Zariski dense surface subgroups in $\mathrm{SL}(4,\mathbb{Z})$

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An infinite family of Zariski dense surface groups of fixed genus is exhibited inside $\mathrm{SL}(4,\mathbb{Z})$, and an account is given of the computational method.

22E40; 20H10

1 Introduction and main results

This note continues in the vein of [6], with a view to both extending the results and clarifying in some detail how the method first introduced in [2] was used to obtain them.

The result of [6] is the existence of an infinite family of Zariski dense surface subgroups of fixed genus inside $\mathrm{SL}(3,\mathbb{Z})$; here we exhibit such subgroups inside $\mathrm{SL}(4,\mathbb{Z})$ and symplectic groups. In this setting the power of such a result comes in large part from the conclusion that the groups are Zariski dense - the existence of surface groups inside $\mathrm{SL}(4,\mathbb{Z})$ can be proved fairly easily, since it's not hard to see that there is a faithful representation of the figure eight knot group into $\mathrm{SL}(4,\mathbb{Z})$ (see the end of §4) and surface groups follow immediately. However this image lies in a six dimensional Lie subgroup of $\mathrm{SL}(4,\mathbb{R})$ and so in particular is not Zariski dense in $\mathrm{SL}(4,\mathbb{R})$. The main result of this paper is the following:

Theorem 1.1 *The family of representations of the triangle group given below are discrete and faithful for every $k \in \mathbb{R}$.*

$$\rho_k : \Delta(3,3,4) = \left\langle a, b \ \mid \ a^3 = b^3 = (a\,b)^4 = 1 \right\rangle \to \mathrm{PSL}(4,\mathbb{R})$$

$$\rho_k(a) = \begin{bmatrix} k(3-4k+4k^2) & -1-4k-8k^2+16k^3-16k^4 & 0 & 0 \\ 1-k+k^2 & -1-3k+4k^2-4k^3 & 0 & 0 \\ k(1-2k+2k^2) & -3-4k-2k^2+8k^3-8k^4 & 1 & 0 \\ 2(1-k+k^2) & -2(1+2k-4k^2+4k^3) & 0 & 1 \end{bmatrix}$$

$$\rho_k(b) = \begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We briefly give an overview of the ingredients involved in constructing such representations. The starting point is a classical theorem of Hitchin [4]. One begins with the discrete faithful representation

$$\rho_\infty : \Delta(3,3,4) \to \mathrm{PSL}(2,\mathbb{R})$$

Now Hitchin's theorem applies to genuine surfaces (i.e. not orbifolds) but we recall that it is well-known (see [9] p. 313) that a discrete faithful representation of a closed surface group into $\mathrm{PSL}(2,\mathbb{R})$ lifts to a representation into $\mathrm{SL}(2,\mathbb{R})$. It follows that if one restricts to the ρ_∞ representation of $\Delta(3,3,4)$ to a torsion-free subgroup K of finite index, then $\rho_\infty : K \to \mathrm{PSL}(2,\mathbb{R})$ will lift to $\mathrm{SL}(2,\mathbb{R})$. This representation is then composed with the usual representation $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(4,\mathbb{R})$ (coming for example from the action on homogeneous polynomials of degree three) to obtain representations

$$\rho : \Delta(3,3,4) \to \mathrm{PSL}(4,\mathbb{R})$$

together with the restriction $\rho : K \to \mathrm{SL}(4,\mathbb{R})$. By definition, the representation ρ of K lies on the Hitchin component. We note in passing that this representation has image in $\mathrm{Sp}(4,\mathbb{R})$ and is in particular not Zariski dense in $\mathrm{SL}(4,\mathbb{R})$, but this does not play any role in the sequel.

Now, using the methods of [2], one can construct a two dimensional real family of representations of the triangle group $\Delta(3,3,4)$ into $\mathrm{PSL}(4,\mathbb{R})$ by deforming ρ (for technical reasons explained in §2, we work with representations of a $\mathbb{Z}/2$ central extension of $\Delta(3,3,4)$ into $\mathrm{SL}(4,\mathbb{R})$ which have $\rho((ab)^4) = -I$). From the discussion above, these representations have the property that restricted to K, they lie on the Hitchin component and therefore are genuine representations of the surface group into $\mathrm{SL}(4,\mathbb{R})$. The results of Labourie [5] now imply that all of these deformations are discrete and faithful on K, and therefore discrete and faithful on $\Delta(3,3,4)$. The application of some linear algebra finds the (discrete faithful) subfamily of Theorem 1.1. As a corollary we have:

Corollary 1.2 *For k a rational integer, ρ_k is a faithful representation of a surface group into* $\mathrm{SL}(4, \mathbb{Z})$.

To prove that these integral families are Zariski dense in $\mathrm{SL}(4, \mathbb{R})$, we appeal to the following remarkable theorem of Lubotzky ([8] Proposition 1 with $n = 4$).

Proposition 1.3 *Let $H < \mathrm{SL}(4, \mathbb{Z})$ and assume that for some odd prime p, H surjects $\mathrm{SL}(4, p)$ under the reduction modulo p.*

Then H is a Zariski dense subgroup of $\mathrm{SL}(4, \mathbb{R})$; moreover, H surjects $\mathrm{SL}(4, q)$ for all but finitely many odd primes q.

One may verify by computer program (e.g. GAP) that for each of the residue classes of k modulo 3, the image groups $\rho_k(\Delta(3, 3, 4))$ surject $\mathrm{SL}(4, 3)$, whence each $\rho_k(\Delta(3, 3, 4))$ is Zariski dense in $\mathrm{SL}(4, \mathbb{R})$. For the image of a surface group Γ of finite index in $\Delta(3, 3, 4)$ we argue as follows. Let N be a finite-index normal subgroup of $\Delta(3, 3, 4)$ contained in Γ. From Proposition 1.3 we may choose an odd prime q such that $|\mathrm{PSL}(4, q)| > |\Delta(3, 3, 4) : N|$ and $\rho_k(\Delta(3, 3, 4))$ surjects $\mathrm{PSL}(4, q)$. Then N maps onto a non-trivial normal subgroup of $\mathrm{PSL}(4, q)$, which equals $\mathrm{PSL}(4, q)$ on account of $\mathrm{PSL}(4, q)$ being simple. Therefore N and $\Gamma > N$ surject $\mathrm{SL}(4, q)$ and we conclude that $\rho_k(\Gamma)$ is Zariski dense.

Since distinct positive integers k give rise to representations ρ_k with distinct characters (as evidenced for example by $\mathrm{trace}(\rho_k(a^{-1}b)) = 8k^4 + 6k^2 + 12$), the family of representations $\{\rho_k : k \geq 0\}$ are pairwise non-conjugate. This alone does not guarantee that the surface group images $\rho_k(\Gamma)$ are pairwise non-conjugate subgroups of $\mathrm{SL}(4, \mathbb{Z})$; however, we can exploit Proposition 1.3 together with special properties of the ρ_k to find a subsequence of representations ρ_{k_i} for which the subgroups $\rho_{k_i}(\Gamma)$ are indeed pairwise non-conjugate in $\mathrm{SL}(4, \mathbb{Z})$. Details are given in § 1.1.

To summarize, we have

Corollary 1.4

(i) For all non-negative integral values of k the image groups $\rho_k(\Delta(3, 3, 4))$ are Zariski dense in $\mathrm{SL}(4, \mathbb{R})$;

(ii) There exists a surface subgroup Γ of $\Delta(3, 3, 4)$ and a subsequence (ρ_{k_i}) such that the images $\rho_{k_i}(\Gamma)$ are pairwise non-conjugate surface subgroups of $\mathrm{SL}(4, \mathbb{Z})$.

We include two further applications: In § 3, we exhibit an infinite family of Zariski dense surface groups in $\mathrm{SL}(5, \mathbb{Z})$ and in § 4, surface groups which are Zariski dense in $\mathrm{Sp}(4, \mathbb{Z})$; the question of whether there infinitely many representations in this last class is as yet unsettled.

1.1 The subsequence of non-conjugate surface subgroups of $\mathrm{SL}(4,\mathbb{Z})$

It is a simple exercise to construct an orientable genus 3 surface as a 48–sheeted covering space of S^2 with cone points of orders $3, 3, 4$; therefore $\Delta(3,3,4)$ contains a genus 3 surface subgroup of index 48.

The purpose of this section is to prove the following.

Theorem 1.5 *Let Γ be a subgroup of finite index in $\Delta(3,3,4)$. There is a subsequence of representations ρ_{k_i} with the property that the images $\rho_{k_i}(\Gamma)$ are pairwise non-conjugate subgroups of $\mathrm{SL}(4,\mathbb{Z})$.*

The crucial lemma is the following:

Lemma 1.6 *There is a strictly increasing sequence of primes (p_j), each with associated value k_{p_j}, with the property that the composition of $\rho_{k_{p_j}} : \Delta(3,3,4) \to \mathrm{SL}(4,\mathbb{Z})$ with the reduction $\mathrm{SL}(4,\mathbb{Z}) \to \mathrm{SL}(4,\mathbb{Z}/p_j)$ is not surjective.*

We note that the sequence (k_{p_j}) must tend to infinity, since it follows from Proposition 1.3 that for each fixed k, the group $\rho_k(\Delta(3,3,4))$ is Zariski dense in $\mathrm{SL}(4,\mathbb{Z})$, and moreover that it can only fail to surject for finitely many primes. So we may assume that the strictly increasing sequence (p_j) of Lemma 1.6 is chosen to that the associated sequence (k_{p_j}) is also strictly increasing.

Proof of Theorem 1.5 assuming Lemma 1.6.

Suppose that representations $\rho_{k_1}, \rho_{k_2}, \ldots, \rho_{k_n}$ have been found so that the images $\rho_{k_i}(\Gamma)$ $(1 \le i \le n)$ are pairwise non-conjugate subgroups of $\mathrm{SL}(4,\mathbb{Z})$. From the discussion immediately following Proposition 1.3, for each i the composition

$$\Gamma \underset{\rho_{k_i}}{\to} \mathrm{SL}(4,\mathbb{Z}) \to \mathrm{SL}(4,\mathbb{Z}/p)$$

fails to be surjective for only finitely many primes p. Let p_0 be the greatest such prime occurring amongst all ρ_{k_i}, $1 \le i \le n$. From Lemma 1.6 we may choose a prime $p_j > p_0$ with $k_{p_j} > k_n$; then

$$\Gamma \underset{\rho_{k_i}}{\to} \mathrm{SL}(4,\mathbb{Z}) \to \mathrm{SL}(4,\mathbb{Z}/p_j)$$

is surjective for all $1 \le i \le n$, whereas

$$\Delta(3,3,4) \underset{\rho_{k_{p_j}}}{\to} \mathrm{SL}(4,\mathbb{Z}) \to \mathrm{SL}(4,\mathbb{Z}/p_j) \ ,$$

hence also the restriction to Γ, is not surjective. This is sufficient to establish that $\rho_{k_{p_j}}(\Gamma)$ is not conjugate in SL$(4,\mathbb{Z})$ to any of the subgroups $\rho_{k_i}(\Gamma)$ $(1 \leq i \leq n)$, and we set $k_{n+1} = k_{p_j}$. $\square$

Proof of Lemma 1.6

The proof of the Lemma involves a matrix computation, which we performed in Mathematica. With modest expertise in Mathematica programming the reader can verify this computation; a Mathematica notebook containing it is available at [7].

Let π_p denote the canonical map $\mathrm{SL}(4,\mathbb{Z}) \to \mathrm{SL}(4,\mathbb{Z}/p)$. Our claim is that if $p \equiv 1 \pmod{24}$ and if $k \in \mathbb{N}$ satisfies $16k^4 + 8k^2 + 25 \equiv 0 \pmod{p}$, then the image of $\Delta(3,3,4)$ under $\pi_p \circ \rho_k$ has an invariant two-dimensional subspace, hence cannot be the whole of $\mathrm{SL}(4,\mathbb{Z}/p)$. We note that by Dirichlet's theorem there are infinitely many primes p of form $p \equiv 1 \pmod{24}$; for the remainder of the proof we assume that p is of this form.

We first check that the claim is not vacuous, i.e. that $16x^4 + 8x^2 + 25$ has a root in $\mathbb{Z}/p$. Let ξ be a primitive element of $\mathbb{Z}/p$; then $\eta = \xi^{(p-1)/8}, \omega = \xi^{(p-1)/3}$ are primitive eighth and cube roots of 1 respectively. Motivated by the arithmetic of complex numbers, we see that $\eta - \eta^3$, $\omega - \omega^2$ are square roots of $2, -3$ respectively. The roots of $16x^4 + 8x^2 + 25$ in the complex numbers are $\frac{1}{2}(\pm\sqrt{2} \pm \sqrt{3}i)$, and these correspond now to obvious roots in $\mathbb{Z}/p$ of $16x^4 + 8x^2 + 25 \in (\mathbb{Z}/p)[x]$.

It is more convenient to work with matrices over $\mathbb{C}$ than over $\mathbb{Z}/p$. Let S be the subring of $\mathbb{C}$ generated by $\frac{1}{2}$, $e^{2\pi i/8}$, $e^{2\pi i/3}$. The assignment

$$\frac{1}{2} \mapsto \frac{p+1}{2} \ , \ e^{2\pi i/8} \mapsto \eta \ , \ e^{2\pi i/3} \mapsto \omega$$

defines an epimorphism of S onto $\mathbb{Z}/p$.

The characteristic polynomial of the matrix $\rho_k(ab)$ is $Q^4 + 1$, so it is conjugate to a diagonal matrix with diagonal entries the primitive eighth roots of 1. When the same conjugation is applied to $\rho_k(b)$, one finds that for an appropriate ordering of basis vectors the entries of the lower left 2×2 submatrix all vanish for k equal to any root in $\mathbb{C}$ of $16x^4 + 8x^2 + 25$. Projecting from $\mathrm{SL}(4,S)$ to $\mathrm{SL}(4,\mathbb{Z}/p)$, we see that likewise, given that k is a root in $\mathbb{Z}/p$ of $16x^4 + 8x^2 + 25 \in (\mathbb{Z}/p)[x]$, we may simultaneously conjugate the image under $\pi_p \circ \rho_k$ of ab to a diagonal matrix, and the image of b to a matrix with zero 2×2 lower left submatrix. Therefore the entire image of $\pi_p \circ \rho_k$ leaves invariant the subspace spanned by the first two vectors of the corresponding basis, and $\pi_p \circ \rho_k$ cannot be a surjection. $\square$

2 The Method

Here we give an outline of the method for finding the $\mathrm{SL}(4,\mathbb{Z})$ representations of Theorem 1, our aim being to provide sufficient detail for an interested reader to perform a similar computation. We consider only the "simplest" hyperbolic triangle group $\Delta(3,3,4) = \langle a,b \mid a^3 = b^3 = (a\,b)^4 = 1\rangle$, but the method is applicable to other triangle groups, and indeed to 3–manifold groups.

In what follows, A, B denote the inverses of a, b respectively. As is customary, we consider two representations to be equivalent if they are conjugate.

2.1 Computing the Hitchin component

Our starting point is the representation $\rho_0 : \Delta(3,3,4) \;\to\; \mathrm{PSL}(4,\mathbb{R})$ obtained by composing the $\mathrm{PSL}(2,\mathbb{R})$ representation, given by the action of $\Delta(3,3,4)$ on the hyperbolic plane, with the irreducible representation of $\mathrm{PSL}(2,\mathbb{R})$ into $\mathrm{PSL}(4,\mathbb{R})$ given by action on homogeneous polynomials of degree 3 in two variables. Here there is a minor technical detail, to which the Introduction has already alluded. Our representations of $\Delta(3,3,4)$ into $\mathrm{PSL}(n,\mathbb{R})$ do not lift into $\mathrm{SL}(n,\mathbb{R})$ for n even (this issue does not arise for odd n). Therefore, in order to work with matrices in the case where n is even, we are obliged to consider representations into $\mathrm{SL}(n,\mathbb{R})$ of the $\mathbb{Z}/2$ central extension of $\Delta(3,3,4)$ coming initially from the pullback of the representation $\rho_\infty : \Delta(3,3,4) \to \mathrm{PSL}(2,\mathbb{R})$ given by the short exact sequence

$$1 \to \mathbb{Z}/2 \to \mathrm{SL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R}) \to 1 \;.$$

Denote this group and the pulled back representation by $\widetilde{\rho}_\infty : \widetilde{\Delta}(3,3,4) \to \mathrm{SL}(2,\mathbb{R})$. Then $\widetilde{\Delta}(3,3,4)$ is generated by elements a, b of order 3, whose product ab has order 8; by continuity, all representations we construct will map the central element $(ab)^4$ to the negative of the identity matrix. Composing with the irreducible representation of $\mathrm{SL}(2,\mathbb{R})$ into $\mathrm{SL}(4,\mathbb{R})$ produces a corresponding pullback $\widetilde{\rho}_0 : \widetilde{\Delta}(3,3,4) \to \mathrm{SL}(4,\mathbb{R})$ of $\rho_0 : \Delta(3,3,4) \;\to\; \mathrm{PSL}(4,\mathbb{R})$.

As observed in the introduction, the restriction of the representation ρ_∞ to any surface group $K < \Delta(3,3,4)$ can be lifted to a genuine representation of K into $\mathrm{SL}(4,\mathbb{R})$ factoring through $\widetilde{\Delta}(3,3,4)$. By continuity this holds over the algebraic set of representations that we construct, a set which restricts to lie inside the Hitchin component of K. In this way it suffices to work with representations of $\widetilde{\Delta}(3,3,4)$. Henceforth, for notational simplicity we will suppress the "tilde" notation, without fear of causing confusion.

Small perturbations are applied to the generating matrices $\rho_0(a)$, $\rho_0(b)$, thus destroying the group relations, and Newton's method is applied to converge to a numerical representation ρ_1 of high accuracy, close to but distinct from ρ_0. It is observed that the coefficients of the characteristic polynomial of $\rho_1(Ab)$ are related as follows:

$$\text{charpoly}\,(\rho_1(Ab)) = 1 - vx + (u+v)x^2 - ux^3 + x^4 \ ,$$

but that u , v can apparently vary independently. Noting that for our "base" representation $\widetilde{\rho}_0$ we have $u = v = 5+3\sqrt{2}$, we run the Newton process again, this time directing the iteration to converge to nearby rational values of u , v, say $(u\,,\,v) = (19/2\,,\,10)$. This "directed" Newton iteration is implemented in the obvious way by inserting extra constraints, but in order for it to converge successfully it is usually necessary to control each step-length dynamically in terms of the magnitudes of the residuals of the previous step. With rational values of u , v it is found that traces of words all appear to be algebraic (of degree at most 2 as it happens), and we reasonably infer that the Hitchin component has dimension 2, and that we can use $u = \mathrm{tr}(Ab)$, $v = \mathrm{tr}(Ba)$ as parameters. We note that u, v are interchanged by precomposition with the automorphism of $\Delta(3,3,4)$ that exchanges the generators a, b.

The program for performing the above Newton process was written in the Mathematica language [11], the QR decomposition of a matrix being used for finding a least-squares solution of the overdetermined linear system at each iteration. We also use the symbolic computational abilities of Mathematica to check that generating matrices of a tautological representation satisfy the group relations; all other computations are performed by scripts in the PARI language [10], with the exception of the use of GAP [3] in Proposition 1.3.

Our immediate aim is to obtain a tautological representation for the two-dimensional $\mathrm{SL}(4,\mathbb{R})$ variety, *i.e.* matrices $a(u,v)$, $b(u,v)$ satisfying the group relations, and with entries in an extension of (hopefully small) finite degree over the transcendental extension $\mathbb{Q}(u,v)$. Assigning suitable values to the parameters u, v will then yield a point of the Hitchin component. The expressions for the matrix entries are obtained by polynomial interpolation over a 20×20 "grid" of representations, corresponding to rational points of the (u,v)–plane. This is essentially the technique described in [2], but here we used the "directed Newton iteration" approach to obtain these 400 representations, as less than an hour of computer time was needed.

In order to apply polynomial interpolation, the generating matrices for the grid representations must be conjugated in a consistent manner to some normalized form, so that the matrix entries become algebraic numbers. Typically this is achieved by choosing a basis for $\mathbb{R}^4$ involving eigenvectors of the generating matrices a , b, although experimentation may be needed to obtain a desirable result. We did not find a single "best" normalization, but instead obtained several tautological representations with complementary features. In each case the matrix entry field is of degree 4 over $\mathbb{Q}(u,v)$.

The trace field $\mathbb{T}$ is of degree 2 over $\mathbb{Q}(u,v)$:

$$\mathbb{T} \;=\; \mathbb{Q}(u,v)(\alpha)\ ,\ \text{where}$$
$$\alpha \;=\; \sqrt{u^2v^2 - 4(u^3+v^3) - 2uv(u+v) + 16uv + 5(u^2+v^2) - 2(u+v) - 7}\ .$$

The Galois automorphism $\tau : \alpha \mapsto -\alpha$ of $\mathbb{T}$ induces an involution of the Hitchin component, indeed of the whole representation variety.

We mention an important linear-algebraic property of τ. Let us fix for the moment a tautological representation $\rho(u,v)$; given a matrix $g(u,v)$ in the image of $\rho(u,v)$, let $G(u,v)$ denote $g(u,v)^{-1}$, and let $g(u,v)^*$ denote the result of transposing $g(u,v)$ and then applying τ to the entries (we could call $g(u,v)^*$ the alpha-conjugate transpose of $g(u,v)$). Then there exists a non-singular matrix $c(u,v)$, depending only on the parameters u, v, satisfying

$$G(u,v) \;=\; c(u,v)^{-1}\, g(v,u)^*\, c(u,v)\ .$$

Along the line $u = v$, we have $c(u,u)^* = c(u,u)$, and we see that $c(u,u)$ is analogous to an Hermitian matrix representing a sesquilinear form, and that the image of $\rho(u,u)$ consists of matrices respecting this form.

We remark that the simplest word in the generators a, b with trace involving α is the commutator $abAB$, with trace $(1/2)(uv - u - v - 1 - \alpha)$. At the "base" representation ρ_0, $\alpha = 0$.

The curve $\alpha = 0$ is shown in Figure 1. The base representation ρ_0 is situated at $u = v = 5+3\sqrt{2}$, the tip of the "northeasternmost" component of $\alpha = 0$. The Hitchin component is homeomorphic to $\mathbb{R}^2$, and can be considered as two half-planes glued together along the component of $\alpha = 0$ containing ρ_0. These half-planes correspond to the two choices of sign of the radical expression for α, and they project to the upper right-hand region of the figure. Henceforth, for convenience our representations will always be taken in the half-plane corresponding to $\alpha \geq 0$.

We note that in the region of Figure 1 immediately "southwest" of ρ_0, α is not real, so representations in this region cannot map into $\mathrm{SL}(4,\mathbb{R})$. Representations in the far southwest region have $\alpha \in \mathbb{R}$, as do those in the small oval, but they cannot take advantage of the special properties of the Hitchin component.

Here are the two simplest tautological representations that we found. For economy, we keep the notation a, b for the images of these generators.

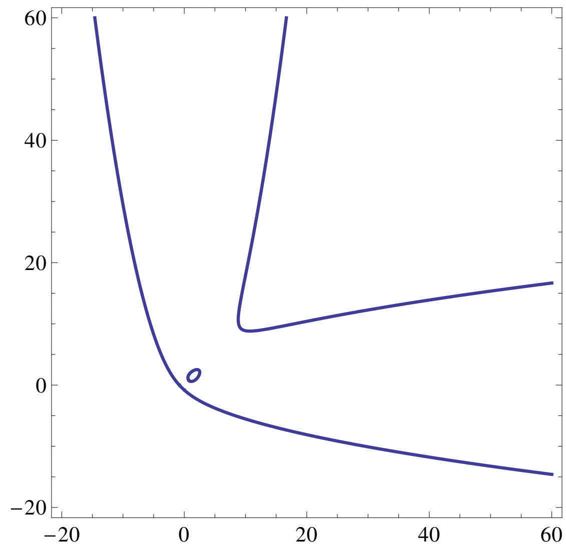

Figure 1: The locus of $\alpha = 0$

First tautological representation. Each of the generators a, b has a two-dimensional eigenspace with eigenvalue 1, and two one-dimensional eigenspaces with eigenvalues $\omega, \overline{\omega}$, where $\omega = e^{2\pi i/3}$. For this representation we make a careful choice of eigenvectors v_a, v_b of a, b respectively, each with eigenvalue 1. Our basis for $\mathbb{R}^4$ is then $\{v_a\,,\, b(v_a)\,,\, v_b\,,\, a(v_b)\}$; writing for the moment a', b' for the generating matrices with respect to the new basis, the eigenvectors v_a, v_b are chosen so that $a'_{22} = b'_{44} = 1\,,\, a'_{32} = -1$. The matrix entry field is $\mathbb{Q}(u,v)(\alpha,\beta)$, where the new element β is $\sqrt{(u+v)(u+v-20)+28}$; the complete result is

$$a = \begin{bmatrix} 1 & \frac{r_1}{s_1} & 0 & \frac{r_2}{2s_1} \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & \frac{(-u+v+\beta)r_3}{s_2} & -1 & -1 \end{bmatrix} \quad , \quad b = \begin{bmatrix} 0 & -1 & 0 & \frac{r_4}{s_1} \\ 1 & -1 & 0 & \frac{(u-v+\beta)\tau(r_3)}{2s_1} \\ 0 & -\frac{\tau(r_2)}{s_2} & 1 & -\frac{\tau(r_1)}{s_1} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ,$$

where

$$\begin{aligned}
r_1 &= -uv(u+v) + 7(u^2+v^2) + 6uv - 7(u+v) + \alpha\beta \\
r_2 &= -2u^3 + uv(5u+v) - 19u^2 - 29v^2 + 47u + 61v - 56 \\
&\qquad + (2u^2 - uv - 3u - 7v + 7)\beta + 3(-u+v-\beta)\alpha \\
r_3 &= 2u^2 - uv - 3u - 7v + 7 + 3\alpha \\
r_4 &= -uv(u+v) + 11(u^2+v^2) + 2uv - 27(u+v) + 28 + 3(u-v)\alpha \\
s_1 &= 2((u^2+v^2) - uv - 5(u+v) + 7) \\
s_2 &= 4(u^3 - 2u^2v + 4u^2 + 9v^2 - uv - 10u - 17v + 14)
\end{aligned}$$

This representation has the advantages (i) it is written over the reals, (ii) all but seven matrix entries of a, b are from the set $\{-1\,,\,0\,,\,1\}$, and (iii) the quantity β generating the matrix entry field as an extension of the trace field is a simple function of u, v. On the negative side, the coefficients r_i, s_i are mildly (though not overly) complicated. Furthermore, the curve $s_2 = 0$ cuts a swath through the Hitchin component, resulting in singularities along that curve. This curve of singular points includes ρ_0, but otherwise the situation is mitigated by the fact that one can avoid the curve if necessary by interchanging $u\,,\,v$.

Second tautological representation. For this representation our basis for $\mathbb{R}^4$ consists of eigenvectors of $a\,,\,b$ with eigenvalues $\omega\,,\,\overline{\omega}$. It was possible to choose these eigenvectors so that the resulting generators are lower triangular, upper triangular respectively:

$$a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{2(u^2+v^2)+uv-7(u+v)-1+3\alpha}{128} & -1 & \overline{\omega} & 0 \\ -1 & -\frac{64}{u^2+v^2-uv-2u-2v+4} & 0 & \omega \end{bmatrix},$$

$$b = \begin{bmatrix} \overline{\omega} & 0 & \frac{-32\,\overline{\omega}\,(2(u^2+v^2)+uv-7(u+v)-1-3\alpha)}{3\,(-u^3-v^3+6\,uv-8)} & -\frac{1}{3}(2+\overline{\omega}u+\omega v) \\ 0 & \omega & -\frac{1}{3}\,(2+\omega u+\overline{\omega} v) & \frac{\omega\,(u^3+v^3-6\,uv+8)}{192} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This representation has the clear advantages (i) the matrix entry field is merely $\mathbb{T}(i\sqrt{3})$, (ii) coefficients are simpler than those of the previous representation, and

(iii) denominators do not vanish anywhere in the Hitchin component. However, some matrix entries are no longer real.

2.2 Finding representations over $\mathbb{Q}$

An obvious necessary condition for the entries of a, b to be integers is that $\alpha = \sqrt{u^2v^2 - 4(u^3+v^3) - 2uv(u+v) + 16uv + 5(u^2+v^2) - 2(u+v) - 7}$ should be an integer. Figure 2 shows a scatterplot of points (u, v) satisfying this condition.

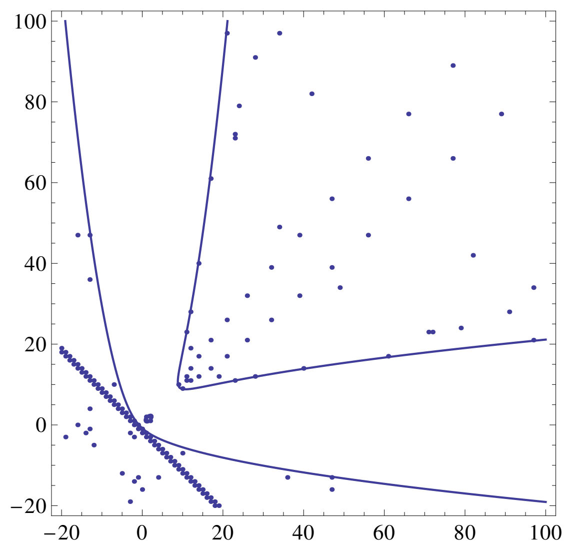

Figure 2: Points in the (u, v)–plane where $\alpha \in \mathbb{Z}$

With modest effort one notices that some of the points in the Hitchin component are organized into a slim parabola-like pattern, and with this cue it is then easy to determine by polynomial interpolation that the substitution

$$u = \frac{1}{2}n^2 + \frac{1}{2}n + 11 \ , \ v = \frac{1}{2}n^2 - \frac{1}{2}n + 11 \quad (n \in \mathbb{Z})$$

in the expression for α yields the integer value

$$\alpha \ = \ \frac{1}{4}(7+n^2)(24+n^2) \ .$$

It is incumbent on us to try to conjugate representations in this special family to representations over $\mathbb{Q}$, as the integrality of the character will then guarantee that they can be written over $\mathbb{Z}$, see [6], Proposition 2.1.

In light of the frequent need for finding the minimal field over which a given representation can be written, we continue the discussion in a slightly more general context, and provide a computational tool that is often effective in achieving a field reduction.

Let L_1 , $L_2 \subset \mathbb{C}$ be distinct number fields with compositum M, and let

$$\rho_1 : G \to \mathrm{SL}(n, L_1) \quad , \quad \rho_2 : G \to \mathrm{SL}(n, L_2)$$

be absolutely irreducible group representations that are conjugate, *i.e.* there exists $c \in \mathrm{GL}(n, M)$, such that $c^{-1}(\rho_1(g))\, c = \rho_2(g)$ for each $g \in G$. Schur's Lemma implies that the centralizer of $\rho_i(G)$ consists only of scalar matrices, whereby the conjugating matrix c is determined up to multiplication by non-zero scalars.

By hypothesis, G acts via ρ_i on the L_i–vector space L_i^n $(i = 1, 2)$. Let $K = L_1 \cap L_2$.

Observation. *In addition to acting via* ρ_1 *on* L_1^n, G *also acts via* ρ_1 *on* $c\,(L_2^n)$. *Therefore* G *acts via* ρ_1 *on the* K*–vector space* $V_1 = L_1^n \cap c\,(L_2^n)$.

Lemma 2.0.1 *The* K*-vector space* $V_1 = L_1^n \cap c\,(L_2^n)$ *has dimension at most* n.

Proof. We begin with a simple observation: Let L be a subfield of $\mathbb{C}$, and let S be a linearly independent subset of the L-vector space L^n. Then S is also a linearly independent subset of the $\mathbb{C}$-vector space $\mathbb{C}^n$. The reason is that if a homogeneous system of linear equations, with coefficients in L has a non-trivial solution in $\mathbb{C}$, then it also has a non-trivial solution in L.

We now argue as follows. Suppose that $\{v_1, \ldots, v_r\}$ lie in $L_1^n \cap c\,(L_2^n)$ and are linearly independent over K. Setting $w_j = c^{-1}(v_j) \in L_2^n$, we claim that these are linearly independent over L_2, thus proving the result.

To see this, suppose to the contrary that they are linearly dependent; choose the shortest such dependence. By renumbering we may suppose it involves the vectors $\{w_1, \ldots, w_s\}$ for some $s \le r$. Necessarily the dependence $\sum_{j \le s} \lambda_j w_j = 0$, $\lambda_j \in L_2$ has all coefficients non-zero. With no loss of generality, $\lambda_s = 1$. By minimality and the simple observation, this is the unique nontrivial linear dependence up to scaling amongst these vectors, even over $\mathbb{C}$, in particular, it is now uniquely defined by the property that $\lambda_s = 1$.

Applying the map c, we obtain $\sum_{j \le s} \lambda_j v_j = 0$. Up to scaling, this linear dependence must be the unique such expression amongst the set $\{v_1, \ldots, v_s\}$ even over $\mathbb{C}$, else we find a shorter expression and apply c to it, to give a shorter dependence amongst the w_j's.

Recalling that $\lambda_s = 1$, follows from another application of the simple observation that $\lambda_j \in L_1$, hence $\lambda_j \in K$. This is a contradiction and the result follows. □

Of course, the typical situation will be that V_1 is the zero subspace, however different choices of the conjugating matrix c within their (1–dimensional) equivalence class will potentially produce different dimensions for the K–vector space V_1. It is this fact that we propose to exploit.

Suppose that there is some scaling of c with the property that there is a non-zero vector $v \in V_1$ say. The orbit of v under the action ρ_1 is contained in the K-vector space V_1, so gives an invariant K-subspace W of dimension at most n in V_1. Considering $W \otimes L_1$, this is a nonzero ρ_1-invariant subspace of L_1^n and hence all of L_1^n, by irreducibility. It follows that with this choice of v, we construct a conjugacy of the representation ρ_1 into $\mathrm{GL}(n,K)$.

The following proposition provides a practical method for searching for suitable c and v.

Proposition 2.1 *Suppose that some K–linear combination S of matrices in $\rho_1(G)$ has nullspace $N \subset L_1^n$ of dimension precisely 1. Let $v \in N$ be any non-zero vector.*

Then v is contained in the K–vector space $V_1 = L_1^n \cap c\,(L_2^n)$ associated to some matrix c conjugating ρ_1 to ρ_2.

Proof Let $v \in N$, and let $c_1 \in \mathrm{GL}(n,M)$ be any matrix conjugating ρ_1 to ρ_2. Then $w = c_1^{-1}(v)$ is in the kernel of $c_1^{-1} S\, c_1$, viewed as a matrix in $\mathrm{End}(M^n)$. Since this kernel has dimension 1 over M and $c_1^{-1} S\, c_1 \in \mathrm{End}(L_2^n)$, there exists non-zero $\mu \in M$ such that $\mu w \in L_2^n$ (for example we could take μ to be the reciprocal of any non-zero coordinate of w). It follows that if we take $c = (1/\mu)\, c_1$, then $v \in c\,(L_2^n)$ as desired. □

Let us return to the context at hand, namely $G = \Delta(3,3,4)$, $n = 4$. In our search for infinitely many representations $G \to \mathrm{SL}(4,\mathbb{Z})$, it seems propitious to focus on the special family of representations in the Hitchin component lying on the parabola of Figure 2, and defined by $u = \frac{1}{2}n^2 + \frac{1}{2}n + 11$, $v = \frac{1}{2}n^2 - \frac{1}{2}n + 11$ $(n \in \mathbb{Z})$. For each n, we take ρ_1 , ρ_2 to be the representations obtained from these values of the parameters u, v in the first and second tautological representations, respectively. Thus

$$L_1 = \mathbb{Q}\left(\sqrt{n^4 + 24n^2 + 72}\right) \quad , \quad L_2 = \mathbb{Q}\left(i\sqrt{3}\right)$$

and

$$M = \mathbb{Q}\left(\sqrt{n^4 + 24n^2 + 72}\,,\, i\sqrt{3}\right) \quad , \quad K = \mathbb{Q} \ .$$

Here is an example of Proposition 2.1 at work for the case $n = 1$, *i.e.* $(u, v) = (12, 11)$. Where space is a concern we denote elements of the field M by vectors relative to the $\mathbb{Q}$–basis $\{1\,,\ \sqrt{97}\,,\ i\sqrt{3}\,,\ i\sqrt{291}\}$. Solving a linear system in the matrix entries of $\rho_i(a)\,,\ \rho_i(b)\ \ (i = 1, 2)$, we find that the matrix

$$\begin{bmatrix} \frac{1}{2}(1,0,-1,0) & \frac{1}{2275}(144,-16,-16,-48) & \frac{1}{975}(1552,-112,-112,16) & \frac{1}{12}(35,-5,7,-1) \\ 1 & \frac{1}{2275}(48,-80,-80,-16) & 0 & 0 \\ \frac{1}{24}(-15,-3,5,1) & \frac{1}{6825}(2088,72,1672,104) & \frac{1}{975}(-24,-24,-24,8) & \frac{1}{6}(3,0,2,0) \\ 0 & 0 & \frac{1}{975}(-24,24,24,8) & \frac{1}{12}(-9,0,1,0) \end{bmatrix}$$

conjugates ρ_1 to ρ_2, and we take c_1 to be this matrix.

A computer search of a few seconds reveals that the matrix $S = 3I + \rho_1(-2a + b - B)$ has 1–dimensional kernel, containing the vector

$$v = \left(-10 + 2\sqrt{97}\ ,\ 4\ ,\ -9 - \sqrt{97}\ ,\ 4\right)\ \in\ L_1^4\ .$$

One can then compute that

$$c_1^{-1}(v) = \left(\frac{1}{3}(6,0,20,2)\,,\ \frac{1}{192}(2925,195,3055,325)\,,\ -\frac{1}{32}(845,65,585,65)\,,\ \frac{1}{7}(2,2,-62,-6)\right)$$

and dividing by the last component of this vector yields

$$\left(\frac{1}{12}\left(-9 + i\sqrt{3}\right)\ ,\ \frac{1}{768}\left(-1365 + 455i\sqrt{3}\right)\ ,\ \frac{1}{64}\left(130 - 65i\sqrt{3}\right)\ ,\ 1\right)\ \in\ L_2^4\ .$$

We therefore choose

$$v = \left(-10 + 2\sqrt{97}\ ,\ 4\ ,\ -9 - \sqrt{97}\ ,\ 4\right) \quad , \quad c = \frac{1}{7}(2,2,-62,-6)\, c_1\ .$$

We observe that the $\mathbb{Q}$–vector space with basis $\{\rho_1(a)(v)\,,\ \rho_1(b)(v)\,,\ \rho_1(A)(v)\,,\ \rho_1(B)(v)\}$ is invariant under the action of $\rho_1(\Delta(3,3,4))$, providing us with the $\mathbb{Q}$–representation

$$a \mapsto \begin{bmatrix} 0 & -\frac{2}{3} & \frac{2}{3} & \frac{7}{3} \\ 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 & -2 \\ 0 & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \quad , \quad b \mapsto \begin{bmatrix} -\frac{1}{3} & 0 & 2 & \frac{2}{3} \\ \frac{5}{3} & 0 & -4 & -\frac{1}{3} \\ 0 & 0 & 1 & 0 \\ \frac{1}{3} & 1 & 1 & \frac{1}{3} \end{bmatrix}$$

Remark When implementing Proposition 2.1, one does not need to perform any of the calculations regarding c and ρ_2. One merely needs to find suitable S and then compute the action of $\rho_1(G)$ on the K–vector space spanned by the orbit of some non-zero $v \in \operatorname{Ker} S$. We performed all the calculations for this example so as to illustrate the workings of the proposition.

There is certainly no guarantee in general that one can reduce the matrix entry field of a representation ρ_1 from L_1 to $K = L_1 \cap L_2$. For our subfamily $(u,v) = (\frac{1}{2}n^2 + \frac{1}{2}n + 11\ ,\ \frac{1}{2}n^2 - \frac{1}{2}n + 11)$ it seems that reduction to $\mathbb{Q}$ is not possible for n a multiple of 3. However, for $n = k^2+1$ $(k \in \mathbb{Z}\,,\, k \geq 0)$ reduction was achieved with $S = kI - 2\rho_1(a) + (k-2)(\rho_1(B) - \rho_1(b))$; application of Proposition 2.1, with k kept general, produced the following formula:

$$a \mapsto \begin{bmatrix} 0 & \frac{-8+4k+2k^2}{-8k+k^4} & \frac{2}{k} & \frac{-8+4k-2k^2-2k^3-k^4}{-8k+k^4} \\ 0 & \frac{-4}{4k+2k^2+k^3} & \frac{-2+k}{k} & \frac{-4}{4k+2k^2+k^3} \\ 1 & \frac{-8-2k+k^3}{-8+k^3} & 0 & \frac{2k+2k^2+k^3}{-8+k^3} \\ 0 & \frac{4+4k+2k^2+k^3}{4k+2k^2+k^3} & \frac{2-k}{k} & \frac{4+4k+2k^2+k^3}{4k+2k^2+k^3} \end{bmatrix}$$

$$b \mapsto \begin{bmatrix} \frac{-2+k}{k} & 0 & \frac{14-2k^3}{k} & \frac{2}{k} \\ \frac{2-2k+k^2}{k} & 0 & \frac{-28+16k-6k^2+3k^3-2k^4+k^5}{2k} & \frac{-2+k}{k} \\ 0 & 0 & 1 & 0 \\ \frac{-2+3k-k^2}{k} & 1 & \frac{28-30k+6k^2-3k^3+4k^4-k^5}{2k} & \frac{2-k}{k} \end{bmatrix}$$

2.3 The final conjugation

We now have an infinite family of representations $\Delta(3,3,4) \to \mathrm{SL}(4,\mathbb{Q})$ in the Hitchin component, and it is guaranteed by [6], Proposition 2.1 that they are conjugate to representations over $\mathbb{Z}$, on account of their characters being integral.

As mentioned above, a "good basis" for the purpose of simplifying a representation often involves eigenvectors of the generators. Let $a_Q\,,\, b_Q$ be the $\mathbb{Q}$–matrices obtained at the end of §2.2. Our initial observation is that the eigenspaces with eigenvalue 1 of the *transposes* of $a_Q\,,\, b_Q$ are very much simpler than those of $a_Q\,,\, b_Q$. Indeed, the eigenspace in question for a_Q^T has basis $\{(0,1,0,1)\,,\ (1,1,1,0)\}$, and that for b_Q^T has basis $\{(\frac{k}{2},1,0,1)\,,\ (0,0,1,0)\}$ (vectors relative to the standard basis for $\mathbb{R}^4$). We therefore replace the generators a_Q , b_Q by their transposes; the group relations are still satisfied and the character is unchanged. Let us write a_1 , b_1 for a_Q^T , b_Q^T, respectively.

Next, the representation $\langle a_1 , b_1\rangle$ is conjugated so as to put b_1 into rational canonical form, the motivation being that the new matrix for b will certainly have integer entries, and that with good luck the matrix for a will also be simplified. A small amount of experimentation is performed with bases of form $\{v , b_1(v) , b_1(b_1(v)) , w\}$, where v , w are candidates chosen from the above eigenspaces of a_1 , b_1 respectively. It is rapidly found that the winning pair is $v = (0,1,0,1)$, $w = (0,0,k-2,0)$. The results for a, b are:

$$\begin{bmatrix} 1 & -4+k^2+k^3 & \frac{1}{2}(6-5k-k^2+k^3+k^4) & \frac{-4+k^2+k^3}{4+2k+k^2} \\ 0 & -3-2k-k^2 & -\frac{1}{2}k(1+k)^2 & -1 \\ 0 & 4+2k+k^2 & \frac{1}{2}(-2+3k+2k^2+k^3) & 1 \\ 0 & -\frac{1}{2}(-2+k)(4+2k+k^2)(3+2k+k^2)) & -\frac{1}{4}(4+2k+k^2)(1-4k-2k^2+k^4) & \frac{1}{2}(8+k-k^3) \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The denominator $p = 4+2k+k^2$ can conveniently be removed by conjugation by the diagonal matrix whose $(4,4)$–entry is p and whose other diagonal entries are 1; the effect is to multiply the last columns by p and divide the last rows by p:

$$\begin{bmatrix} 1 & -4+k^2+k^3 & \frac{1}{2}(6-5k-k^2+k^3+k^4) & -4+k^2+k^3 \\ 0 & -3-2k-k^2 & -\frac{1}{2}k(1+k)^2 & -4-2k-k^2 \\ 0 & 4+2k+k^2 & \frac{1}{2}(-2+3k+2k^2+k^3) & 4+2k+k^2 \\ 0 & \frac{1}{2}(2-k)(3+2k+k^2)) & -\frac{1}{4}(1-4k-2k^2+k^4) & \frac{1}{2}(8+k-k^3) \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For odd k, all matrix entries are now integers. For even k, the only entry that is not an integer is the $(4,3)$–entry of a, which has denominator 4. This can be fixed in the same manner as the previous step, conjugating by the diagonal matrix whose $(4,4)$–entry is $\frac{1}{4}$, all other diagonal entries being 1. The result for even k is:

$$\begin{bmatrix} 1 & -4+k^2+k^3 & \frac{1}{2}(6-5k-k^2+k^3+k^4) & \frac{1}{4}(-4+k^2+k^3) \\ 0 & -3-2k-k^2 & -\frac{1}{2}k(1+k)^2 & -\frac{1}{4}(4+2k+k^2) \\ 0 & 4+2k+k^2 & \frac{1}{2}(-2+3k+2k^2+k^3) & \frac{1}{4}(4+2k+k^2) \\ 0 & 2(2-k)(3+2k+k^2)) & -1+4k+2k^2-k^4 & \frac{1}{2}(8+k-k^3) \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For this last family we have

$$u = \mathrm{tr}(\rho_k(Ab)) = \frac{1}{2}(k^4 + 3k^2 + 24) \quad , \quad v = \mathrm{tr}(\rho_k(Ba)) = \frac{1}{2}(k^4 + k^2 + 22) \ .$$

Writing $k = 2r$, we see that this family, parametrized by r instead of k, is conjugate to the family given in the announcement at the beginning of the article. We chose the alternate version for the announcement, as the matrix entries are a little simpler.

3 The case $n = 5$.

Using exactly the same method applied with more fortitude, one obtains the analogous:

Theorem 3.1 *The family of representations of the triangle group*

$$\rho_k : \Delta(3,3,4) = \langle a, b \mid a^3 = b^3 = (a\,b)^4 = 1 \rangle \to \mathrm{SL}(5,\mathbb{R})$$

given by

$$\rho_k(a) = \begin{bmatrix} 1 & 0 & -3-2k-8k^2 & -1+10k+32k^3 & -5-16k^2 \\ 0 & 4(-1+k) & -13-4k & 3+16(1+k)^2 & -4+16k \\ 0 & 1-k+4k^2 & 3-2k+8k^2 & -2(1+3k+16k^3) & 3+16k^2 \\ 0 & k & 2k & 1-2k-8k^2 & 1+4k \\ 0 & 0 & 3k & 3(-1+k-4k^2) & -2 \end{bmatrix}$$

$$\rho_k(b) = \begin{bmatrix} 0 & 0 & -3-2k-8k^2 & -1+10k+32k^3 & -5-16k^2 \\ 0 & 1 & 3+4k & -13-8k-16k^2 & 4-16k \\ 0 & 0 & -2(1+k+4k^2) & 6k+32k^3 & -3-16k^2 \\ 1 & 0 & -2(1+k) & -1+2k+8k^2 & -1-4k \\ 2k & 0 & 1-2k & -4k & 1 \end{bmatrix}$$

are discrete and faithful for every $k \in \mathbb{R}$.

We remark that the discussion concerning projective representations is unnecessary here, since $n = 5$ is odd. As before, a calculation shows that the image groups are Zariski dense in $\mathrm{SL}(5,\mathbb{R})$. Futhermore, the method of § 1.1 shows that there is a subsequence (ρ_{k_i}) whose images contain pairwise nonconjugate surface subgroups of $\mathrm{SL}(5,\mathbb{Z})$. Here we require $\mathbb{Z}/p$ to contain square roots of -1, so that the matrix ab can be diagonalized, and we obtain an invariant two dimensional subspace for the image in $\mathrm{SL}(5,\mathbb{Z}/p)$ if k is set to either square root of $-\frac{19}{48}$ (see [7]). A simple calculation involving quadratic reciprocity shows that $\mathbb{Z}/p$ contains square roots of $-1, -\frac{19}{48}$ for p in any one of several congruence classes modulo 228, in particular for $p \equiv 1 \pmod{228}$.

Corollary 3.2

(i) For all non-negative integral values of k the image groups $\rho_k(\Delta(3,3,4))$ are Zariski dense in $\mathrm{SL}(5,\mathbb{R})$;

(ii) There exists a surface subgroup Γ of $\Delta(3,3,4)$ and a subsequence (ρ_{k_i}) such that the images $\rho_{k_i}(\Gamma)$ are pairwise non-conjugate surface subgroups of $\mathrm{SL}(5,\mathbb{Z})$.

4 Symplectic Representations

We close with some examples of a somewhat different nature. These will be investigated more systematically in future work.

For even n, the image of $\mathrm{SL}(2,\mathbb{R})$ under the irreducible representation $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R})$ is (up to conjugacy) contained in the symplectic group $\mathrm{Sp}(n,\mathbb{R})$. One can then look at those representations in the Hitchin component that map into $\mathrm{Sp}(n,\mathbb{R})$, and consider the possibility of representations into $\mathrm{Sp}(n,\mathbb{Z})$. For our representation variety of the triangle group $\Delta(3,3,4)$, those representations for which the parameters u, v are equal have image in $\mathrm{Sp}(n,\mathbb{R})$; however, the only representation in the Hitchin component on the line $u = v$ and with integral character appears to be at $u = v = 11$, and this seems not to be conjugate to a representation over $\mathbb{Z}$.

It is proved in [1] that hyperbolic triangle groups of type (p,q,r) (*i.e.* satisfying $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$) are rigid in $\mathrm{SL}(3,\mathbb{R})$ for $p = 2$, and otherwise have a two-dimensional Hitchin component. Experiments suggest strongly that hyperbolic groups of type $(2,3,r)$ are still rigid in $\mathrm{SL}(4,\mathbb{R})$, but that those of type $(2,q,r)$ with $4 \leq q \leq r$ (excluding $(2,4,4)$ which is Euclidean) have non-trivial Hitchin components in $\mathrm{SL}(4,\mathbb{R})$. In particular, computations show that each of the groups $\Delta(2,4,5)$, $\Delta(2,5,5)$ has a two-dimensional Hitchin component, surprisingly consisting entirely of representations into $\mathrm{Sp}(4,\mathbb{R})$.

In the case of $\Delta(2,5,5)$, we found an infinite sequence of representations into $\mathrm{Sp}(4,\mathbb{R})$ with integral character, but were unable to conjugate these to representations over $\mathbb{Z}$ (although in partial mitigation we were able to find an infinite sequence of representations into $\mathrm{Sp}(4,\mathbb{Q})$).

For $\Delta(2,4,5)$ we were able to find numerous representations into $\mathrm{Sp}(4,\mathbb{Z})$. It seems likely that our family is an infinite one, but it is a more subtle question in this setting, apparently relying upon some questions involving elliptic curves. Specifically, we have trace field $\mathbb{T} = \mathbb{Q}(u,v)(\alpha)$, where

$$\alpha = \sqrt{u^2v^2 - 4u^4 + 20u^2v - 4v^3 - 16u^2 - 12v^2 + 16} \ ,$$

and for representations over $\mathbb{Z}$ we require the expression under the radical sign to be the square of an integer, say

$$u^2v^2 - 4u^4 + 20u^2v - 4v^3 - 16u^2 - 12v^2 + 16 = y^2 \ .$$

For each integer value of u this defines an elliptic curve, and the question is whether these curves have integer points for infinitely many u.

Here is an example of a representation of $\Delta(2,4,5)$ into $\mathrm{Sp}(4,\mathbb{Z})$:

$$a \mapsto \begin{bmatrix} 1 & 0 & 1 & 0 \\ -7 & 5 & -20 & 20 \\ 0 & 0 & 0 & 1 \\ 4 & -2 & 9 & -6 \end{bmatrix} \quad , \quad b \mapsto \begin{bmatrix} 6 & -2 & 10 & -5 \\ -28 & 13 & -60 & 39 \\ 5 & -2 & 10 & -5 \\ 27 & -11 & 53 & -30 \end{bmatrix}$$

The reader can check that after the usual projectivization this is a representation of the group $\langle a, b \,|\, a^4 = b^5 = (ab)^2 = 1\rangle$, and that the above two matrices respect the skew-symmetric form given by

$$\begin{bmatrix} 0 & 2 & 1 & 1 \\ -2 & 0 & 3 & 2 \\ -1 & -3 & 0 & 0 \\ -1 & -2 & 0 & 0 \end{bmatrix}$$

Since this representation lies in the Hitchin component, it is discrete and faithful. Moreover, application of Lubotsky's theorem [8] with $p = 5$ shows that it is Zariski dense in $\mathrm{Sp}(4,\mathbb{R})$.

We remark that there is a "cheap" way of creating faithful $\mathrm{Sp}(4,\mathbb{Z})$–representations of surface groups by taking suitable representations into $\mathrm{SL}(2,\mathbb{C})$ and then representing each complex entry by a 2×2 matrix. For example, we can do this with the parabolic representation of the figure-eight knot group:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \ , \ \begin{bmatrix} 1 & 0 \\ -e^{2\pi i/3} & 1 \end{bmatrix} \ \longrightarrow \ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \ , \ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

However, these representations are not Zariski dense in the 10-dimensional Lie group $\mathrm{Sp}(4,\mathbb{R})$ as they lie in a Zariski closed subgroup of dimension 6, namely the "inflated" 4×4 version of $\mathrm{SL}(2,\mathbb{C})$.

Finally, investigations show that the Hitchin component for representations of $\Delta(3,3,4)$ into $\mathrm{SL}(6,\mathbb{R})$ is four-dimensional, and that it has a two-dimensional subvariety consisting of representations into the symplectic group $\mathrm{Sp}(6,\mathbb{R})$. Owing to the relative complexity of these representations, it seems unlikely that they will yield representations of $\Delta(3,3,4)$ into $\mathrm{Sp}(6,\mathbb{Z})$.

5 Acknowledgments

The first author is partially supported by the National Science Foundation, and the second author is partially supported by the Simons Foundation.

We are indebted to the referee for helpful comments.

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