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Subgroup separability and 3-manifold groups

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§ 1 Introduction

The purpose of this paper is to describe positive and negative results on subgroup separability. This is largely in the context of 3-manifold groups, though several of the examples we will give answer longstanding questions about subgroup separability and amalgamated free products. These may be of independent group theoretic interest.

Definition. Let H be a subgroup of a group G. H is separable in G if given any element g in $G-H$, there is a finite index subgroup $K<G$, which contains H but $g \notin K$.

G is said to be *subgroup separable*, or, for historical reasons, *Lerf*, if every finitely generated subgroup of G is separable in G.

See [12] for an outline of the connection between subgroup separability and geometric topology.

Our main theorem answers a question raised by Jaco in [7] V.22:

Theorem 1 *Suppose that M is an orientable, irreducible compact 3-manifold and that X is an incompressible connected subsurface of a component of $\partial(M)$. If we put the basepoint, p, of M on X then $\pi_1(X, p)$ is a separable subgroup of $\pi_1(M, p)$.*

We call the above restricted form of subgroup separability, *peripheral (subgroup) separability*. It is known that there are Haken 3-manifolds with fundamental groups which are not subgroup separable, examples are given in Sect. 3. Whether hyperbolic 3-manifold groups are actually subgroup separable is an important unsolved problem in the theory.

Part of the interest in this property comes from the following construction. Let M_1 and M_2 be a pair of manifolds each with a single incompressible boundary component, of genus g, say. Suppose that when we glue M_1 to M_2 we obtain a hyperbolic manifold M. The manifold M is Haken via the separating surface F, but in general may be a homology sphere, however it is conjectured

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that there is a finite covering in which some component of the lift of F becomes nonseparating. Peripheral separability can be regarded as a first step towards solving this conjecture.

Some results in the direction of Theorem 1 are already known; for example, if M is hyperbolic and contains no essential annulus, and X is an entire boundary component which is not a torus, then there is a hyperbolic structure on M in which X is totally geodesic, and the result follows from [9]. It is also pointed out in [7] that Thurston has proved Theorem 1 in the case that X is a boundary torus.

In fact in the case of the torus, we can say a little more:

Theorem 2 *Suppose that M is an orientable Haken* 3*-manifold, possibly without boundary. Then if* i: $T^2 \to M$ *is an incompressibly embedded torus,* $i_*(\pi_1(T))$ *is a separable subgroup in* $\pi_1(M)$.

The proof of Theorem 1 uses the fact that doubling a 3-manifold along its boundary preserves residual finiteness. In contrast, the second section is devoted to an example where glueing two 3-manifolds along their boundary produces bad behaviour for subgroup separability and it is perhaps of independent interest group-theoretically. We now briefly describe this.

It is known that not all 3-manifold groups are subgroup separable; there is an example given in [3] which is the basis of our construction. It had previously seemed possible that pathological behaviour could be avoided by invoking the finitely generated intersection property. (Peripheral subgroups of a 3-manifold group are known to have the f.g.i.p. See [7].) The philosophy in dealing with amalgamated free products has been that the failure of subgroup separability could be traced to the presence of finitely generated subgroups of the ambient group meeting the edge group in an infinitely generated subgroup. By this reasoning amalgamated free products of two subgroup separable groups along a $\mathbf{Z}$ would again be subgroup separable; we give an example to show that this is false. Denoting the free group of rank n by F_n we show:

Theorem 3 *There is an amalgamated free product* $(F_2 \times \mathbf{Z}) *_{\mathbf{Z}} (F_3 \times \mathbf{Z})$ *which is not subgroup separable.*

Since $F_n \times \mathbf{Z}$ is subgroup separable, Theorem 3 gives the first explicit example of two subgroup separable groups which can be amalgamated over $\mathbf{Z}$ to give a non-Lerf group. Recently Rips [11] has shown that examples of this phenomenon must exist, however his construction is neither geometric nor explicit.

More surprisingly the following is true:

Theorem 4 *There is a double of the* (4, 4, 2) *triangle group along an infinite cyclic subgroup which yields a non-Lerf group.*

This is to be contrasted with [2] and [10] where it is shown that any $\mathbf{Z}$-amalgamation of two finitely generated Fuchsian groups does result in a subgroup separable group. This shows that hyperbolic groups are better behaved than the Euclidean crystallographic groups with respect to amalgamated free products.

In terms of 3-manifolds, our main example is constructed by glueing two Seifert fibred spaces along their boundary:

Theorem 5 *Let P be a punctured torus. Then we may glue $P \times S^1$ to $P \times S^1$ along the boundary torus to obtain a closed 3-manifold with non-subgroup separable fundamental group.*

Since proving Theorem 1 we have been informed that G.P. Scott and G. Mess have recently proved a related result, by somewhat different methods.

§ 2 Main results

In this section, we prove Theorem 1 and Theorem 2. The proof of Theorem 1 depends on the following simple algebraic lemma.

Lemma. *Let θ: $G \to G$ be an automorphism of a residually finite group G. Then* $\mathrm{Fix}(\theta)$ *is separable in G.*

Proof. Let H be the subgroup of G fixed by θ, and $g \in G - H$; since g is not fixed by τ, $g^{-1}\tau(g) \neq 1$. Residual finiteness now implies that there is a homomorphism ϕ: $G \to F$, (where F is some finite group) with the property that $\phi(g^{-1} \cdot \tau(g))$ is nontrivial, in particular $\phi(\tau(g)) \neq \phi(g)$. We now have a representation, ξ: $G \to F \times F$ given by $h \to (\phi(h), \phi(\tau(h)))$, let K denote the kernel of ξ. Since τ fixes H elementwise, the image of H is diagonal, that is, its image only contains elements of the form (f, f). On the other hand, the homomorphism ϕ was chosen so that element $\xi(g)$ is not of this form, so $\xi(g) \notin \xi(H)$. We see that HK is a finite index subgroup of G, containing H but avoiding g, as required. □

Proof of Theorem 1 Let D be the 3-manifold obtained by doubling M along the surface X. Notice that D need not be hyperbolic even if M were, since M may contain essential annuli, which give rise to tori in D. However, it is clear that D is Haken, since it contains the incompressible surface X. Set G to be the fundamental group of D based at the point p; by van-Kampen's theorem G is isomorphic to the group $A_+ *_B A_-$, where $A_\pm \cong \pi_1(M, p)$, $B \cong \pi_1(X, p)$. The crucial fact from this point of view is that the group G is residually finite, by the result of [5]. We will identify $\pi_1(M, p)$ with the subgroup A_+, and $\pi_1(X, p)$ with the subgroup B of G.

Observe that the manifold D has an involution τ: $D \to D$ which fixes the basepoint, given by reflecting in the surface X. This therefore induces an automorphism of the fundamental group, which we also denote by τ. We note that the subgroup $\mathrm{Fix}(\tau)$ is precisely the subgroup $\pi_1(X, p)$. To see this let $h \in \mathrm{Fix}(\tau)$, let $\omega = a_1 \dots a_n b$ be its unique reduced word form in the amalgamated free product $A_+ *_B A$ and assume that $a_1 \in A_+ - B$. Then $\tau(\omega) = \tau(a_1) \dots \tau(a_n) b$ is also a reduced word, but $\tau(a_1) \in A_- - B$. This is a contradiction.

We may now apply the lemma to deduce that $\pi_1(X, p)$ is separable in G. Hence given an element $g \in \pi_1(M, p) - \pi_1(X, p) = A_+ - B$ we can find a finite index subgroup K in G containing $\pi_1(X, p)$ but not g. We see that $K \cap A_+$ is a finite index subgroup of $A_+ = \pi_1(M, p)$ which contains $\pi_1(X, p)$ but not g, as required. □

Remark. We need not assume the whole boundary of M is incompressible; only that the subsurface X with which we are dealing is π_1-injective. The doubled manifold D may now have compressible boundary, but by compressing maximally, we see that the resulting fundamental group is constructed from residually finite groups (i.e. Haken 3-manifold groups) by free products and is therefore still residually finite.

To prove Theorem 2, we use a folklore lemma, a proof was given in [9], but for the convenience of the reader we sketch it here. This requires the following definitions:

Definition. Let $\{W_1, W_2, \dots W_n\}$ be a collection of abstract monomials in some set of symbols $\{x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_k, x_k^{-1}\}$. Then a group H is *verbal* with respect to this collection, if for every substitution of elements of H into the words W_i we obtain the trivial element. For example, if $W = XYX^{-1}Y^{-1}$ then any abelian subgroup is verbal with respect to W.

If G is a group containing a subgroup H, set

$$H^* = \cap\{K \mid H \leqq K \leqq G \quad \text{and} \quad [G:K] < \infty\}.$$

Lemma. *Let G be a finitely generated residually finite group. Let $\{W_1, W_2, \dots W_n\}$ be a collection of abstract words, and suppose that H is a subgroup which is maximal subject to being verbal with respect to this set. Then $H^* = H$.*

Proof. Since G is finitely generated and residually finite there is a countable collection $\{K_i\}$ of normal subgroups of finite index in G with the properties that:

(a) If Q is any subgroup of finite index in G then $Q \geqq K_j$ for some j.
(b) $\cap K_i = \{e\}$.

Observe that for each fixed i the subgroup $H \cdot K_i$ has the property that every substitution of its elements into one of the W's yields an element of K_i. The reason is that such an element projects trivially under the map $G \to G/K_i$. Clearly the subgroup $H \leqq H^* = \cap H \cdot K_i$, and the above observation shows that H^* is verbal for the Ws. Whence by maximality, $H = H^*$ as required. □

Proof of theorem 2. The possibilities for abelian subgroups of a 3-manifold group are well known [6], in particular, if we set $A = i_*(\pi_1(T))$, then the only abelian subgroups of $\pi_1(M)$ which can contain A are $\mathbf{Z} \oplus \mathbf{Z}$ and $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$. In the latter case, the manifold is finitely covered by the 3-torus, which has subgroup separable fundamental group and we are done. In the former case, we must have that the index $[\mathbf{Z} \oplus \mathbf{Z}: A]$ is finite, and since the torus is embedded, it follows from [6], chapter 10, that this index must be 1 or 2; the latter case cannot occur since this gives rise to a twisted I-bundle. Therefore A is a maximal abelian subgroup, and the result follows from the lemma. □

Exactly as in [9] we may deduce:

Corollary. [8] *If a 3-manifold contains an incompressible torus, then it has virtually positive first Betti number.*

§ 3 An example

We now describe a geometric example. This illustrates that glueing 3-manifolds along their boundaries can yield bad groups. The basis of this is the construction of [3]. There it is shown that the group

$$B = \langle a, b, t, \mid t.b.t^{-1} = b, t.a.t^{-1} = b.a \rangle$$

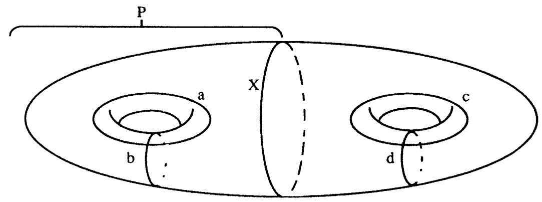

Fig. 1. The surface S

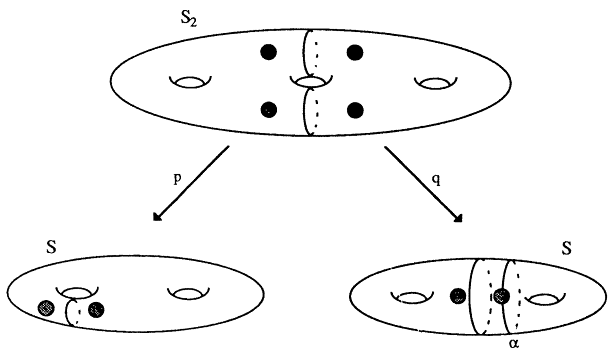

Fig. 2. The two covering maps p and q

is not subgroup separable. This is a 3-manifold group via the mapping torus construction; since we shall need to use this construction, we briefly recall the details. (For fuller explanation of mapping classes and Dehn twists, etc. we refer to [4].) If S is an orientable surface and θ: $S \to S$ is a mapping class, then the *mapping torus* of θ is the 3-manifold obtained from $S \times I$ by identifying $(x, 0)$ with $(\theta x, 1)$. This is an S-bundle over S^1, we shall denote it by $M(\theta)$. The map θ is the *monodromy* of the bundle.

Consider the left hand punctured torus P of Fig. 1, with the basis a and b for its fundamental group as shown there. Then B is the fundamental group of the mapping torus of P with monodromy given by the Dehn twist in b. We shall call this particular mapping torus M.

We now turn our attention to the genus two surface S shown in in Fig. 1, with the given homology classes as defined by that diagram. Let T_b be the Dehn twist in the curve b, and consider $M(T_b)$. Observe that this manifold decomposes along the vertical incompressible torus lying over X as M together with a $P \times S^1$. This splits the fundamental group of $M(T_b)$ as an amalgameted free product $B *_{Z \times Z}(F_2 \times \mathbf{Z})$, where F_2 is the free group of rank 2. In particular, $\pi_1(M(T_b))$ contains the non-subgroup separable group B, so that it is itself non-subgroup separable.

Define a covering of S defined by the map $H_1(S) \to \mathbf{Z}_2$ given by $b, c, d \to 0$ and $a \to 1$. This defines a double covering p: $S_2 \to S$ shown in Fig. 2. (The shaded discs and arc α will be referred to in a later example.) The preimage of the curve b is the two curves shown in Fig. 2, so it follows that there is a lift of the map T_b which we shall denote by τ_b. (Alternatively, one can compute this directly at the fundamental group level.)

Our convention in S is that the action of T_b is to twist any curve which crosses b to the left. The crucial observation from our point of view is that the lifted map is a Dehn twist *to the left* in each of the components of $p^{-1}(b)$. This is because Dehn twists only notice the orientation on the surface.

If we form the mapping torus $M(\tau_b)$, one sees that since τ_b commutes with the action of the covering group, there is a double covering map $M(\tau_b) \to M(T_b)$. In particular, it follows that $\pi_1(M(\tau_b))$ is not separable, since subgroup separability is preserved by finite extensions.

We now consider another mapping torus, this time using the Dehn twist in the element X, denoted T_X. There is another double covering $q: S_2 \to S$ defined by the map $H_1(S) \to \mathbf{Z}_2$ given by $b, d \to 0$ and $a, c \to 1$.

One checks that the pre-image of the curve X is again the pair of curves shown in Fig. 2, and the above observations show that if we denote the lifted map by τ_X then it is the Dehn twist to the left in the curves shown in Fig. 2. Thus the manifolds $M(\tau_X)$ and $M(\tau_b)$ have the same monodromy and are therefore homeomorphic. The conclusion is that the group $\pi_1(M(T_X))$ contains a subgroup of index 2 which is not subgroup separable, and therefore cannot itself be subgroup separable.

Theorem 5 now follows since if we split $\pi_1(M(T_X))$ along the vertical torus $X \times S^1$, we obtain two product bundles $P \times S^1$.

To obtain the example of Theorem 3 we now consider bounded surfaces obtained by removing the interiors of the shaded discs shown in Fig. 2. Using the same construction as above we see that the two 3-manifolds M_1 and M_2 obtained by the mapping torus construction on the surfaces of the diagram, given by the analogous Dehn twists, have commensurate fundamental groups.

The above argument is unaltered and we see that $\pi_1(M_1)$ cannot be subgroup separable. Now M_1 can be cut open along the vertical annulus A lying over the separating arc α, see Fig. 2, to obtain 3-manifolds homeomorphic to $P \times S^1$ and $K \times S^1$ where P is a punctured torus and K is a twice punctured torus. This gives a splitting of $\pi_1(M)$ as an amalgamated free product $(F_2 \times \mathbf{Z}) *_Z (F_3 \times \mathbf{Z})$ as required. Observe that this example is geometrically described as the union of two Seifert fibred spaces glued along an annulus.

The example of Theorem 4 uses very similar ideas so we only sketch the method. We must regard the group B in a different light; as in [3] we can

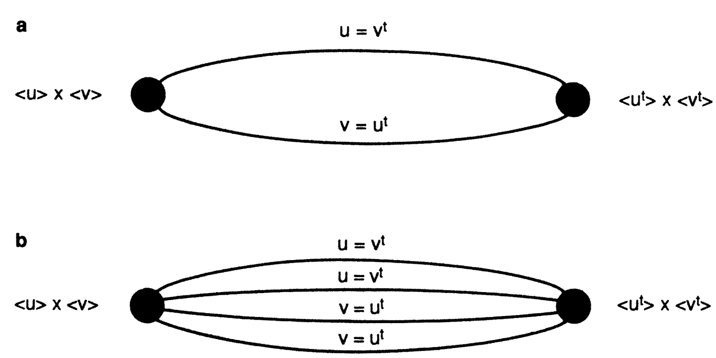

Fig. 3

give it the presentation $\langle u,v,t \mid [u,v]=1, t.u.t^{-1}=v\rangle$, and view it as an HNN extension of $\mathbf{Z}\times\mathbf{Z}$ with the two infinite cyclic subgroups $\langle u\rangle$ and $\langle v\rangle$ associated by the conjugating element t. Consider the map from B to $\mathbf{Z}_2$ given by sending u and v to 0, and t to 1. The kernel of this map is the index two subgroup H given by the presentation:

$$\langle u,v,t^2, t.u.t^{-1}, t.v.t^{-1} \mid v=t.u.t^{-1}, t^2\cdot u.t^{-2}=t.v.t^{-1}, [u,v] = [t.u.t^{-1}, t.v.t^{-1}]=1\rangle,$$

and since it is of finite index in B it cannot be subgroup separable. H splits as the graph of groups given in Fig. 3a.

We now consider the group L obtained from H by two successive HNN extensions; the first will associate $\langle v\rangle$ to $\langle t.u.t^{-1}\rangle$ and the second will associate $\langle u\rangle$ to $\langle t.v.t^{-1}\rangle$. L splits as the graph of groups given in Fig. 3.

Finally we note that L has an extension by $\mathbf{Z}_4$ which is the double of the (4,4,2) triangle group (given by the presentation $\langle x, y \mid x^4=y^4=(xy)^2=1\rangle$) amalgamating the two copies of the infinite cyclic subgroup $\langle x.y^{-1}\rangle$.

To see this note that the triangle group has an index 4 subgroup which is free abelian of rank 2 and generated by the conjugates of $\langle x.y^{-1}\rangle$; the four cosets of $\langle x.y^{-1}\rangle$ given by this subgroup correspond to the four edge groups in the splitting of L. Topologically we have an action of $\mathbf{Z}_4$ on the torus which yields the three coned sphere with cone angles $\pi/4$, $\pi/4$, and $\pi/2$. The action identifies the two loops carrying the generators u and v on the torus.

It follows that this doubled group is not subgroup separable.

This example contrasts strongly with the result in [10] where it is shown that any $\mathbf{Z}$-amalgamated free product of two finitely generated Fuchsian groups is always subgroup separable. Clearly the hyperbolic groups are better behaved than the euclidean crystallographic groups with respect to amalgamated free products. Indeed one easily sees that the group B is not commensurable with a subgroup of a hyperbolic 3-manifold group.

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