

# Virtually spinning hyperbolic manifolds

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## 1 Introduction

Let  $M$  be an orientable Riemannian manifold and let  $w_2(M)$  denote its second Stiefel-Whitney class. Then  $M$  admits a spin structure or said to be *spinnable* if  $w_2(M) = 0$ ; we refer the reader to §2 and §3 for more details, definitions and background.

It is well-known that all compact orientable surfaces, and compact orientable 3-manifolds are spinnable, however, the situation in higher dimensions is a good deal more subtle. For example, a well-known consequence of the Hirzebruch Signature Theorem is that the signature of any closed orientable hyperbolic 4-manifold is 0, and so according to Rochlin's Theorem (see [9]) there is no obstruction to having a spin structure. On the other hand, whether a closed hyperbolic manifold of dimension 4 (or more) is spinnable, is much harder to establish.

However, on p. 553 of his paper [20], Sullivan notes that his previous work with Deligne [5], can be used to show that if  $M^n$  is a finite volume hyperbolic  $n$ -manifold, then  $M^n$  has a finite cover that is stably parallelizable, and hence spinnable (see Remark 2.2 below).

The aim of this note is to give a simple proof of Sullivan's virtually spinning result that seems not to have been noticed previously. Furthermore, in the setting of arithmetic hyperbolic manifolds of simplest type we will provide a sharper version of Sullivan's result in many cases (see §5 for definitions).

**Theorem 1.1.** *Assume that  $n \geq 4$  and let  $M^n = \mathbb{H}^n/\Gamma$  be a well-located arithmetic hyperbolic  $n$ -manifold of simplest type from the admissible quadratic form  $f$  defined over the totally real field  $k$ . Then  $M^n$  admits a finite cover of degree  $C(k, f)$  that is spinnable. The constant  $C(k, f)$  is an effectively computable constant depending on  $k$  and  $f$ .*

The question as to whether all orientable hyperbolic manifolds of finite volume in dimensions  $\geq 4$  are spinnable, seems to be open. To that end, in §7 we point out that in each dimension  $n \geq 5$  there are infinitely many finite volume (non-compact) orientable hyperbolic  $n$ -manifolds that are not spinnable.

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## 2 Spin groups and spin structures

We assume  $n \geq 4$  throughout this section. General references for what follows are [9] and [12]. We will restrict our discussion to spin structures associated with the Tangent Bundle of a Riemannian manifold, rather than an arbitrary vector bundle.

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The spin group in dimension  $n$  will be denoted by  $\text{Spin}(n)$ . Since  $n \geq 4$ ,  $\text{Spin}(n)$  is the universal 2-fold covering group of the special orthogonal group  $\text{SO}(n)$  (with covering map  $j$ ), and determines a short exact sequence:

$$1 \rightarrow \{\pm 1\} \rightarrow \text{Spin}(n) \rightarrow \text{SO}(n) \rightarrow 1.$$

Now let  $M$  be a connected orientable Riemannian manifold of dimension  $\geq 4$  with Tangent Bundle  $TM$ . Denote by  $\text{SO}(TM) \rightarrow M$ , the  $\text{SO}(n)$ -principal bundle of oriented orthonormal frames on  $TM$ .  $M$  admits a spin structure or is *spinable* if there is a principal  $\text{Spin}(n)$ -bundle  $\text{Spin}(TM) \rightarrow M$  together with a 2-fold covering map  $\eta : \text{Spin}(TM) \rightarrow \text{SO}(TM)$  such that  $\eta(pg) = \eta(p)j(g)$  for all  $p \in \text{Spin}(TM)$  and  $g \in \text{Spin}(n)$ .

Not every orientable Riemannian manifold admits a spin structure, the obstruction to this is the second Stiefel-Whitney class. With  $M$  as above, we let  $w_2(M) \in H^2(M, \mathbb{Z}/2\mathbb{Z})$  denote the second Stiefel-Whitney class of  $TM$ . We summarize what we need in the following proposition.

**Proposition 2.1.**  *$M$  admits a spin structure if and only if  $w_2(M) = 0$ .*

**Remark 2.2.** *A smooth orientable manifold  $M^n$  is stably parallelizable if its tangent bundle is stably trivial; i.e.  $TM \oplus E_1 = E_2$  where  $E_i$  are trivial vector bundles for  $i = 1, 2$ . When this is the case, since Stiefel-Whitney classes are invariants of the stable equivalence class of a vector bundle, it follows that  $w_2(M) = 0$ , and so this proves Sullivan's result mentioned in §1.*

**Remark 2.3.** *If an oriented 4-manifold  $X$  is spinable, then the intersection form of  $X$  is even. The converse holds if  $H_1(X, \mathbb{Z})$  has no 2-torsion (see [9, Chapter II.4]).*

## 3 Hyperbolic manifolds and spin structures

### 3.1 Some notation

Let  $J_n$  be the diagonal matrix associated to the quadratic form  $x_0^2 + x_1^2 + \dots + x_{n-1}^2 - x_n^2$ . We identify hyperbolic space  $\mathbb{H}^n$  with  $\{x \in \mathbb{R}^{n+1} : J_n x = -1, x_{n+1} > 0\}$ , and letting

$$\text{O}(n, 1) = \{X \in \text{GL}(n+1, \mathbb{R}) : X^t J_n X = J_n\}.$$

we can identify  $\text{Isom}(\mathbb{H}^n)$  with the connected component of the identity of  $\text{O}(n, 1)$ , denoted by  $\text{O}_0(n, 1)$ . This is also the subgroup of  $\text{O}(n, 1)$  preserving the hyperboloid  $\{x \in \mathbb{R}^{n+1} : J_n x = -1, x_{n+1} > 0\}$ . Equivalently,  $\text{O}_0(n, 1) = \text{PO}(n, 1)$  (the central quotient of  $\text{O}(n, 1)$ ). With this notation,  $\text{Isom}^+(\mathbb{H}^n) = \text{SO}_0(n, 1)$ , the index 2 subgroup in  $\text{O}_0(n, 1)$  which is the connected component of the identity of  $\text{SO}(n, 1)$ .

Although we will not make explicit use of this, if  $M = \mathbb{H}^n/\Gamma$  is a finite volume orientable hyperbolic  $n$ -manifold, then it is a well-known consequence of Mostow-Prasad Rigidity that we can conjugate  $\Gamma$  to be a subgroup of  $\text{SO}_0(n, 1)$  where the elements all have matrix entries in a real number field  $k$  (which we can take to be minimal).

### 3.2 Spin structures on hyperbolic manifolds

The recent preprint [19] contains a particularly useful discussion of spin structures on hyperbolic manifolds, and we refer the reader there for a fuller discussion. We continue to assume that  $n \geq 4$ . As with  $\text{SO}(n)$ , the group  $\text{SO}_0(n, 1)$  has a universal 2-fold cover, which following [19], we denote by  $\text{Spin}^+(n, 1)$  (with covering map  $\phi$ ) and as above there is an exact sequence:

$$1 \rightarrow \{\pm 1\} \rightarrow \text{Spin}^+(n, 1) \rightarrow \text{SO}_0(n, 1) \rightarrow 1.$$

Now if  $M = \mathbb{H}^n/\Gamma$  is an orientable finite volume hyperbolic manifold, it is well-known that the  $\mathrm{SO}(n)$ -principal bundle of oriented orthonormal frames on  $TM$  can be identified with  $\Gamma\backslash\mathrm{SO}_0(n, 1)$  (see [19, Section 2] for example). Via the exact sequence given above, we can construct an extension  $\bar{\Gamma} < \mathrm{Spin}^+(n, 1)$  with  $\phi(\bar{\Gamma}) = \Gamma$ . Note that  $\bar{\Gamma}\backslash\mathrm{Spin}^+(n, 1) \cong \Gamma\backslash\mathrm{SO}_0(n, 1)$  since:

$$\Gamma\backslash\mathrm{SO}_0(n, 1) \cong (\{\pm 1\}\backslash\bar{\Gamma})\backslash(\{\pm 1\}\backslash\mathrm{Spin}^+(n, 1)) \cong \bar{\Gamma}\backslash\mathrm{Spin}^+(n, 1).$$

The following is implicit in the proof of [19, Theorem 2.1], we include a brief sketch.

**Lemma 3.1.** *We follow the notation established above. Let  $M = \mathbb{H}^n/\Gamma$  be an orientable finite volume hyperbolic manifold with  $n \geq 4$ .*

1. *Suppose that there is a subgroup  $H < \bar{\Gamma}$  of index 2 so that  $\phi$  maps  $H$  isomorphically onto  $\Gamma$  (or equivalently,  $H \cap \{\pm 1\} = 1$ ), then  $M$  is spinnable.*
2. *Let  $D < \bar{\Gamma}$  be of finite index containing an index 2 subgroup  $D_0$  such that  $D_0 \cap \{\pm 1\} = 1$ . Let  $\Delta = \phi(D) < \Gamma$ , then  $\mathbb{H}^n/\Delta$  is a finite cover of  $M$  that is spinnable.*

*Proof.* From above we have an exact sequence:

$$1 \rightarrow \{\pm 1\} \rightarrow \bar{\Gamma} \rightarrow \Gamma \rightarrow 1.$$

If  $H$  is a subgroup of index 2 as claimed, then  $H\backslash\mathrm{Spin}^+(n, 1)$  is a principal  $\mathrm{Spin}(n)$ -bundle that double covers  $\bar{\Gamma}\backslash\mathrm{Spin}^+(n, 1)$  which from the discussion above is  $\cong \Gamma\backslash\mathrm{SO}_0(n, 1)$ . Moreover, as described in the proof of [19, Theorem 2.1], the right action of  $\mathrm{SO}(n)$  lifts to the right action of  $\mathrm{Spin}(n)$ . That is to say  $M$  is spinnable. This proves the first part.

The second part follows from the first part on noting that  $\mathbb{H}^n/\Delta$  is a finite cover of  $M$ .  $\square$

## 4 A proof of Sullivan's theorem

We prove the following.

**Theorem 4.1** (Sullivan). *Let  $M^n = \mathbb{H}^n/\Gamma$  be a finite volume orientable hyperbolic  $n$ -manifold. Then  $M$  is virtually spinnable.*

The proof of Theorem 4.1 will follow from the next proposition (notation as in §3.2).

**Proposition 4.2.** *Let  $M^n = \mathbb{H}^n/\Gamma$  be a finite volume orientable hyperbolic  $n$ -manifold and  $\bar{\Gamma} < \mathrm{Spin}^+(n, 1)$  with  $\phi(\bar{\Gamma}) = \Gamma$ . Then  $\bar{\Gamma}$  is residually finite.*

Given this, Theorem 4.1 is proved as follows. Residual finiteness implies that there exists a finite quotient  $\psi : \bar{\Gamma} \rightarrow Q$  so that  $\psi$  is injective on  $\{\pm 1\}$ . Let  $D < \bar{\Gamma}$  be the subgroup of finite index given by  $\psi^{-1}(\psi(\{\pm 1\}))$ . Then  $D$  contains a subgroup  $D_0$  of index 2 such that  $D_0 \cap \{\pm 1\} = 1$ . Let  $\Delta = \phi(D)$ , then  $\Delta$  is a finite index subgroup of  $\Gamma$  and  $\mathbb{H}^n/\Delta$  is spinnable by Lemma 3.1.  $\square$

The proof of Proposition 4.2 requires some additional material which is described below.

### 4.1 Linear representations of spin groups

We follow [12] and [19, Section 3] which provides a very helpful detailed account of the general framework we describe below.

Let  $V$  be a  $m$ -dimensional vector space over  $\mathbb{R}$  and  $q$  a non-degenerate quadratic form on  $V$ . The Clifford algebra  $\mathcal{Cl}(V, q)$  associated to  $(V, q)$  is the associative algebra with 1 obtained from the

free tensor algebra on  $V$  by adding relations  $v \otimes v = -q(v).1$  for each  $v \in V$ . Note that  $V$  embeds naturally into  $\mathcal{C}\ell(V, q)$ , and  $\mathcal{C}\ell(V, q)$  has the structure of a real vector space of dimension  $2^m$  with a basis  $\mathcal{B}$  constructed naturally from  $V$  (we will not dwell on this). Following [19], let  $P(V, q)$  denote the multiplicative group of  $\mathcal{C}\ell(V, q)$  generated by all  $v \in V$  such that  $q(v) \neq 0$ . Then, the *Spin group* of  $(V, q)$  is the subgroup of  $P(V, q)$  defined as:

$$\text{Spin}(V, q) = \{v_1 \dots v_k : v_i \in V, q(v_i) = \pm 1 \text{ for each } i, \text{ and } k \text{ even}\}.$$

In the case when  $q = J_n$  (from §2),  $\mathcal{C}\ell(V, q)$  is denoted  $\mathcal{C}\ell(n, 1)$ , the group  $P(V, q)$  is denoted  $P(n, 1)$ , and  $\text{Spin}(V, q) = \text{Spin}(n, 1)$ . The group  $\text{Spin}^+(n, 1)$  is the connected component of the identity in  $\text{Spin}(n, 1)$ .

Now the group  $\text{Spin}(n, 1)$  acts on the vector space  $\mathcal{C}\ell(n, 1)$  by left multiplication (on the basis  $\mathcal{B}$ ) thereby determining a *faithful* linear representation of  $L : \text{Spin}(n, 1) \rightarrow \text{GL}(2^{n+1}, \mathbb{R})$ .

The proof of Proposition 4.2 is now complete since  $L(\bar{\Gamma})$  is a finitely generated subgroup of  $\text{GL}(2^{n+1}, \mathbb{R})$  and hence is residually finite by Malcev's theorem.  $\square$

**Remark 4.3.** *It is important that the extensions considered above are of arithmetic groups in  $\text{SO}_0(n, 1)$ , since [16] constructs arithmetic groups in  $\text{SL}(n, \mathbb{R}) \times \text{SL}(n, \mathbb{R})$  that have extensions by  $\mathbb{Z}/2\mathbb{Z}$  that are not residually finite.*

## 5 Arithmetic hyperbolic manifolds of simplest type

### 5.1 Quadratic forms and arithmetic lattices

Let  $k$  be a totally real number field of degree  $d$  over  $\mathbb{Q}$  equipped with a fixed embedding into  $\mathbb{R}$  which we refer to as the identity embedding, and denote the ring of integers of  $k$  by  $R_k$ . Let  $V$  be an  $(n + 1)$ -dimensional vector space over  $k$  equipped with a non-degenerate quadratic form  $f$  defined over  $k$  which has signature  $(n, 1)$  at the identity embedding, and signature  $(n + 1, 0)$  at the remaining  $d - 1$  embeddings. Given this, the quadratic form  $f$  is equivalent over  $\mathbb{R}$  to the quadratic form  $x_0^2 + x_1^2 + \dots + x_{n-1}^2 - x_n^2$ , and for any galois embedding  $\sigma : k \rightarrow \mathbb{R}$ , the quadratic form  $f^\sigma$  (obtained by applying  $\sigma$  to each entry of  $f$ ) is equivalent over  $\mathbb{R}$  to  $x_0^2 + x_1^2 + \dots + x_{n-1}^2 + x_n^2$ . We call such a quadratic form *admissible*.

Let  $F$  be the symmetric matrix associated to the quadratic form  $f$  and let  $O(f)$  (resp.  $SO(f)$ ) denote the linear algebraic groups defined over  $k$  described as:

$$O(f) = \{X \in \text{GL}(n + 1, \mathbb{C}) : X^t F X = F\} \text{ and } SO(f) = \{X \in \text{SL}(n + 1, \mathbb{C}) : X^t F X = F\}.$$

For a subring  $L \subset \mathbb{C}$ , we denote the  $L$ -points of  $O(f)$  (resp.  $SO(f)$ ) by  $O(f, L)$  (resp.  $SO(f, L)$ ). An *arithmetic subgroup* of  $O(f)$  (resp.  $SO(f)$ ) is a subgroup  $\Gamma < O(f)$  (resp.  $SO(f)$ ) commensurable with  $O(f, R_k)$  (resp.  $SO(f, R_k)$ ). Note that an arithmetic subgroup of  $SO(f)$  is an arithmetic subgroup of  $O(f)$ , and an arithmetic subgroup  $\Gamma < O(f)$  determines an arithmetic subgroup  $\Gamma \cap SO(f)$  in  $SO(f)$ .

### 5.2 Constructing arithmetic groups

To pass to arithmetic subgroups of  $O_0(n, 1)$  and  $SO_0(n, 1)$ , we first note from §5.1, that, given an admissible quadratic form defined over  $k$  of signature  $(n, 1)$ , there exists  $T \in \text{GL}(n + 1, \mathbb{R})$  such that  $TO(f, \mathbb{R})T^{-1} = O(n, 1)$ .

A subgroup  $\Gamma < O_0(n, 1)$  is called *arithmetic of simplest type* if  $\Gamma$  is commensurable with the image in  $O_0(n, 1)$  of an arithmetic subgroup of  $O(f)$  (under the conjugation map described above). An arithmetic hyperbolic  $n$ -manifold  $M = \mathbb{H}^n / \Gamma$  is called *arithmetic of simplest type* if  $\Gamma$  is. The

same set-up using special orthogonal groups constructs orientation-preserving arithmetic groups of simplest type (and orientable arithmetic hyperbolic  $n$ -manifolds of simplest type).

It is known (see [21]) that when  $n$  is even all arithmetic hyperbolic manifolds are of simplest type. Furthermore, when  $n$  is even, the algebraic groups  $\mathrm{SO}(f)$  are centerless, and it follows from a result of Borel [2] that any arithmetic subgroup of  $\mathrm{SO}(f, \mathbb{R})$  commensurable with  $\mathrm{SO}(f, \mathbb{R}_k)$  is contained in  $\mathrm{SO}(f, k)$ .

In addition, we will say that an arithmetic hyperbolic manifold  $M = \mathbb{H}^n/\Gamma$  of simplest type is *well-located* if  $\Gamma$  is conjugate to a subgroup of some  $\mathrm{SO}(f, \mathbb{R}_k)$  for some admissible quadratic form  $f$  as above.

### 5.3 Arithmetic subgroups of $\mathrm{Spin}^+(n, 1)$

Now we can repeat much of the above discussion replacing  $\mathbb{R}$  by a totally real field  $k$  and take for  $q$  an admissible quadratic form  $f$ . In particular, following the construction of §4.1, we can construct a Clifford algebra  $\mathcal{C}\ell(V, k, q)$  with a basis  $\mathcal{B}$  and define an algebraic group  $\mathrm{Spin}(f)$  defined over  $k$ , with  $k$ -points  $\mathrm{Spin}(f, k)$  and an arithmetic subgroup  $\mathrm{Spin}(f, \mathbb{R}_k)$ . Note that  $\mathrm{Spin}(f, k) \rightarrow \mathrm{SO}(f, k)$  with kernel  $\{\pm 1\}$  and similarly  $\mathrm{Spin}(f, \mathbb{R}_k) \rightarrow \mathrm{SO}(f, \mathbb{R}_k)$ .

In addition, on taking the vector space  $V_{\mathbb{R}} = V \otimes_k \mathbb{R}$  we can construct arithmetic subgroups of  $\mathrm{Spin}(n, 1)$ , since by admissibility of  $f$  we have that  $\mathrm{Spin}(f, \mathbb{R})$  is conjugate to  $\mathrm{Spin}(n, 1)$ , and so  $\mathrm{Spin}(f, \mathbb{R}_k)$  can be conjugated into  $\mathrm{Spin}(n, 1)$ .

Again referring to the construction in §4.1, via the basis  $\mathcal{B}$  the following proposition can be proved.

**Proposition 5.1.** *Let  $f$  be an admissible quadratic form of signature  $(n, 1)$  defined over the totally real field  $k$ . The group  $\mathrm{Spin}(f, \mathbb{R}_k)$  admits a faithful representation into  $\mathrm{GL}(2^{n+1}, \mathbb{R}_k)$ .*

## 6 Proof of Theorem 1.1

We assume throughout that  $n \geq 4$ , and  $f$  will be an admissible quadratic form of signature  $(n, 1)$  defined over the totally real field  $k$ . For  $I \subset R_k$  an ideal, we denote the norm of  $I$  by  $NI$ . Denote the principal ideal  $4R_k$  by  $I_4$  and define the constant,  $N_k = \min\{N\mathcal{P}, NI_4\}$  where  $\mathcal{P}$  runs over all prime ideals  $\mathcal{P}$  with odd residue class degree. Note that if  $[k : \mathbb{Q}] = d$  then  $NI_4 \leq 4^d$ . Let  $I$  denote the ideal of norm  $N_k$ .

The key proposition is the following (using the notation established).

**Proposition 6.1.** *Let  $M = \mathbb{H}^n/\Gamma$  be an arithmetic hyperbolic manifold of simplest type that is well-located. Then  $M$  has a finite cover of degree  $\leq |\mathrm{GL}(2^{n+1}, \mathbb{R}_k/I)|/2$  that is spinnable.*

*Proof.* We can assume that  $M$  is not spinnable, otherwise we are done. Following the notation above, we have an extension  $\bar{\Gamma}$  of  $\Gamma$  by  $\{\pm 1\}$ . By Proposition 4.2 and the sharper version Proposition 5.1 we have the following. Let  $I$  denote the ideal of norm  $N_k$  as in the statement and consider the homomorphism  $\psi : \bar{\Gamma} \rightarrow \mathrm{GL}(2^{n+1}, \mathbb{R}_k/I)$  given by restricting the reduction homomorphism  $\mathrm{GL}(2^{n+1}, \mathbb{R}_k) \rightarrow \mathrm{GL}(2^{n+1}, \mathbb{R}_k/I)$ . By choice of ideal  $I$  (i.e. either it is prime of odd residue class degree or the principal ideal generated by 4) we see that  $\{\pm 1\}$  injects under  $\psi$ . Hence,  $K = \ker \psi$  excludes  $-1$  and has index bounded as claimed.

Let  $D$  denote the preimage of  $\{\pm 1\}$  under  $\psi$ , so that  $[D : K] = 2$ . Let  $\Delta = \phi(D)$ , and let  $N$  denote the cover of  $M$  given by  $\mathbb{H}^n/\Delta$ . Then  $N$  is a cover of degree  $\leq |\mathrm{GL}(2^{n+1}, \mathbb{R}_k/I)|/2$  and by construction is spinnable by Lemma 3.1.  $\square$

**Remark 6.2.** *By definition any arithmetic subgroup  $\Gamma < \mathrm{SO}(f, \mathbb{R})$  contains a well-located subgroup of finite index. As discussed in §5.2, when  $n$  is even, all arithmetic subgroups commensurable with*

$\mathrm{SO}(f, \mathbb{R}_k)$  are contained in  $\mathrm{SO}(f, k)$ , and when  $n$  is odd, the group  $\Gamma^{(2)}$  is contained in  $\mathrm{SO}(f, k)$  (see the proof of [6, Lemma 10]). Now  $[\Gamma : \Gamma^{(2)}] = H_1(\Gamma, \mathbb{Z}/2\mathbb{Z})$ , and so to get a sharper version of Sullivan's result (i.e. bounding the index) in the general arithmetic setting reduces to consideration of  $\Gamma < \mathrm{SO}(f, k)$  and getting an effectively computable constant bounding  $[\Gamma : \Gamma \cap \mathrm{SO}(f, \mathbb{R}_k)]$ .

## 6.1 Some examples

**Example 1:** As is shown in [7], the Davis manifold  $D$  [4] is a well located arithmetic hyperbolic 4-manifold. This was proved to be spinnable in [18] by showing that the intersection form of  $D$  is even (c.f. Remark 2.3).

**Example 2:** The closed orientable hyperbolic 4-manifold  $X$  of euler characteristic 16 obtained by Conder and Maclachlan in [3] is not known to be spinnable. Although the homology groups are computed in [3], its intersection form is not known at present.

It is known to be arithmetic and well located (see [3] and [8]) with quadratic form defined over the field  $\mathbb{Q}(\sqrt{5})$ . Since 2 and 3 are inert in  $\mathbb{Q}(\sqrt{5})$ , the ideal  $I$  of Propostion 6.1 is the ideal  $\langle \sqrt{5} \rangle$ . Hence Proposition 6.1 provides a spinnable cover of  $X$  of degree at most  $|\mathrm{GL}(2^5, \mathbb{F}_5)|/2 = \frac{1}{2}(5^{32} - 1)(5^{32} - 5) \dots (5^{32} - 5^{31})$ . Note  $5^{32} = 23283064365386962890625$ .

As is evident in this example, the method given by Propostion 6.1 for producing a spinnable cover gives a gigantic bound for the degree of a spinnable cover!

On the other hand, since  $D$  (of Example 1) and  $X$  are commensurable [8], it seems likely that a smaller degree spinnable cover of  $X$  can be constructed using the fact that the Davis manifold is spinnable and this can be seen to pass to finite sheeted covering spaces.

**Remark 6.3.** *It follows from [1, Theorem 1] that if a closed hyperbolic 4-manifold  $X$  has even intersection form then there is a finite cyclic cover that is spinnable. This finite cover is a power of 2 controlled by the the size of the two-torsion in  $H_1(X, \mathbb{Z})$ .*

## 7 Non-spinnable hyperbolic manifolds

As advertised in §1, we now produce non-spinnable examples of finite volume orientable hyperbolic manifolds in all dimensions  $\geq 5$ .

**Theorem 7.1.** *For every  $n \geq 5$  there exists infinitely many non-spinnable orientable finite volume hyperbolic  $n$ -manifolds.*

*Proof.* As shown in [11] (see also [17]) there exists non-spinnable orientable flat 4-manifolds (although only 3 of the 27 orientable flat, 4-manifolds are not spinnable). Let  $Y$  be one of these non-spinnable orientable flat 4-manifolds. By [14] and the improvement in [15], every flat 4-manifold occurs as some cusp cross-section of a possibly (indeed likely) multi-cusped arithmetic hyperbolic 5-manifold. Hence  $Y$  can be arranged as a cusp cross-section of an arithmetic hyperbolic 5-manifold  $X$ .  $X$  cannot be spin, since it is well-known that a spin structure induces a spin structure on a boundary component (see [12, Chapter II, Proposition 2.15]).

To get higher dimensional examples, the manifold  $Y_n = Y \times S^1 \times S^1 \times \dots \times S^1$  (with  $n - 4$  copies of  $S^1$ ) is a flat  $n$ -manifold. Hence we can repeat the above construction to produce a multi-cusped arithmetic hyperbolic  $n + 1$ -manifold  $X_n$  for which  $Y_n$  is a cusp cross-section. Another application of [12, Chapter II, Proposition 2.15], implies that since  $Y$  is not spinnable,  $Y_n$  is not spinnable for all  $n \geq 4$ . Hence, it follows as above that  $X_n$  is not spinnable.

To get infinitely many in each dimension we note that the fundamental groups  $\pi_1(Y_n)$  are separable subgroups of  $\pi_1(X_n)$  (see [14] for example), and hence for each  $n$ , we can construct infinitely many finite sheeted covers for which  $Y_n$  occurs as a cusp cross-section.  $\square$

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