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Heegaard genus and property τ for hyperbolic 3-manifolds

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Abstract

We show that any finitely generated non-elementary Kleinian group has a co-final family of finite index normal subgroups with respect to which it has Property τ. As a consequence, any closed hyperbolic 3-manifold has a co-final family of finite index normal subgroups for which the infimal Heegaard gradient is positive.

1. *Introduction*

Let M be a finite volume hyperbolic 3-manifold and $\mathcal{L} = \{M_i\}$ some family of finite sheeted regular covers of M. We say that $\mathcal{L}$ is *co-final* if $\bigcap_i \pi_1(M_i) = \{1\}$, where, as usual, the $\pi_1(M_i)$ are all to be regarded as subgroups of $\pi_1(M)$. The *infimal Heegaard gradient* of M with respect to the family $\mathcal{L}$ is defined as:

$$\mathrm{inf_i}\,\frac{\chi_-^{\mathrm{h}}(M_i)}{[\pi_1(M):\pi_1(M_i)]},$$

where $\chi_-^{\mathrm{h}}(M_i)$ denotes the minimal value for the negative of the Euler characteristic of a Heegaard surface in M_i.

In [**12**], Lackenby showed that if $\pi_1(M)$ is an arithmetic lattice in $\mathrm{PSL}(2, \mathbf{C})$, then M has a co-final family of covers (namely, those arising from congruence subgroups) with positive infimal Heegaard gradient. The main point of the current note is to show that the same applies for every finite volume hyperbolic 3-manifold.

Theorem 1.1. *Let M be a finite volume hyperbolic 3-manifold. Then M has a co-final family of finite sheeted covers for which the infimal Heegaard gradient is positive.*

It is interesting to recall that if M has a finite sheeted cover which fibers over a circle (and a well-known conjecture, due to Thurston, asserts that every hyperbolic M has such a covering), then M has a (co-final) family of finite sheeted covers whose infimal Heegaard gradient is zero.

Theorem 1.1 is a consequence of Theorem 1.2 below. To state this theorem, we need some preliminary definitions. Let Γ be a group generated by some finite symmetric set S and let $\mathcal{L} = \{N_i\}$ be a family of finite index normal subgroups of Γ. Then the group Γ is said to have Property τ with respect to $\mathcal{L}$ if the family of Cayley graphs $X(\Gamma/N_i, S)$ forms a family of expanders (see [**14**, Chapter 4] for various equivalent forms of Property τ).

In [**12**], Lackenby showed for M, a finite volume hyperbolic 3-manifold and $\mathcal{L} = \{M_i\}$, a family of finite sheeted covers (but not necessarily a cofinal family), that if $\pi_1(M)$ has Property τ with respect to $\pi_1(M_i)$, $M_i \in \mathcal{L}$, then the infimal Heegaard gradient of M with respect to $\mathcal{L}$ is positive. Thus, Theorem 1.1 follows immediately from our next result, which can be viewed as providing a first step towards a generalization of Clozel's result [**8**] to non-arithmetic lattices.

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THEOREM 1.2. *Let Γ be a finitely generated non-elementary Kleinian group. Then Γ has a co-final family of finite index normal subgroups $\mathcal{L} = \{N_i\}$ with respect to which Γ has Property τ.*

In § 2, we review Property τ and prove a lifting result about Property τ from smaller subgroups (see Proposition 2.1) and in § 3, we prove Theorem 1.2. The proof of Theorem 1.2 uses a recent result of Bourgain and Gamburd [**3**] on expanding properies of the finite groups SL(2,p) when the generating set makes the Cayley graph have large girth. This result can then be combined with Lemma 3.2, which generalizes a result of Margulis [**18**] and shows that in our context, we can ensure that the Cayley graph does have large girth. It is perhaps worth emphasizing that our main result has a purely topological conclusion, but the methods of [**3**] are those of additive combinatorics, and in particular recent work of Tao and Vu [**20**], work of Helfgott [**10**], and sum-product estimates in finite fields ([**5**,**2**]).

The key tool in Theorem 1.2 is proving a somewhat stronger result, namely Proposition 3.1, which shows that co-final families of the required type exist for every non-virtually soluble subgroup of SL(2,k), where k is an arbitrary number field. Theorem 1.2 will now follow: if the Kleinian group Γ has finite co-covolume, then Mostow–Prasad rigidity implies that Γ admits a discrete faithful representation into $\mathrm{SL}(2, \mathbf{C})$ where the entries lie in some number field. The general case of a finitely generated Kleinian group follows by applying results of Brooks, Scott and Thurston.

Finally, in § 4 we discuss some possible generalizations of Theorem 1.2 to finding nested co-final families and to more general linear groups.

2. *Promoting τ*

Let Γ be a group, S a finite symmetric set of generators of Γ, $\mathcal{L} = \{N_i\}$ a family of finite index normal subgroups of Γ, and $X(\Gamma/N_i, S)$ the quotient Cayley graphs. Recall the definition that $X(\Gamma/N_i, S)$ form a family of expanders (see [**14**]).

DEFINITION. A finite k-regular graph $X = X(V, E)$ with a set V of n vertices is called an (n, k, c)-expander if for every subset $A \subset V$:

$$|\partial A| \geqslant c\left(1 - \frac{|A|}{n}\right)|A|,$$

where $\partial A = \{v \in V : d(v, A) = 1\}$ and d the distance function on the graph.

A family of k-regular graphs (k fixed) is called an expander family if there is a $c > 0$ such that all of the graphs are (n, k, c)-expanders.

In this notation the family of Cayley graphs $X(\Gamma/N_i, S)$ are $([\Gamma : N_i], |S|, c)$-expanders for some $c > 0$.

There are various methods of lifting Property τ from smaller subgroups of a larger group to the larger group (see [**7**] and [**15**]). These have been used in the context of arithmetic groups. We now provide another method (inspired by [**14**, p. 52, Example E]) which applies to both arithmetic and non-arithmetic groups.

PROPOSITION 2.1. *Let Γ be a finitely generated group, and $\mathcal{L} = \{N_i\}$ a family of finite index normal subgroups of Γ. Suppose that $H < \Gamma$ (not necessarily of finite index) is finitely generated, and assume that H surjects onto the finite quotients Γ/N_i for all but a finite number of i. Then if H has Property τ with respect to the family $\{H \cap N_i\}$, Γ has Property τ with respect to $\mathcal{L}$.*

Proof. Fix a symmetric generating set S_H for H and extend this to a generating set S for Γ. By assumption, the quotient Cayley graphs $X(H/H\cap N_i, S_H)$ form a family of expanders. Now H surjects onto the finite quotient Γ/N_i for all but a finite number of i and so we can ignore this finite number for our considerations since throwing away a finite number of graphs will not affect the fact that a family forms a set of expanders. It follows that $X(\Gamma/N_i, S)$ is also a family of expanders, since for every subset A of Γ/N_i, the computation of the ratio $|\partial A|/((1-|A|/n_i)|A|)$ (where $n_i = [\Gamma : N_i] = [H : H\cap N_i]$) can only be increased by the addition of the extra edges coming from enlarging the generating set. □

3. *Proof of Theorem* 1.2

The key result in the proving of Theorem 1.2 is our next proposition.

PROPOSITION 3.1. *Let k be a number field and let Γ be a finitely generated subgroup of* SL(2,k) *which is not virtually soluble. Then Γ has a co-final family of finite index normal subgroups $\mathcal{L} = \{N_i\}$ with respect to which Γ has Property τ.*

Proof. Since Γ is finitely generated, we can assume that $\Gamma <$ SL(2,A) where A is a ring of S-integers in a number field k, of degree n say, over $\mathbf{Q}$. We fix some notation. Let $\mathcal{P}$ be a prime ideal of A with residue class field $\mathbf{F}$, and let

$$\pi_{\mathcal{P}} : \mathrm{SL}(2, A) \longrightarrow \mathrm{SL}(2, \mathbf{F}),$$

be the reduction homomorphism.

Since Γ is not virtually soluble, it contains a non-abelian free subgroup $F =< a^{\pm 1}, b^{\pm 1} >$. By the Cebotarev density theorem, we can find infinitely many rational primes p which split completely in k, and it follows from Strong Approximation (see [**13**] for an elementary argument in the case of SL(2)) that for all but finitely many of the rational primes p that split completely in k, F surjects SL(2,p) under the homomorphisms $\pi_{\mathcal{P}}$. Also, note that ker $\pi_{\mathcal{P}} \cap \Gamma$ (respectively ker $\pi_{\mathcal{P}} \cap \mathrm{F}$) forms a co-final family of normal subgroups of finite index in Γ (respectively F).

We need the following lemma (cf. [**18**]). Recall that the *girth* of a finite graph X is the length of the shortest non-trivial closed path in X.

LEMMA 3.2. *There is a constant $C = C(a,b)$ so that the girth of the Cayley graph of* SL(2,p) *with respect to the generating set $\{\pi_{\mathcal{P}}(a^{\pm 1}), \pi_{\mathcal{P}}(b^{\pm 1})\}$ is at least $C\log(p)$.*

This lemma will be proved below. Assuming Lemma 3.2, the proof of Proposition 3.1 is completed by the following result of Bourgain and Gamburd (see [**3**, Theorem 3]).

THEOREM 3.3. *Suppose that for each p, S_p is some symmetric generating set for* SL(2,p), *of fixed size independent of p, such that the girth of the Cayley graph $X(\mathrm{SL}(2,p), S_p)$ is at least $C\log(p)$ (where C is independent of p). Then X(SL(2,p),S_p) forms a family of expanders.*

Thus to complete the proof, applying Lemma 3.2 to the free subgroup F of Γ (with generating set as before), we can apply Theorem 3.3 to SL(2,p) with these generating sets and the result now follows from Proposition 2.1 and our previous discussion.

Proof of Lemma 3.2. Denote the ring of integers of k by R_k. If $\alpha \in R_k$, define:

$$\mu(\alpha) = \max\{|\alpha'| \ : \ \alpha' \text{ is a Galois conjugate of } \alpha\}.$$

Here, $|*|$ denotes the complex absolute value. Note that since α is an algebraic integer, $\mu(\alpha) \geqslant 1$ with equality if and only if α is a root of unity.

It follows easily from the definition that $\mu(\alpha+\beta) \leqslant \mu(\alpha)+\mu(\beta)$ and $\mu(\alpha\cdot\beta) \leqslant \mu(\alpha)\cdot\mu(\beta)$, since for example, in computing the maximum for $|\alpha'+\beta'|$, one clearly cannot do better than maximize the two terms of $|\alpha'|+|\beta'|$.

Given a matrix $t \in \mathrm{SL}(2,A)$, we may write t as $1/\alpha \cdot t^*$, where $t^* \in M(2,R_k)$ and $\alpha \in R_k$. Take M to be the biggest value of μ taken over the entries of the matrices a^*, $(a^{-1})^*$, b^* and $(b^{-1})^*$, together with the four denominators of those matrices.

Claim. If w is a word of length r in matrices taken from a^*, $(a^{-1})^*$, b^* and $(b^{-1})^*$, then the entries of w cannot have their μ value be larger than $(2M)^r$.

The proof of the claim is by induction on r: Consider $X\cdot a^*$, for example, where X is a word of length $r-1$. The entries of the product $X\cdot a^*$ have the form $x_1a_1+x_2a_2$, where x_i is an entry in X, etc. Then by the above remarks $\mu(x_1a_1+x_2a_2) \leqslant \mu(x_1a_1)+\mu(x_2a_2) \leqslant \mu(x_1)\mu(a_1)+\mu(x_2)\mu(a_2)$ and by induction, this is at most $(2M)^{r-1}M+(2M)^{r-1}M=(2M)^r$. This completes the proof of the claim.

Now suppose that $p >> 0$ is a rational prime that splits completely in k and $\mathcal{P}$ is a k-prime dividing p. Let r denote the girth, and let $w(a,b) \in F$ be a reduced word of length r which projects to a cycle of length r under $\pi_{\mathcal{P}}$. Clearing denominators in the congruence $w(a,b) = \mathrm{id} \bmod \mathcal{P}$, we obtain a congruence between R_k-integral matrices

$$w(a^*,(a^{-1})^*,b^*,(b^{-1})^*) = Z\cdot \mathrm{id} \bmod \mathcal{P}.$$

By the claim, the entries on the left-hand side have their μ values bounded above by $(2M)^r$, and Z is a product of r integers with μ value at most M, so $\mu(Z) \leqslant M^r$.

Now the integral matrix $w(a^*,(a^{-1})^*,b^*,(b^{-1})^*) - Z\cdot\mathrm{id}$ is not identically zero, since a and b generate a free group of rank two. Let β be one of its non-zero entries. The above remarks show that $\mu(\beta)$ is bounded above by $(2M)^r+M^r < (3M)^r$, say. Notice that $\beta \in \mathcal{P}$, and the $k/\mathbf{Q}$-norm of β is a non-zero integer which is divisible by p, since we take the product of all the conjugates of β which lie in conjugates of $\mathcal{P}$. Recalling that n is the degree of k over the rationals, it follows from the definition of μ that this integer is bounded above by $\mu(\beta)^n \leqslant ((3M)^r)^n$.

We therefore deduce that in order to be divisible by p, r must be large enough, so that $(3M)^{rn} > p$; that is to say, $r \geqslant C\log(p)$ with $C = 1/(n\log(3M))$ as required.

Proof of Theorem 1.2. Without any loss of generality, we may assume that Γ is torsion free. In the case that Γ has finite co-volume, then as remarked in § 1, it follows from local rigidity that Γ (or more precisely a lift to $\mathrm{SL}(2,\mathbf{C})$) can be conjugated into $\mathrm{SL}(2,k)$ for some number field k (indeed k can be chosen to be a quadratic extension of the trace-field; see [**17**] Corollary 3.2.4), and Proposition 3.1 applies.

Now assume that Γ has infinite co-volume. Since Γ is non-elementary, it is not virtually soluble, and is either geometrically finite or geometrically infinite. In the former case, we can apply a result of Brooks [**6**] that produces a quasi-conformal conjugate Γ' of Γ that is a subgroup of a Kleinian group of finite co-volume. Hence, as in the previous paragraph, this implies $\Gamma' < \mathrm{SL}(2,k)$ for some number field k. In the case when Γ is geometrically infinite, as pointed out in [**1**] for example, it is a consequence of the Scott Core Theorem and Thurston's Hyperbolization Theorem for Haken manifolds that there exists a group Γ' isomorphic to Γ with Γ' geometrically finite. We now argue as before. □

4. *Final remarks*

4.1. Although Theorem 1.2 provides a co-final family, it does not provide a family which is co-final and *nested*. In the proof of Theorem 1.2, we used [**3**] which does not provide a nested family. Recent work of Bourgain, Gamburd and Sarnak [**4**] extends some of [**3**] to products of primes, and may yet eventually lead to a co-final nested family in groups Γ as above.

However, even in the absence of this result, we can produce a rich class of non-arithmetic Kleinian groups that do contain a co-final nested family.

Proposition 4.1. *Let Γ be a Kleinian group of finite co-volume that contains an arithmetic Fuchsian subgroup. Then Γ contains a co-final nested family $\mathcal{L} = \{N_i\}$ of normal subgroups of finite index, such that Γ has Property τ with respect to $\mathcal{L}$.*

Proof. We begin with a preliminary remark. Let F be an arithmetic Fuchsian group. As is pointed out in [**14**, pp. 51–52] for example, it follows from a result of Selberg and an application of the Jacquet–Langlands correspondence, that F has Property τ with respect to the entire family of its congruence subgroups.

Thus, if F is an arithmetic Fuchsian subgroup of Γ, then after possibly discarding perhaps finitely many prime ideals, we can form 'congruence subgroups of Γ' obtained via reduction homomorphisms. Furthermore, by considering only the primes that split completely in the invariant trace-field of Γ (and hence, also in the invariant trace-field of F), we can arrange that there are infinitely many congruence quotients of Γ for which F surjects.

In fact, an easy argument using Strong Approximation and the Chinese Remainder Theorem shows that we can find a sequence of completely split primes, so that F surjects each of the reductions modulo the descending sequence of ideals $I_n = \mathcal{P}_1 \dots \mathcal{P}_n$. These now form a co-final nested family of congruence subgroups. The result now follows from Proposition 2.1 with $F = H$. □

Examples of non-arithmetic Kleinian groups Γ satisfying the hypothesis of Proposition 4.1 are plentiful as we now discuss.

Construction of examples

Example 1. A thrice-punctured sphere has a unique hyperbolic structure arising as $\mathbf{H}^2/\Gamma(2)$, where $\Gamma(2)$ is the principal congruence subgroup of level 2 in the modular group. In particular, $\Gamma(2)$ is an arithmetic Fuchsian group. Many non-arithmetic link complements contain an immersed (or embedded) thrice-punctured sphere. For example, all the non-arithmetic hyperbolic twist knot complements. By [**11**], these are all mutually incommensurable.

Example 2. The examples of non-arithmetic hyperbolic manifolds of Gromov and Piatetski-Shapiro [**9**] as hybrids of arithmetic ones. By construction, these contain an arithmetic totally geodesic surface.

Example 3. One can also easily obtain closed examples by only 3-dimensional methods using the construction of [**19**]. There, a non-compact hyperbolic orbifold with a torus cusp is constructed, the boundary of which consists of two totally geodesic isometric copies of a hyperbolic 2-orbifold $\mathbf{H}^2/\Delta$ for some hyperbolic triangle group Δ. Doubling this orbifold along the totally geodesic boundary gives a two-cusped hyperbolic orbifold $\mathbf{H}^3/\Gamma$ for which Γ contains triangle groups. This construction can be done where the triangle group is chosen arithmetic (see [**19**]). These are rigid, so any surgeries on the cusps will leave them as totally geodesic and arithmetic. Sufficiently high order Dehn surgeries will produce non-arithmetic hyperbolic 3-orbifolds (for example by using Borel's result on the discreteness of the set of co-volumes of arithmetic Kleinian groups; see [**17**, Chapter 11.2.1]).

4.2. It seems quite possible that Theorem 1.2 and Proposition 3.1 hold for any finitely generated subgroup of $\mathrm{SL}(2,\mathbf{C})$ which is not virtually soluble. At present, our methods only prove the following.

THEOREM 4.2. *Let Γ be a finitely generated subgroup of $\mathrm{SL}(2,\mathbf{C})$ which is not virtually soluble. Then Γ has an infinite family $\mathcal{L} = \{N_i\}$ of finite index normal subgroups, such that Γ has Property τ with respect to $\mathcal{L}$.*

This result follows by noting that Proposition 3.1 implies that we are done, unless Γ contains an element whose trace is transcendental; in this case, we may choose an algebraic specialization where the image group is not virtually soluble (see [**16**, Proposition 2.2] for a more general version of this specialization result).

At present our argument works only for SL(2), since there is no analogue of the result of [**3**] yet known for groups $\mathrm{SL}(n)$, $n > 2$. However, it seems reasonable to expect the following stronger conjecture to hold.

CONJECTURE. Let Γ be a finitely generated subgroup of $\mathrm{GL}(n,\mathbf{C})$ whose Zariski closure is semi-simple. Then Γ has a co-final (nested) family $\mathcal{L} = \{N_i\}$ of finite index normal subgroups, for which Γ has Property τ with respect to $\mathcal{L}$.

This Conjecture would provide a far-reaching generalization of Clozel's work [**8**] mentioned in § 1.

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