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SIMPLE QUOTIENTS OF HYPERBOLIC 3-MANIFOLD GROUPS

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ABSTRACT. We show that hyperbolic 3-manifolds have residually simple fundamental group.

1. INTRODUCTION

Let G be a finitely generated group and X a property of groups, e.g. finite, simple, p-group. G is said to be *residually* X, if for any element $g \neq 1$, there is a group H with property X and a *surjective* homomorphism $\phi : G \to H$ such that $\phi(g) \neq 1$.

Of interest to us are residual properties of groups $\pi_1(M)$ where M is a compact orientable 3-manifold with infinite fundamental group. Now it is well-known that if M is a hyperbolic 3-manifold, that is the quotient of hyperbolic 3-space by a torsion-free Kleinian group, then $\pi_1(M)$ is residually finite. In this note we prove a much stronger result which seems to have been unnoticed previously. First we make a definition.

Definition 1.1. Let M be a compact orientable 3-manifold with infinite fundamental group and $\rho : \pi_1(M) \to SL(2, \mathbb{C})$ a faithful representation whose image lies in $SL(2, \overline{\mathbb{Q}})$, where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$. In this case we define ρ to be an algebraic representation.

With this we can state:

Theorem 1.2. *Let M be a compact orientable 3-manifold such that $\pi_1(M)$ admits an algebraic representation. Then $\pi_1(M)$ is residually simple.*

A particular case of this is:

Corollary 1.3. *Let M be a finite volume hyperbolic 3-manifold. Then $\pi_1(M)$ is residually simple.*

In fact we shall show more; the simple groups will all be of the type $PSL(2, \mathbb{F})$ for finite fields $\mathbb{F}$ of prime cardinality. A corollary of Theorem 1.2 is a new proof of a result originally observed by Magnus, which follows by taking M to be a handlebody:

Corollary 1.4. *Nonabelian free groups are residually simple.*

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2. General remarks

We start with some general remarks about algebraic representations of 3-manifold groups, and groups $PSL(2,\mathbb{F})$ which do not necessarily surject. This is just a reformulation of classical notions about linear groups; cf. [6]. For details on number fields and their completions, see [2] for instance.

Let M be a compact orientable 3-manifold, and ρ an algebraic representation of $\pi_1(M)$ into $SL(2,\mathbb{C})$. Denote the image of the group $\pi_1(M)$ under ρ by Γ. Let $k = \mathbb{Q}(tr\gamma : \gamma \in \Gamma)$ denote the trace-field of Γ. Since ρ is an algebraic representation, k is a finite extension of $\mathbb{Q}$. Let $A\Gamma$ be the algebra

$$\{\sum a_i\gamma_i \ : \ a_i \in k, \gamma_i \in \Gamma\}.$$

This is a quaternion algebra over k as follows from [1]. We remark that for finite volume hyperbolic 3-manifolds k is always a finite extension of $\mathbb{Q}$; cf. [4], Proposition 6.7.4.

By the classification theorem for quaternion algebras $A\Gamma$ is unramified at all but a finite number of places of k, [5]. In particular, for all but a finite number of prime ideals $\wp$ of k, $A \otimes_k k_\wp \cong M(2, k_\wp)$. Thus on specifying an isomorphism between $A \otimes_k k_\wp$ and $M(2, k_\wp)$, we induce a representation of Γ into $SL(2, k_\wp)$.

Since M is compact, Γ is finitely generated and finitely presented, therefore for all but finitely many prime ideals, we actually induce a representation of Γ into $SL(2, O_\wp)$ where $O_\wp$ are the $\wp$-adic integers in $k_\wp$, since only finitely many k-primes can divide denominators of elements of Γ.

Denote by $\pi_\wp$ a local uniformizing parameter for $O_\wp$. The unique maximal ideal of $O_\wp$ is $\pi O_\wp$, and $O_\wp/\pi O_\wp$ is a finite field with p^n elements where $\wp$ divides p for a rational prime p and n is the inertial degree of $\wp$. Therefore reduction induces a homomorphism (so far, not necessarily surjective) of Γ into $SL(2, \mathbb{F}_{p^n})$ where $\mathbb{F}_{p^n}$ is the finite field with p^n elements. By composing the map $\pi_1(M) \to \Gamma$, with the above, and then projectivising, we get a homomorphism $\phi_\wp$ of $\pi_1(M)$ into $PSL(2,\mathbb{F})$, for infinitely many finite fields $\mathbb{F}$.

Lemma 2.1. *There are infinitely many k-primes $\wp$ such that the homomorphisms $\phi_\wp$ constructed above are nontrivial and map $\pi_1(M)$ into $PSL(2,\mathbb{F})$ where $\mathbb{F}$ has prime cardinality.*

Proof. It is a well-known consequence of how prime ideals behave in finite extensions of $\mathbb{Q}$, that there are infinitely many rational primes that split completely in the finite extension $k/\mathbb{Q}$; see [2] Theorem 4.12 for example. Now a rational prime p splits completely if and only if the inertial degree of the k-prime divisors of p are all equal to 1. In particular we deduce that there are infinitely many rational primes p with $\wp | p$ such that $\phi_\wp$ maps $\pi_1(M)$ into $PSL(2,\mathbb{F}_p)$ where $\mathbb{F}_p$ has p elements.

It is also easy to see that infinitely many of these homomorphisms are non-trivial. For if $\gamma \in \Gamma$ is given, reduction of, for example, the $(1, 2)$-entry of γ will be non-zero for all but a finite number of k-primes—since entries of γ will be $\wp$-adic units for all but a finite number of $\wp$. Hence, the number of k-primes such that $\phi_\wp(\gamma)$ is trivial is finite. □

In particular Lemma 2.1 implies that we may construct infinitely many non-trivial representations of $\pi_1(M)$ into groups $PSL(2,\mathbb{F})$ where the cardinalities of $\mathbb{F}$ are distinct primes.

3. Proof of Theorem 1.2

To prove Theorem 1.2 we shall make use of the description of subgroups of the groups $PSL(2,\mathbb{F})$ where $|\mathbb{F}|$ is of odd prime cardinality. The following is deduced from [3], Theorem 6.25, together with the observation that all abelian subgroups of such $PSL(2,\mathbb{F})$ are cyclic.

Theorem 3.1. *Let p be an odd rational prime; then a complete list of subgroups of $PSL(2,\mathbb{F})$ where $|\mathbb{F}| = p$ is*

1. *Cyclic groups of order p and order n where n divides $\frac{(p\pm1)}{2}$.*
2. *Dihedral groups of order n where n is as in 1.*
3. *Semi-direct products of cyclic groups of order p with cyclic groups of order $(p-1)/2$.*
4. *A_4, S_4 or A_5.*

We now show that the homomorphisms $\phi_\wp$ constructed above actually surject infinitely many groups $PSL(2,\mathbb{F})$ as in the statement of Lemma 2.1.

The group Γ is never soluble of any finite degree. This follows for example by the fact that they contain free non-abelian groups, as they are non-elementary subgroups of $SL(2,\mathbb{C})$. Thus for the remainder of the proof, we fix some nontrivial element α which lies deep in the solubility series of $\pi_1(M)$.

Now suppose then that we are given some element $\gamma \in \Gamma$. As observed above, we can find (infinitely many) k-primes $\wp$ so that $\phi_\wp(\gamma) \neq 1$ and $\phi_\wp(\alpha) \neq 1$.

Since the homomorphic image of a term in the solubility series for a group lies in the same term of the solubility series of the image, the fact that the element α maps nontrivially means that the image of the group $\pi_1(M)$ cannot be of type 1, 2, 3 nor A_4 or S_4 in the list provided by Theorem 3.1 since these are all soluble of small fixed degree.

Thus the map $\phi_\wp$ will be shown to have been a surjection if we show that $\phi_\wp(\Gamma) \neq A_5$ for infinitely many $\wp$. Now if infinitely many of the homomorphisms constructed surject A_5, then since there are only finitely many normal subgroups in Γ of index 60, it follows that for infinitely many of these homomorphisms the kernels coincide. However this is impossible. These homomorphisms were constructed by reducing Γ modulo $\pi_\wp O_\wp$, hence if infinitely many homomorphisms had the same kernel this would mean that the elements in this matrix group were congruent to the identity modulo infinitely many $\pi_\wp O_\wp$, which is clearly false.

Thus we have $\phi_\wp$ that surjects $\pi_1(M)$ onto some $PSL(2,\mathbb{F})$ and which maps γ non-trivially.

In fact we may conclude that under the homomorphisms $\phi_\wp$ constructed above, $\pi_1(M)$ surjects infinitely many of the simple groups $PSL(2,\mathbb{F})$, with $|\mathbb{F}|$ of odd prime cardinality. □

4. Application

A motivation for this result arises from trying to show that covers of hyperbolic 3-manifolds have positive first Betti number. With this in mind, an application of this result stems from the following question raised by D. Cooper. Here $inj(M)$ denotes the injectivity radius of M, which is simply half the length of the shortest closed geodesic in M:

Question. Is there a number $K > 0$ so that if M is a closed hyperbolic 3-manifold and $inj(M) > K$, then $rank(H_1(M;\mathbb{Q})) > 0$?

An affirmative answer to this question, taken with the fact that $\pi_1(M)$ is residually finite, implies in particular that every closed hyperbolic 3-manifold has a finite sheeted covering with positive first Betti number. However our main theorem shows that actually more would be true.

Corollary 4.1. *If the above question has an affirmative answer, then every rational hyperbolic homology 3-sphere has infinite virtual Betti number.*

Proof. We recall that M is said to have infinite virtual Betti number if given any integer N, one can find a finite sheeted covering of M whose first Betti number is larger than N. Equivalently, M has infinite virtual Betti number if the rank of $H_2(\widetilde{M};\mathbb{Z})$ is unbounded as $\widetilde{M}$ ranges over all finite covers of M.

Since, given any constant C, M has only finitely many geodesics of length at most C, our main theorem implies that one can find regular covers of M having arbitrarily large injectivity radius for which the group of covering transformations has the form $PSL(2,\mathbb{F})$. An affirmative answer to the question implies that we may assume that these manifolds all have $H_2(M_{\mathbb{F}};\mathbb{Z})$ having rank at least one. The action of the covering group gives a series of representations

$$\alpha_{\mathbb{F}} : PSL(2,\mathbb{F}) \longrightarrow GL(H_2(M_{\mathbb{F}};\mathbb{Z}))$$

Suppose to the contrary that the ranks of the groups $H_2(M_{\mathbb{F}};\mathbb{Z})$ were bounded, by P say. Then since there is a bound on the size of the finite subgroups in $GL(P,\mathbb{Z})$, and the sizes of the groups $PSL(2,\mathbb{F})$ are going to infinity, we would eventually see that some $\alpha_{\mathbb{F}}$ is nonfaithful, hence trivial. It follows that the fixed homology $H_1(M_{\mathbb{F}};\mathbb{Q})^{PSL(2,\mathbb{F})} \cong H_1(M_{\mathbb{F}};\mathbb{Q})$. However using the transfer map we see that the left hand side of this isomorphism is $H_1(M;\mathbb{Q})$, a contradiction, since we assumed that M was a rational homology sphere. □

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