

Virtually spinning hyperbolic manifolds

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1 Introduction

Let M be an orientable Riemannian manifold and let $w_2(M)$ denote its second Stiefel-Whitney class. Then M admits a spin structure or said to be *spinnable* if $w_2(M) = 0$; we refer the reader to §2 and §3 for more details, definitions and background.

It is well-known that all compact orientable surfaces, and compact orientable 3-manifolds are spinnable, however, the situation in higher dimensions is a good deal more subtle. For example, a well-known consequence of the Hirzebruch Signature Theorem is that the signature of any closed orientable hyperbolic 4-manifold is 0, and so according to Rochlin's Theorem (see [9]) there is no obstruction to having a spin structure. On the other hand, whether a closed hyperbolic manifold of dimension 4 (or more) is spinnable, is much harder to establish.

However, on p. 553 of his paper [20], Sullivan notes that his previous work with Deligne [5], can be used to show that if M^n is a finite volume hyperbolic n -manifold, then M^n has a finite cover that is stably parallelizable, and hence spinnable (see Remark 2.2 below).

The aim of this note is to give a simple proof of Sullivan's virtually spinning result that seems not to have been noticed previously. Furthermore, in the setting of arithmetic hyperbolic manifolds of simplest type we will provide a sharper version of Sullivan's result in many cases (see §5 for definitions).

Theorem 1.1. *Assume that $n \geq 4$ and let $M^n = \mathbb{H}^n/\Gamma$ be a well-located arithmetic hyperbolic n -manifold of simplest type from the admissible quadratic form f defined over the totally real field k . Then M^n admits a finite cover of degree $C(k, f)$ that is spinnable. The constant $C(k, f)$ is an effectively computable constant depending on k and f .*

The question as to whether all orientable hyperbolic manifolds of finite volume in dimensions ≥ 4 are spinnable, seems to be open. To that end, in §7 we point out that in each dimension $n \geq 5$ there are infinitely many finite volume (non-compact) orientable hyperbolic n -manifolds that are not spinnable.

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2 Spin groups and spin structures

We assume $n \geq 4$ throughout this section. General references for what follows are [9] and [12]. We will restrict our discussion to spin structures associated with the Tangent Bundle of a Riemannian manifold, rather than an arbitrary vector bundle.

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The spin group in dimension n will be denoted by $\text{Spin}(n)$. Since $n \geq 4$, $\text{Spin}(n)$ is the universal 2-fold covering group of the special orthogonal group $\text{SO}(n)$ (with covering map j), and determines a short exact sequence:

$$1 \rightarrow \{\pm 1\} \rightarrow \text{Spin}(n) \rightarrow \text{SO}(n) \rightarrow 1.$$

Now let M be a connected orientable Riemannian manifold of dimension ≥ 4 with Tangent Bundle TM . Denote by $\text{SO}(TM) \rightarrow M$, the $\text{SO}(n)$ -principal bundle of oriented orthonormal frames on TM . M admits a spin structure or is *spinable* if there is a principal $\text{Spin}(n)$ -bundle $\text{Spin}(TM) \rightarrow M$ together with a 2-fold covering map $\eta : \text{Spin}(TM) \rightarrow \text{SO}(TM)$ such that $\eta(pg) = \eta(p)j(g)$ for all $p \in \text{Spin}(TM)$ and $g \in \text{Spin}(n)$.

Not every orientable Riemannian manifold admits a spin structure, the obstruction to this is the second Stiefel-Whitney class. With M as above, we let $w_2(M) \in H^2(M, \mathbb{Z}/2\mathbb{Z})$ denote the second Stiefel-Whitney class of TM . We summarize what we need in the following proposition.

Proposition 2.1. *M admits a spin structure if and only if $w_2(M) = 0$.*

Remark 2.2. *A smooth orientable manifold M^n is stably parallelizable if its tangent bundle is stably trivial; i.e. $TM \oplus E_1 = E_2$ where E_i are trivial vector bundles for $i = 1, 2$. When this is the case, since Stiefel-Whitney classes are invariants of the stable equivalence class of a vector bundle, it follows that $w_2(M) = 0$, and so this proves Sullivan's result mentioned in §1.*

Remark 2.3. *If an oriented 4-manifold X is spinable, then the intersection form of X is even. The converse holds if $H_1(X, \mathbb{Z})$ has no 2-torsion (see [9, Chapter II.4]).*

3 Hyperbolic manifolds and spin structures

3.1 Some notation

Let J_n be the diagonal matrix associated to the quadratic form $x_0^2 + x_1^2 + \dots + x_{n-1}^2 - x_n^2$. We identify hyperbolic space \mathbb{H}^n with $\{x \in \mathbb{R}^{n+1} : J_n x = -1, x_{n+1} > 0\}$, and letting

$$\text{O}(n, 1) = \{X \in \text{GL}(n+1, \mathbb{R}) : X^t J_n X = J_n\}.$$

we can identify $\text{Isom}(\mathbb{H}^n)$ with the connected component of the identity of $\text{O}(n, 1)$, denoted by $\text{O}_0(n, 1)$. This is also the subgroup of $\text{O}(n, 1)$ preserving the hyperboloid $\{x \in \mathbb{R}^{n+1} : J_n x = -1, x_{n+1} > 0\}$. Equivalently, $\text{O}_0(n, 1) = \text{PO}(n, 1)$ (the central quotient of $\text{O}(n, 1)$). With this notation, $\text{Isom}^+(\mathbb{H}^n) = \text{SO}_0(n, 1)$, the index 2 subgroup in $\text{O}_0(n, 1)$ which is the connected component of the identity of $\text{SO}(n, 1)$.

Although we will not make explicit use of this, if $M = \mathbb{H}^n/\Gamma$ is a finite volume orientable hyperbolic n -manifold, then it is a well-known consequence of Mostow-Prasad Rigidity that we can conjugate Γ to be a subgroup of $\text{SO}_0(n, 1)$ where the elements all have matrix entries in a real number field k (which we can take to be minimal).

3.2 Spin structures on hyperbolic manifolds

The recent preprint [19] contains a particularly useful discussion of spin structures on hyperbolic manifolds, and we refer the reader there for a fuller discussion. We continue to assume that $n \geq 4$. As with $\text{SO}(n)$, the group $\text{SO}_0(n, 1)$ has a universal 2-fold cover, which following [19], we denote by $\text{Spin}^+(n, 1)$ (with covering map ϕ) and as above there is an exact sequence:

$$1 \rightarrow \{\pm 1\} \rightarrow \text{Spin}^+(n, 1) \rightarrow \text{SO}_0(n, 1) \rightarrow 1.$$

Now if $M = \mathbb{H}^n/\Gamma$ is an orientable finite volume hyperbolic manifold, it is well-known that the $\mathrm{SO}(n)$ -principal bundle of oriented orthonormal frames on TM can be identified with $\Gamma\backslash\mathrm{SO}_0(n, 1)$ (see [19, Section 2] for example). Via the exact sequence given above, we can construct an extension $\bar{\Gamma} < \mathrm{Spin}^+(n, 1)$ with $\phi(\bar{\Gamma}) = \Gamma$. Note that $\bar{\Gamma}\backslash\mathrm{Spin}^+(n, 1) \cong \Gamma\backslash\mathrm{SO}_0(n, 1)$ since:

$$\Gamma\backslash\mathrm{SO}_0(n, 1) \cong (\{\pm 1\}\backslash\bar{\Gamma})\backslash(\{\pm 1\}\backslash\mathrm{Spin}^+(n, 1)) \cong \bar{\Gamma}\backslash\mathrm{Spin}^+(n, 1).$$

The following is implicit in the proof of [19, Theorem 2.1], we include a brief sketch.

Lemma 3.1. *We follow the notation established above. Let $M = \mathbb{H}^n/\Gamma$ be an orientable finite volume hyperbolic manifold with $n \geq 4$.*

1. *Suppose that there is a subgroup $H < \bar{\Gamma}$ of index 2 so that ϕ maps H isomorphically onto Γ (or equivalently, $H \cap \{\pm 1\} = 1$), then M is spinnable.*
2. *Let $D < \bar{\Gamma}$ be of finite index containing an index 2 subgroup D_0 such that $D_0 \cap \{\pm 1\} = 1$. Let $\Delta = \phi(D) < \Gamma$, then \mathbb{H}^n/Δ is a finite cover of M that is spinnable.*

Proof. From above we have an exact sequence:

$$1 \rightarrow \{\pm 1\} \rightarrow \bar{\Gamma} \rightarrow \Gamma \rightarrow 1.$$

If H is a subgroup of index 2 as claimed, then $H\backslash\mathrm{Spin}^+(n, 1)$ is a principal $\mathrm{Spin}(n)$ -bundle that double covers $\bar{\Gamma}\backslash\mathrm{Spin}^+(n, 1)$ which from the discussion above is $\cong \Gamma\backslash\mathrm{SO}_0(n, 1)$. Moreover, as described in the proof of [19, Theorem 2.1], the right action of $\mathrm{SO}(n)$ lifts to the right action of $\mathrm{Spin}(n)$. That is to say M is spinnable. This proves the first part.

The second part follows from the first part on noting that \mathbb{H}^n/Δ is a finite cover of M . \square

4 A proof of Sullivan's theorem

We prove the following.

Theorem 4.1 (Sullivan). *Let $M^n = \mathbb{H}^n/\Gamma$ be a finite volume orientable hyperbolic n -manifold. Then M is virtually spinnable.*

The proof of Theorem 4.1 will follow from the next proposition (notation as in §3.2).

Proposition 4.2. *Let $M^n = \mathbb{H}^n/\Gamma$ be a finite volume orientable hyperbolic n -manifold and $\bar{\Gamma} < \mathrm{Spin}^+(n, 1)$ with $\phi(\bar{\Gamma}) = \Gamma$. Then $\bar{\Gamma}$ is residually finite.*

Given this, Theorem 4.1 is proved as follows. Residual finiteness implies that there exists a finite quotient $\psi: \bar{\Gamma} \rightarrow Q$ so that ψ is injective on $\{\pm 1\}$. Let $D < \bar{\Gamma}$ be the subgroup of finite index given by $\psi^{-1}(\psi(\{\pm 1\}))$. Then D contains a subgroup D_0 of index 2 such that $D_0 \cap \{\pm 1\} = 1$. Let $\Delta = \phi(D)$, then Δ is a finite index subgroup of Γ and \mathbb{H}^n/Δ is spinnable by Lemma 3.1. \square

The proof of Proposition 4.2 requires some additional material which is described below.

4.1 Linear representations of spin groups

We follow [12] and [19, Section 3] which provides a very helpful detailed account of the general framework we describe below.

Let V be a m -dimensional vector space over \mathbb{R} and q a non-degenerate quadratic form on V . The Clifford algebra $\mathcal{Cl}(V, q)$ associated to (V, q) is the associative algebra with 1 obtained from the

free tensor algebra on V by adding relations $v \otimes v = -q(v).1$ for each $v \in V$. Note that V embeds naturally into $\mathcal{C}\ell(V, q)$, and $\mathcal{C}\ell(V, q)$ has the structure of a real vector space of dimension 2^m with a basis \mathcal{B} constructed naturally from V (we will not dwell on this). Following [19], let $P(V, q)$ denote the multiplicative group of $\mathcal{C}\ell(V, q)$ generated by all $v \in V$ such that $q(v) \neq 0$. Then, the *Spin group* of (V, q) is the subgroup of $P(V, q)$ defined as:

$$\text{Spin}(V, q) = \{v_1 \dots v_k : v_i \in V, q(v_i) = \pm 1 \text{ for each } i, \text{ and } k \text{ even}\}.$$

In the case when $q = J_n$ (from §2), $\mathcal{C}\ell(V, q)$ is denoted $\mathcal{C}\ell(n, 1)$, the group $P(V, q)$ is denoted $P(n, 1)$, and $\text{Spin}(V, q) = \text{Spin}(n, 1)$. The group $\text{Spin}^+(n, 1)$ is the connected component of the identity in $\text{Spin}(n, 1)$.

Now the group $\text{Spin}(n, 1)$ acts on the vector space $\mathcal{C}\ell(n, 1)$ by left multiplication (on the basis \mathcal{B}) thereby determining a *faithful* linear representation of $L : \text{Spin}(n, 1) \rightarrow \text{GL}(2^{n+1}, \mathbb{R})$.

The proof of Proposition 4.2 is now complete since $L(\bar{\Gamma})$ is a finitely generated subgroup of $\text{GL}(2^{n+1}, \mathbb{R})$ and hence is residually finite by Malcev's theorem. \square

Remark 4.3. *It is important that the extensions considered above are of arithmetic groups in $\text{SO}_0(n, 1)$, since [16] constructs arithmetic groups in $\text{SL}(n, \mathbb{R}) \times \text{SL}(n, \mathbb{R})$ that have extensions by $\mathbb{Z}/2\mathbb{Z}$ that are not residually finite.*

5 Arithmetic hyperbolic manifolds of simplest type

5.1 Quadratic forms and arithmetic lattices

Let k be a totally real number field of degree d over \mathbb{Q} equipped with a fixed embedding into \mathbb{R} which we refer to as the identity embedding, and denote the ring of integers of k by R_k . Let V be an $(n + 1)$ -dimensional vector space over k equipped with a non-degenerate quadratic form f defined over k which has signature $(n, 1)$ at the identity embedding, and signature $(n + 1, 0)$ at the remaining $d - 1$ embeddings. Given this, the quadratic form f is equivalent over \mathbb{R} to the quadratic form $x_0^2 + x_1^2 + \dots + x_{n-1}^2 - x_n^2$, and for any galois embedding $\sigma : k \rightarrow \mathbb{R}$, the quadratic form f^σ (obtained by applying σ to each entry of f) is equivalent over \mathbb{R} to $x_0^2 + x_1^2 + \dots + x_{n-1}^2 + x_n^2$. We call such a quadratic form *admissible*.

Let F be the symmetric matrix associated to the quadratic form f and let $O(f)$ (resp. $SO(f)$) denote the linear algebraic groups defined over k described as:

$$O(f) = \{X \in \text{GL}(n + 1, \mathbb{C}) : X^t F X = F\} \text{ and } SO(f) = \{X \in \text{SL}(n + 1, \mathbb{C}) : X^t F X = F\}.$$

For a subring $L \subset \mathbb{C}$, we denote the L -points of $O(f)$ (resp. $SO(f)$) by $O(f, L)$ (resp. $SO(f, L)$). An *arithmetic subgroup* of $O(f)$ (resp. $SO(f)$) is a subgroup $\Gamma < O(f)$ (resp. $SO(f)$) commensurable with $O(f, R_k)$ (resp. $SO(f, R_k)$). Note that an arithmetic subgroup of $SO(f)$ is an arithmetic subgroup of $O(f)$, and an arithmetic subgroup $\Gamma < O(f)$ determines an arithmetic subgroup $\Gamma \cap SO(f)$ in $SO(f)$.

5.2 Constructing arithmetic groups

To pass to arithmetic subgroups of $O_0(n, 1)$ and $SO_0(n, 1)$, we first note from §5.1, that, given an admissible quadratic form defined over k of signature $(n, 1)$, there exists $T \in \text{GL}(n + 1, \mathbb{R})$ such that $T O(f, \mathbb{R}) T^{-1} = O(n, 1)$.

A subgroup $\Gamma < O_0(n, 1)$ is called *arithmetic of simplest type* if Γ is commensurable with the image in $O_0(n, 1)$ of an arithmetic subgroup of $O(f)$ (under the conjugation map described above). An arithmetic hyperbolic n -manifold $M = \mathbb{H}^n / \Gamma$ is called *arithmetic of simplest type* if Γ is. The

same set-up using special orthogonal groups constructs orientation-preserving arithmetic groups of simplest type (and orientable arithmetic hyperbolic n -manifolds of simplest type).

It is known (see [21]) that when n is even all arithmetic hyperbolic manifolds are of simplest type. Furthermore, when n is even, the algebraic groups $\mathrm{SO}(f)$ are centerless, and it follows from a result of Borel [2] that any arithmetic subgroup of $\mathrm{SO}(f, \mathbb{R})$ commensurable with $\mathrm{SO}(f, \mathbb{R}_k)$ is contained in $\mathrm{SO}(f, k)$.

In addition, we will say that an arithmetic hyperbolic manifold $M = \mathbb{H}^n/\Gamma$ of simplest type is *well-located* if Γ is conjugate to a subgroup of some $\mathrm{SO}(f, \mathbb{R}_k)$ for some admissible quadratic form f as above.

5.3 Arithmetic subgroups of $\mathrm{Spin}^+(n, 1)$

Now we can repeat much of the above discussion replacing \mathbb{R} by a totally real field k and take for q an admissible quadratic form f . In particular, following the construction of §4.1, we can construct a Clifford algebra $\mathcal{C}\ell(V, k, q)$ with a basis \mathcal{B} and define an algebraic group $\mathrm{Spin}(f)$ defined over k , with k -points $\mathrm{Spin}(f, k)$ and an arithmetic subgroup $\mathrm{Spin}(f, \mathbb{R}_k)$. Note that $\mathrm{Spin}(f, k) \rightarrow \mathrm{SO}(f, k)$ with kernel $\{\pm 1\}$ and similarly $\mathrm{Spin}(f, \mathbb{R}_k) \rightarrow \mathrm{SO}(f, \mathbb{R}_k)$.

In addition, on taking the vector space $V_{\mathbb{R}} = V \otimes_k \mathbb{R}$ we can construct arithmetic subgroups of $\mathrm{Spin}(n, 1)$, since by admissibility of f we have that $\mathrm{Spin}(f, \mathbb{R})$ is conjugate to $\mathrm{Spin}(n, 1)$, and so $\mathrm{Spin}(f, \mathbb{R}_k)$ can be conjugated into $\mathrm{Spin}(n, 1)$.

Again referring to the construction in §4.1, via the basis \mathcal{B} the following proposition can be proved.

Proposition 5.1. *Let f be an admissible quadratic form of signature $(n, 1)$ defined over the totally real field k . The group $\mathrm{Spin}(f, \mathbb{R}_k)$ admits a faithful representation into $\mathrm{GL}(2^{n+1}, \mathbb{R}_k)$.*

6 Proof of Theorem 1.1

We assume throughout that $n \geq 4$, and f will be an admissible quadratic form of signature $(n, 1)$ defined over the totally real field k . For $I \subset R_k$ an ideal, we denote the norm of I by NI . Denote the principal ideal $4R_k$ by I_4 and define the constant, $N_k = \min\{N\mathcal{P}, NI_4\}$ where \mathcal{P} runs over all prime ideals \mathcal{P} with odd residue class degree. Note that if $[k : \mathbb{Q}] = d$ then $NI_4 \leq 4^d$. Let I denote the ideal of norm N_k .

The key proposition is the following (using the notation established).

Proposition 6.1. *Let $M = \mathbb{H}^n/\Gamma$ be an arithmetic hyperbolic manifold of simplest type that is well-located. Then M has a finite cover of degree $\leq |\mathrm{GL}(2^{n+1}, \mathbb{R}_k/I)|/2$ that is spinnable.*

Proof. We can assume that M is not spinnable, otherwise we are done. Following the notation above, we have an extension $\bar{\Gamma}$ of Γ by $\{\pm 1\}$. By Proposition 4.2 and the sharper version Proposition 5.1 we have the following. Let I denote the ideal of norm N_k as in the statement and consider the homomorphism $\psi : \bar{\Gamma} \rightarrow \mathrm{GL}(2^{n+1}, \mathbb{R}_k/I)$ given by restricting the reduction homomorphism $\mathrm{GL}(2^{n+1}, \mathbb{R}_k) \rightarrow \mathrm{GL}(2^{n+1}, \mathbb{R}_k/I)$. By choice of ideal I (i.e. either it is prime of odd residue class degree or the principal ideal generated by 4) we see that $\{\pm 1\}$ injects under ψ . Hence, $K = \ker \psi$ excludes -1 and has index bounded as claimed.

Let D denote the preimage of $\{\pm 1\}$ under ψ , so that $[D : K] = 2$. Let $\Delta = \phi(D)$, and let N denote the cover of M given by \mathbb{H}^n/Δ . Then N is a cover of degree $\leq |\mathrm{GL}(2^{n+1}, \mathbb{R}_k/I)|/2$ and by construction is spinnable by Lemma 3.1. \square

Remark 6.2. *By definition any arithmetic subgroup $\Gamma < \mathrm{SO}(f, \mathbb{R})$ contains a well-located subgroup of finite index. As discussed in §5.2, when n is even, all arithmetic subgroups commensurable with*

$\mathrm{SO}(f, \mathbb{R}_k)$ are contained in $\mathrm{SO}(f, k)$, and when n is odd, the group $\Gamma^{(2)}$ is contained in $\mathrm{SO}(f, k)$ (see the proof of [6, Lemma 10]). Now $[\Gamma : \Gamma^{(2)}] = H_1(\Gamma, \mathbb{Z}/2\mathbb{Z})$, and so to get a sharper version of Sullivan's result (i.e. bounding the index) in the general arithmetic setting reduces to consideration of $\Gamma < \mathrm{SO}(f, k)$ and getting an effectively computable constant bounding $[\Gamma : \Gamma \cap \mathrm{SO}(f, \mathbb{R}_k)]$.

6.1 Some examples

Example 1: As is shown in [7], the Davis manifold D [4] is a well located arithmetic hyperbolic 4-manifold. This was proved to be spinnable in [18] by showing that the intersection form of D is even (c.f. Remark 2.3).

Example 2: The closed orientable hyperbolic 4-manifold X of euler characteristic 16 obtained by Conder and Maclachlan in [3] is not known to be spinnable. Although the homology groups are computed in [3], its intersection form is not known at present.

It is known to be arithmetic and well located (see [3] and [8]) with quadratic form defined over the field $\mathbb{Q}(\sqrt{5})$. Since 2 and 3 are inert in $\mathbb{Q}(\sqrt{5})$, the ideal I of Propostion 6.1 is the ideal $\langle \sqrt{5} \rangle$. Hence Proposition 6.1 provides a spinnable cover of X of degree at most $|\mathrm{GL}(2^5, \mathbb{F}_5)|/2 = \frac{1}{2}(5^{32} - 1)(5^{32} - 5) \dots (5^{32} - 5^{31})$. Note $5^{32} = 23283064365386962890625$.

As is evident in this example, the method given by Propostion 6.1 for producing a spinnable cover gives a gigantic bound for the degree of a spinnable cover!

On the other hand, since D (of Example 1) and X are commensurable [8], it seems likely that a smaller degree spinnable cover of X can be constructed using the fact that the Davis manifold is spinnable and this can be seen to pass to finite sheeted covering spaces.

Remark 6.3. *It follows from [1, Theorem 1] that if a closed hyperbolic 4-manifold X has even intersection form then there is a finite cyclic cover that is spinnable. This finite cover is a power of 2 controlled by the the size of the two-torsion in $H_1(X, \mathbb{Z})$.*

7 Non-spinnable hyperbolic manifolds

As advertised in §1, we now produce non-spinnable examples of finite volume orientable hyperbolic manifolds in all dimensions ≥ 5 .

Theorem 7.1. *For every $n \geq 5$ there exists infinitely many non-spinnable orientable finite volume hyperbolic n -manifolds.*

Proof. As shown in [11] (see also [17]) there exists non-spinnable orientable flat 4-manifolds (although only 3 of the 27 orientable flat, 4-manifolds are not spinnable). Let Y be one of these non-spinnable orientable flat 4-manifolds. By [14] and the improvement in [15], every flat 4-manifold occurs as some cusp cross-section of a possibly (indeed likely) multi-cusped arithmetic hyperbolic 5-manifold. Hence Y can be arranged as a cusp cross-section of an arithmetic hyperbolic 5-manifold X . X cannot be spin, since it is well-known that a spin structure induces a spin structure on a boundary component (see [12, Chapter II, Proposition 2.15]).

To get higher dimensional examples, the manifold $Y_n = Y \times S^1 \times S^1 \times \dots \times S^1$ (with $n - 4$ copies of S^1) is a flat n -manifold. Hence we can repeat the above construction to produce a multi-cusped arithmetic hyperbolic $n + 1$ -manifold X_n for which Y_n is a cusp cross-section. Another application of [12, Chapter II, Proposition 2.15], implies that since Y is not spinnable, Y_n is not spinnable for all $n \geq 4$. Hence, it follows as above that X_n is not spinnable.

To get infinitely many in each dimension we note that the fundamental groups $\pi_1(Y_n)$ are separable subgroups of $\pi_1(X_n)$ (see [14] for example), and hence for each n , we can construct infinitely many finite sheeted covers for which Y_n occurs as a cusp cross-section. \square

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