z})$. $\square$

Remark. For Σ_∞ inside $\Delta(5/7, 6)$, one finds that a bound is 6.

We see immediately that:

Corollary 3.4 *For each fixed $\gamma \in \Delta(5/7, 6)$, the set $\{N(p/q) \mid p/q \in \gamma\Sigma_\infty(\infty)\}$ is bounded.*

This analysis has another consequence. At each rational number, it makes sense to consider two horoballs, namely the image of the maximal horoball coming from the modular torus and the image of the horoball coming from the pseudomodular torus. Suppose that S is some set of rationals then we say that the modular torus and a pseudomodular torus are *quasicomparable* on S if there is a distance D so that for every rational $p/q \in S$, the modular horoball at p/q is within distance D of the pseudomodular horoball at p/q.

Theorem 3.5 *The modular torus and $\Delta(5/7, 6)$ are quasi-comparable on $\Sigma_\infty(\infty)$.*

Proof. It follows from Theorem 3.3 that there is a subgroup $N^* \leq \mathrm{SL}(2, \mathbf{Z})$ which is normal and of finite index in Σ_∞. Choosing some finite set of coset representatives we see that there is a uniform bound on the size of denominators which can appear in Σ_∞.

Note that if $p/q \in \Sigma_\infty(\infty)$, say $M(\infty) = p/q$, then M must have the shape

$$\begin{pmatrix} rp & * \\ rq & * \end{pmatrix}$$

for some rational r. The denominator of r is bounded by the above paragraph. Moreover, writing $M = \xi \cdot \mu^*$, where $\mu^* \in N^*$ and ξ is one of the fixed coset representatives, we see that it is also possible to bound the numerator of r, for example by the determinant of the integral matrix that one obtains by clearing fractions in ξ.

The image of a height 1 horoball centred at ∞ under the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has radius $1/|c|^2$, so that the modular horoball centred at p/q has radius $1/|q|^2$ and the pseudomodular horoball has radius $1/|rq|^2$; however the above argument establishes bounds on r and the result follows. $\square$

3.3 A pseudo-Farey tessellation of the hyperbolic plane.

We begin by recalling the Farey tessellation of the hyperbolic plane, see [17] for more details. Let $\mathcal{T}$ denote the ideal triangle in the hyperbolic plane with vertices $\{0, 1, \infty\}$. The images of $\mathcal{T}$ under $\mathrm{SL}(2, \mathbf{Z})$ tessellate the hyperbolic plane, giving the so-called *Farey tessellation* of the hyperbolic plane. The vertices are $\mathbf{Q} \cup \{\infty\}$ and two rationals p/q, p'/q' are joined by an edge of $\mathcal{T}$ if and only if

$$\begin{pmatrix} p & p' \\ q & q' \end{pmatrix} \in \mathrm{GL}(2, \mathbf{Z}).$$

The reason for the name is the connection with Farey sequences. The nth-Farey sequence F_n is the set of rationals p/q with $|p|, |q| \leq n$ arranged in increasing order. Thus,

$$F_1 \text{ is } : -\infty, -1, 0, 1, \infty$$

$$F_2 \text{ is } : -\infty, -2, -1, -1/2, 0, 1/2, 1, 2, \infty$$

and so on. If $p/q < p'/q'$ are consecutive rationals in some Farey sequence then $p'q - pq' = 1$. Thus $\mathcal{T}$ can be constructed by drawing the vertical line through 0 and then successively joining adjacent rationals in each Farey sequence.

Now consider a pseudomodular group $\Delta(u^2, 2\tau)$, and the fundamental domain $\mathcal{Q}_{(u^2,2\tau)}$ for such a group pictured in Figure 3.2. By definition of $\Delta(u^2, 2\tau)$, the tessellation of the hyperbolic plane given by the $\Delta(u^2, 2\tau)$-images of $\mathcal{Q}_{(u^2,2\tau)}$ has vertex set precisely $\mathbf{Q} \cup \{\infty\}$. We define this tessellation to be a *pseudo-Farey tessellation*. See Figure 3.2 for part of $\mathcal{Q}_{(5/7,6)}$. Note $\mathcal{Q}_{(1,6)}$ is essentially the Farey tessellation discussed above—the edge $[0, \infty]$ and its images needs inserting. However, it is more natural for the pseudomodular groups to consider the tessellation $\mathcal{Q}_{(u^2,2\tau)}$.

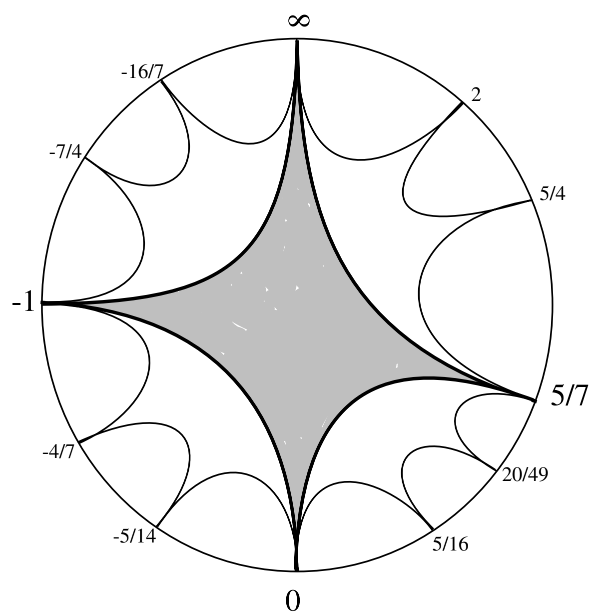


Figure 3.2

Giving a number theoretic description of the tessellation $\mathcal{F}_{(u^2,2\tau)}$, as was done for the Farey tessellation, seems a good deal harder. Indeed an initial question could be: is there such a description?

However, one can use $\mathcal{Q}_{(u^2,2\tau)}$ to formally define *pseudo-Farey sequences* of rationals $\{F_n(u^2,2\tau)\}$, as follows:

First consider the Farey sequence $\{F_n\}$, F_1 can be identified with a sequence of triangles $\{-\infty,-1,0\},\{0,1,\infty\}$, F_2 can be identified with a sequence of triangles

$$\{-\infty,-2,-1\},\{-1,-1/2,0\},\{0,1/2,1\},\{1,2,\infty\},$$

and so on. For pseudomodular $\Delta(5/7,6)$ we can proceed as follows using quadrilaterals (refer to Figure 3.2):

$$F_1(5/7,6) \text{ is } : -\infty,-1,0,5/7,\infty$$
$$F_2(5/7,6) \text{ is } : -\infty,-16/7,-7/4,-1,-4/7,-5/14,0,5/16,20/49,5/7,5/4,2,\infty,$$

and so on.

Remark: If we consider groups I of the type defined in the proof of Corollary 4.1 below, observe that this group preserves **both** the Farey tessellation and any pseudo-Farey tessellation $\mathcal{Q}_{(u^2,2\tau)}$ arising from a pseudomodular group.

4 The finitely generated intersection property.

A group G is said to have the *finitely generated intersection property* if for any pair of finitely generated subgroups H and K, $H\cap K$ is also finitely generated. We abbreviate this and say G has fgip. It is well known that if G is a finitely generated Fuchsian group, then G has fgip.

4.1

We can use pseudomodular surfaces to show:

Corollary 4.1 *The group* $\mathrm{PSL}(2,\mathbf{Q})$ *(and hence* $\mathrm{PSL}(2,\mathbf{R})$*) does not have the finitely generated intersection property.*

Proof. Let Γ denote the pseudomodular group given by Theorem 1.1. We claim that the intersection $I=\Gamma\cap\mathrm{PSL}(2,\mathbf{Z})$ cannot be finitely generated. The reason is this: Notice that given a rational number, each group contains a parabolic element which stabilises it. By Lemma 3.1 some power of the parabolic element in Γ stabilising p/q is in the group $\mathrm{PSL}(2,\mathbf{Z})$, that is to say, there is a parabolic subgroup in I which stabilises p/q. From this it follows that the limit set of the group I is the entire circle at infinity.

From this it follows that if I were finitely generated, $\mathbf{H}^2/I$ would be a finite area surface and this surface obviously covers $\mathbf{H}^2/\Delta$ and $\mathbf{H}^2/\mathrm{PSL}(2,\mathbf{Z})$. Since $\mathbf{H}^2/\Delta$ has finite area both these coverings are finite, so that I must be of finite index in both groups, a contradiction which completes the proof. $\square$

This corollary gives negative answers to questions 11 and 12 of the problem list [2]. In fact the following theorem can be proved in exactly the same way:

Theorem 4.2 *Let* Γ *be a finitely generated Fuchsian group of the first kind containing a parabolic element satisfying the following:*

- *1.* $\Gamma^{(2)}=ker\{\Gamma\to H_1(\Gamma;\mathbf{Z}_2)\}$ *is a subgroup of* $\mathrm{PSL}(2,\mathbf{Q})$*;*
- *2. There exists an element* $\gamma\in\Gamma$ *such that* $\mathrm{tr}(\gamma^2)\notin\mathbf{Z}$*.*

Then $\Gamma \cap \mathrm{PSL}(2, \mathbf{Z})$ *is infinitely generated.*

Proof. We sketch some of the details, since the proof follows the argument in Corollary 4.1. Now Γ contains parabolic elements by assumption, and since $\Gamma^{(2)} < \mathrm{PSL}(2, \mathbf{Q})$, the set of parabolic fix-points of $\Gamma^{(2)}$, and therefore Γ, is contained in $\mathbf{Q} \cup \{\infty\}$. It follows by Lemma 3.1 that any parabolic element of Γ powers into $\mathrm{PSL}(2, \mathbf{Z})$. The set of parabolic fix-points of Γ is dense in $\mathbf{R} \cup \{\infty\}$ (else Γ is not finite coarea) and so it follows that the limit set of $\Gamma \cap \mathrm{PSL}(2, \mathbf{Z})$ is $\mathbf{R} \cup \{\infty\}$. The second condition implies that Γ and $\mathrm{PSL}(2, \mathbf{Z})$ are not commensurable, and so the intersection is infinitely generated as before.□

4.2

The examples above allow us to construct certain S-arithmetic groups that do not have fgip. Little seems known about fgip for lattices in general Lie groups. It is known that $\mathrm{SL}(n, \mathbf{Z})$ does not have fgip for $n \geq 4$, as is easy to see by injecting the fundamental group of a fibered hyperbolic 3-manifold in $\mathrm{SL}(4, \mathbf{Z})$ (e.g. the group of the Borromean rings injects, by taking the obvious injection of $\mathrm{SL}(2, \mathbf{Z}[i])$ into $\mathrm{SL}(4, \mathbf{Z})$.). We do not know the answer for $n = 3$.

However, consider the pseudomodular group $\Delta(5/7, 6)$, and the subgroup Γ of index 4 discussed in the introduction. It is easy to check that Γ is contained in $\mathrm{SL}(2, \mathbf{Z}[\frac{1}{3}, \frac{1}{5}])$. Since $\mathrm{SL}(2, \mathbf{Z})$ is obviously a subgroup of $\mathrm{SL}(2, \mathbf{Z}[\frac{1}{3}, \frac{1}{5}])$, we deduce easily from Corollary 4.1.

Corollary 4.3 $\mathrm{SL}(2, \mathbf{Z}[\frac{1}{3}, \frac{1}{5}])$ *does not have fgip.* □

Indeed, using pseudomodular groups we prove the following more general statement.

Theorem 4.4 *Let V be the collection of all rational primes excluding 2 and 5. Then $\mathrm{SL}(2, \mathbf{Z}[S])$ does not have fgip for any finite set of $S \subset V$ of primes inverted.*

Proof. It suffices to prove the theorem in the case of $\mathrm{SL}(2, \mathbf{Z}[\frac{1}{p}])$ for any prime $p \in V$. Write $p = 2\ell - 1$, and consider the pseudomodular group $\Delta(1/\ell, 6)$. A direct check shows that the group $\Delta^{(2)}(1/\ell, 6)$ is a subgroup of $\mathrm{SL}(2, \mathbf{Z}[1/p])$. Now $\mathrm{SL}(2, \mathbf{Z})$ is also obviously a subgroup of $\mathrm{SL}(2, \mathbf{Z}[1/p])$ and so Theorem 4.2 applies to produce an infinitely generated subgroup. Note that this uses the fact that $p \neq 2, 5$, for by Theorem 2.2, the groups $\Delta(1/2, 6)$ and $\Delta(1/5, 6)$ are arithmetic.□

5 Small values of u^2.

We indicate below a table describing behaviour of the groups $\Delta(u^2, 4)$ and $\Delta(u^2, 6)$ for small values of u^2. All the candidates we know of have prime denominator.

A few comments are in order. Various groups are annotated as "conjecturally pseudomodular"; this is to say that extensive computer checking strongly suggests that such groups are pseudomodular but the rigourous check of the nature of the pseudomodular groups has failed to find a finite covering by killer intervals. Various behaviours might explain these examples. It seems plausible that these groups are pseudomodular but many more killer intervals are required. Alternatively, the killer intervals might form a dense open set which only omits irrational points.

The value $(u^2, 2\tau) = (3/11, 6)$ is the first occurence where the behaviour is not even conjecturally determined; there are rational numbers for which the Euclidean algorithm does not seem to converge, but no special has been found. The first occurence of this behaviour for the other family is at $(u^2, 2\tau) = (2/19, 4)$.

Table 6.1 : $2\tau = 4$

$0 < u^2 \leq 1$	Structure	$0 < u^2 \leq 1$	Structure
1	Arithmetic	1/9	Special fixing 1/3
1/2	Arithmetic	2/9	Special fixing 1/3
1/3	Arithmetic	4/9	Special fixing 2/3
2/3	Arithmetic	5/9	Special fixing 1/3
1/4	Special fixing 1/2	7/9	Special fixing 1/3
3/4	Special fixing 1/2	8/9	Special fixing 2/3
1/5	Arithmetic	1/10	Special fixing 7/2
2/5	**Pseudomodular**	3/10	Special fixing 1/5
3/5	**Pseudomodular**	7/10	Special fixing 1/2
4/5	Arithmetic	9/10	Special fixing 6/5
1/6	Special fixing 3/2	1/11	Conjectural pseudomodular
5/6	Special fixing 1/2	2/11	Conjectural pseudomodular
1/7	Conjectural pseudomodular	3/11	**Pseudomodular**
2/7	Conjectural pseudomodular	4/11	Conjectural pseudomodular
3/7	**Pseudomodular**	5/11	Conjectural pseudomodular
4/7	Conjectural pseudomodular	6/11	Conjectural pseudomodular
5/7	Conjectural pseudomodular	7/11	Conjectural pseudomodular
6/7	Conjectural pseudomodular	8/11	Conjectural pseudomodular
1/8	Special fixing 1/2	9/11	Conjectural pseudomodular
3/8	Special fixing 1/2	10/11	Conjectural pseudomodular
5/8	Special fixing 1/2		
7/8	Special fixing 1/2		

Table 6.2 : $2\tau = 6$

$0 < u^2 \leq 1$	Structure	$0 < u^2 \leq 1$	Structure
1	Arithmetic	1/9	Special fixing $-100/117$
1/2	Arithmetic	2/9	Special fixing 545/1521
1/3	Special fixing 1	4/9	Special fixing $-52/9$
2/3	Special fixing 1/3	5/9	Special fixing $-5/16$
1/4	Special fixing $-5/8$	7/9	Special fixing 29/9
3/4	Special fixing 3/2	8/9	Special fixing $-205/9$
1/5	Arithmetic	1/10	Special fixing 5/52
2/5	Special fixing 1/7	3/10	Special fixing 1/2
3/5	Conjectural pseudomodular	7/10	Special fixing 1/2
4/5	Conjectural pseudomodular	9/10	Special fixing 6/5
1/6	Special fixing $-1/35$	1/11	Conjectural pseudomodular
5/6	Special fixing $-17/24$	2/11	Special fixing $-266/4717$
1/7	Special fixing $-37/14$	3/11	Undecided
2/7	Conjectural pseudomodular	4/11	Special fixing 1/5
3/7	Special fixing 3/4	5/11	Special fixing $-1778/741$
4/7	Special fixing 2/7	6/11	Special fixing 69/11
5/7	**Pseudomodular**	7/11	Special fixing 149/136
6/7	Special fixing 5/3	8/11	Special fixing $-79/93$
1/8	Special fixing 1/14	9/11	Conjectural pseudomodular
3/8	Special fixing $-15/2$	10/11	Special fixing 1/3
5/8	Special fixing 7/4		
7/8	Special fixing 1/2		

6 Open questions.

The existence of pseudomodular groups raises many questions.

1. For which values of $(u^2, 2\tau)$ are the groups $\Delta(u^2, 2\tau)$ pseudomodular?

2. Are there finitely many pseudomodular groups up to commensurability?

3. Can the killer intervals associated to $\Delta(u^2, 2\tau)$ cover $[0, \tau]$ except possibly for some irrational points?

   This suffices to prove $\Delta(u^2, 2\tau)$ is pseudomodular. The example of the covering for $\Delta(3/4, 4)$ (see §7.5) shows that a covering may omit only a finite number of points.

4. Does $\Delta(u^2, 2\tau)$ always contain only finitely many conjugacy classes of special hyperbolic? Is there a way to predict which rationals will be fixed by some special hyperbolic? More generally, for a fixed $\Delta(u^2, 2\tau)$, is every rational equivalent to either a cusp or one of a finite number of points fixed by a special hyperbolic?

   We remark that it seems that this may not always be the case, for example the chaotic example $\Delta(3/11, 6)$. However, the nonchaotic examples do appear to behave in this way.

5. Is there a recursive formula for $\mathcal{Q}_{(u^2, 2\tau)}$?

6. Which quadratic irrationals are the endpoints of hyperbolic elements in $\Delta(u^2, 2\tau)$?

   Our normalization is such that $\Delta^{(2)}(u^2, 2\tau) \subset \mathrm{PSL}(2, \mathbf{Q})$. Hence, if $\gamma \in \Delta(u^2, 2\tau)$ is hyperbolic but **not** a special hyperbolic, it has an axis with endpoints lying in a real quadratic number field. In the case of $\Delta(1, 6)$ since it is of finite index in $\mathrm{PSL}(2, \mathbf{Z})$, every real quadratic number is a fixed point of some hyperbolic element in the group. We believe this is well-known, but we cannot find a suitable reference. A proof is sketched below (see Proposition 6.1). In connection with this question, we raise:

7. Can a nonarithmetic $\Delta(u^2, 2\tau)$ be isoaxial with $\mathrm{SL}(2, \mathbf{Z})$?

   Recall that two Fuchsian groups are defined to be *isoaxial* if they share the same set of axes in $\mathbf{H}^2$. If both the Fuchsian groups are assumed arithmetic, then [12] shows they are commensurable (without conjugating). There is considerable experimental evidence that the pseudomodular groups we know of are not isoaxial with $\mathrm{SL}(2, \mathbf{Z})$; for example it would appear that if A denotes the axis of the element

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

   then $\Delta(5/7, 6)$ contains no non-trivial element leaving A invariant. On the other hand,

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^4 = \begin{pmatrix} 34 & 21 \\ 21 & 13 \end{pmatrix},$$

   lies in $\Delta(5/7, 6)$.

   A puzzling empirical observation is that in all of the cases that $M \in \mathrm{SL}(2, \mathbf{Z}) - \Delta(5/7, 6)$, but M^k is in $\Delta(5/7, 6)$, then $k \leq 36$ and $4|k$. Similar behaviour is exhibited in the other examples.

   Proposition 6.1 *Let q be a real quadratic number. Then q is the fix-point of a hyperbolic element of* $\mathrm{PSL}(2, \mathbf{Z})$.

   Proof. Let D be a square-free positive integer, and write $q = \frac{a+b\sqrt{D}}{c}$, where $a, b, c \in \mathbf{Z}$. We may assume without loss that $(a, c) = 1$ and $(b, c) = 1$. Let q' denote the non-trivial galois conjugate of q, and $A_{[q,q']}$ be the geodesic in $\mathbf{H}^2$ (unoriented) between q and q'. We claim there exists $\gamma \in \mathrm{PSL}(2, \mathbf{Z})$ such that the axis of γ is $A_{[q,q']}$. This will prove the Proposition.

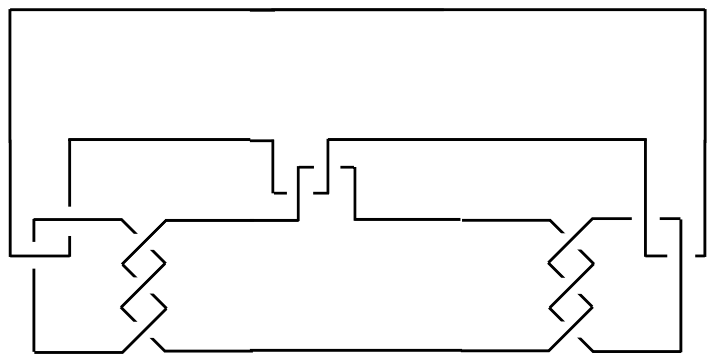
   Associated to this geodesic is an *integral* quadratic form obtained from the quadratic equation $x^2 - \frac{2a}{c}x + \frac{a^2-b^2D}{c^2}$, namely:

$$f(X, Y) = c^2X^2 - 2acXY + (a^2 - b^2D)Y^2.$$

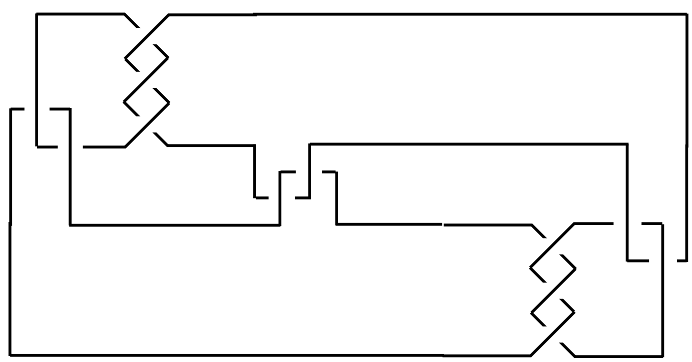
   It follows for example from the theory of automorphs of integral binary forms (cf [5] Chapter 12) or an argument from the theory of orders in quaternion algebras (cf [14]) that the set of elements of $\mathrm{PSL}(2, \mathbf{Z})$ preserving this form is infinite cyclic. Choosing any element of this infinite cyclic subgroup will suffice.□

8. What sort of behaviours are possible in dimension 3?

   An obvious candidate construction to provide examples of such a phenomena is that of mutation. The classical setting for this is the case of a link L in S^3, and one mutates $S^3 \setminus L$



(a)



(b)

Figure 6.1

by cutting-and-pasting along an embedded incompressible and boundary incompressible 4-punctured sphere. However, one can show that the pair of mutant links in Figure 6.1 have different cusp sets as we now briefly describe.

The link in Figure 6.1(a) has arithmetic complement being commensurable with $\mathbf{H}^3/\mathrm{PSL}(2, O_3)$, (where O_3 is the ring of integers in the number field $\mathbf{Q}(\sqrt{-3})$) and so the cusp set (for a suitable representation) of the link group is $\mathbf{Q}(\sqrt{-3}) \cup \{\infty\}$. By [13] mutation does preserve the invariant trace-field, however, it can be shown that the mutant in Figure 6.1(b) has a special fixing $-3/2 + i\sqrt{3}$.

Another interesting feature of this example is that the link group of the mutant has a non-integral trace, and so gives an example where mutation does not preserve integrality of traces.

However, we have computer evidence that suggests there do exist analogues of pseudomodular groups in dimension 3.

7 Appendix : Killer intervals.

For the convenience (?!) of the reader, we provide a list of killer intervals for the groups we know to be pseudomodular. The notation in the list below is (rational number, contraction constant) which specifies the killer interval completely. If this untiring reader wishes to check for himself that these intervals work, the computation is two-fold. First one needs to check that these intervals cover, an easy if tedious computation made somewhat easier by the fact that they are in ascending order.

The second phase takes a little more work, one needs to check that the contraction constants are as claimed; this is not difficult, one writes a short computer program which performs the Euclidean algorithm with the given matrices.

7.1 Killer intervals for $\Delta(2/5, 4)$

$\{(0:5),(1/5:1),(2/5:1),(3/5:9),(7/10:1),(4/5:3),(1:5),(6/5:3),$
$(13/10:1),(7/5:9),(8/5:1),(9/5:1),(2:5)\}$

7.2 Killer intervals for $\Delta(3/7, 4)$

$\{(0:7),(3/14:1),(3/11:7),(1/7:3),(2/7:9),(3/7:1),(4/7:3),(5/7:1),$
$(6/7:1),(1:7),(8/7:1),(9/7:1),(7/5:7),(10/7:3),(11/7:1),(12/7:9),$
$(25/14:1),(13/7:3),(2:7)\}$

7.3 Killer intervals for $\Delta(3/11, 4)$

$\{(0:11),(1/11:3),(3/22:1),(2/11:36),(3/11:1),(4/11:9),$
$(45/121:1),(21/55:6),(711/1859:1),(207/539:1),(13/33:3),$
$(9/22:2),(5/11:3),(255/539:1),(27/55:1),(129/253:1),(17/33:6),$
$(585/1133:1),(177/341:1),(41/77:1),(6/11:4),(7/11:1),(8/11:6),$
$(9/11:1),(10/11:1),(1:11),(12/11:1),(13/11:1),(41/33:1),$
$(14/11:6),(15/11:1),(16/11:4),(113/77:1),(49/33:6),(61/41:11),$
$(377/253:1),(2345/1573:1),(83/55:1),(17/11:3),(11/7:11),(35/22:2),$
$(439/275:1),(53/33:3),(89/55:6),(115/71:11),(197/121:1),(18/11:9),$
$(19/11:1),(20/11:36),(41/22:1),(21/11:3),(2:11)\}$

7.4 Killer intervals for $\Delta(5/7, 6)$

$\{(0:7),(1/7:125),(5/28:1),(5/21:1),(2/7:25),(45/154:5),$
$(40/133:1),(5/16:7),(30/91:1),$
$(55/161:1),(130/371:1),(55/156:7),(3190/9023:1),$
$(255/721:25),(5735/16212:1),$
$(635/1792:1),(5/14:25),(35/97:7),(120/329:1),$
$(235/637:1),(115/308:1),(45/119:5),(155/406:1),(65/168:1),$
$(20/49:1),(3/7:25),(10/21:1),(85/161:1),(15/28:5),(65/119:1),$
$(55/98:1),(4/7:25),(5/7:1),(6/7:125),$
$(25/28:1),(13/14:5),(125/133:1),(20/21:5),(75/77:1),(125/126:1),$
$(1:175),(50/49:1),(270/259:1),(300/287:1),(1550/1477:1),$
$(125/119:25),(2825/2688:1),(2525/2401:1),(325/308:1),$
$(15/14:5),(25/23:7),(100/91:1),(125/112:1),(8/7:5),$
$(25/21:1),(5/4:7),(9/7:5),(10/7:1),(11/7:1),(12/7:5),(13/7:1),$

$(2:7), (15/7:1), (16/7:5), (17/7:1), (18/7:25), (37/14:1), (19/7:5),$
$(58/21:1), (453/161:1), (2033/721:1), (79/28:25), (1917/679:1),$
$(337/119:1), (20/7:5), (3:7)\}$

7.5 Killer intervals for $\Delta(3/4, 4)$

$\{(0:4), (1:4), (2:4), (3/4:1), (5/4:1), (3/8:1), (13/8:1), (1/4:3), (7/4:3)\}$

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