CONSTRUCTING THIN SUBGROUPS OF $\mathrm{SL}(n+1,\mathbb{R})$ VIA BENDING

SAMUEL BALLAS AND D. D. LONG

ABSTRACT. In this paper we use techniques from convex projective geometry to produce many new examples of thin subgroups of lattices in special linear groups that are isomorphic to the fundamental groups of finite volume hyperbolic manifolds. More specifically, we show that for a large class of arithmetic lattices in $\mathrm{SO}(n,1)$ it is possible to find infinitely many non-commensurable lattices in $\mathrm{SL}(n+1,\mathbb{R})$ that contain a thin subgroup isomorphic to a finite index subgroup of the original arithmetic lattice. This class of arithmetic lattices includes all non-cocompact arithmetic lattices and all cocompact arithmetic lattices when n is even.

CONTENTS

Let G be a semi-simple Lie group and let $\Gamma \subset G$ be a lattice. A subgroup $\Delta \subset \Gamma$ is called *a thin group* if Δ has infinite index in Γ and is Zariski dense in G. Over the last several years, there has been a great deal of interest in thin subgroups of lattices in a variety of Lie groups [12, 24, 11]. Much of this interest has been motivated by work of Bourgain, Gamburd, and Sarnak [9] related to expanders and "affine sieves." More generally, there is an increasingly strong sense that thin groups have many properties in common with lattices in G.

Furthermore, there is evidence that suggests that generic discrete subgroups of lattices are thin and free (see [11, 13]). However, there is also great interest in constructing thin groups that are not free (or even decomposable as free products). For instance the seminal work of Kahn and Markovic [16] constructs many thin subgroups contained in any cocompact lattice of $\mathrm{SL}(2,\mathbb{C})$ that are isomorphic to the fundamental group of a closed surface. There are several generalizations of this result that exhibit thin surface groups in a variety of Lie groups. For instance, Cooper and Futer [10] recently proved a similar result for non-compact lattices in $\mathrm{SL}(2,\mathbb{C})$ and Kahn, Labouire and Mozes [15] proved an analogue for cocompact lattices in a large class of Lie groups.

These result naturally lead to the question of which isomorphism types of groups can occur as thin groups. In this paper we provide a partial answer by showing that in each dimension there are infinitely

many finite volume hyperbolic manifolds whose fundamental groups arise as thin subgroups of lattices in special linear groups. Our main result is:

Theorem 0.1. *Let Γ be a cocompact (resp. non-cocompact) arithmetic lattice in $\mathrm{SO}(n,1)$ of orthogonal type then there are infinitely many non-commensurable cocompact (resp. non-cocompact) lattices in $\mathrm{SL}(n+1,\mathbb{R})$ that each contain a thin subgroup isomorphic to a finite index subgroup of Γ.*

It turns out that all non-cocompact arithmetic lattices in $\mathrm{SO}(n,1)$ are of orthogonal type (see the introduction of [18] and §6.4 of [26]), and so we have the following immediate corollary of Theorem 0.1.

Corollary 0.2. *Let Γ be a non-cocompact arithmetic lattice in $\mathrm{SO}(n,1)$ then there are infinitely many non-cocompact lattices in $\mathrm{SL}(n+1,\mathbb{R})$ that contain a thin subgroup isomorphic to a finite index subgroup of Γ.*

In the cocompact setting, there is another construction of arithmetic lattices in $\mathrm{SO}(n,1)$ using quaternion algebras. However, this construction only works when n is odd (again, see [18] and §6.4 of [26]), which implies:

Corollary 0.3. *Let $n \geq 3$ be even and let Γ be a cocompact arithmetic lattice in $\mathrm{SO}(n,1)$ then there are infinitely many cocompact lattices in $\mathrm{SL}(n+1,\mathbb{R})$ that contain a thin subgroup isomorphic to a finite index subgroup of Γ*

Our main result generalizes several previous results regarding the existence of thin groups isomorphic to hyperbolic manifolds in low dimensions. For example, there are examples of thin surface groups in both cocompact and non-cocompact lattices in $\mathrm{SL}(3,\mathbb{R})$ [20, 19]. There are further examples of thin subgroups in $\mathrm{SL}(4,\mathbb{R})$ isomorphic to the fundamental groups of closed hyperbolic 3-manifolds [21] and others isomorphic to the fundamental groups of finite volume hyperbolic 3-manifolds [4].

Organization of the paper. Section 1 provides the necessary background in convex projective geometry. Section 2 describes the relevant arithmetic lattices in both $\mathrm{SO}(n,1)$ and $\mathrm{SL}(n+1,\mathbb{R})$. Section 3 contains the construction of the thin groups in Theorem 0.1. Finally, Section 4 contains the proof that the examples constructed in Section 3 are thin.

1. Convex projective geometry

Let $V = \mathbb{R}^{n+1}$. There is an equivalence relation on the non-zero vectors in V given by $x \sim y$ if there is $\lambda > 0$ such that $\lambda x = y$. The set $S(V)$ of equivalence classes of $\sim$ is called the *projective n-sphere*. Alternatively, $S(V)$ can be regarded as the set of rays through the origin in V. Sending each equivalence class to the unique representative of length 1 gives an embedding of $S(V)$ into V as the unit n-sphere.

The group $\mathrm{GL}(V)$ acts on $S(V)$, however this action is not faithful. The kernel of this action consists of positive scalar multiples of the identity, $\mathbb{R}^+ I$. Furthermore, if $A \in \mathrm{GL}(V)$ then $|\det(A)|^{\frac{-1}{(n+1)}} A$ has determinant ± 1 and as a result we see that there is a faithful action of

$$\mathrm{SL}^{\pm}(V) = \{A \in \mathrm{GL}(V) \mid \det(A) = \pm 1\}$$

on $S(V)$.

The projective sphere is a 2-fold cover of the more familiar *projective space* $P(V)$ consisting of lines through the origin in V. The covering map is given by mapping a ray through the origin to the line through the origin that contains it. There is also a 2-fold covering of Lie groups from $\mathrm{SL}^{\pm}(V)$ to $\mathrm{PGL}(V)$ that maps an element of $\mathrm{SL}^{\pm}(V)$ to its scalar class.

Each (open) hemisphere in $S(V)$ can be identified with $\mathbb{R}^n$ via projection, in such a way that great circles on $S(V)$ are mapped to straight lines in $\mathbb{R}^n$ (see Figure 1). For this reason we refer to (open) hemispheres as *affine patches* of $S(V)$ and refer to great circles as *projective lines*. This identification allows us to define a
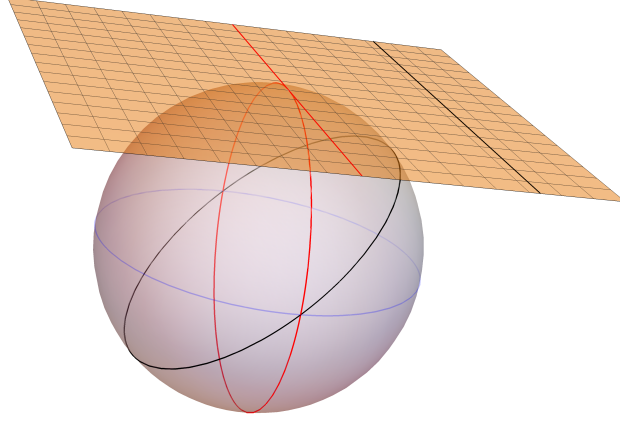
FIGURE 1. The projection to an affine patch

notion of convexity for subsets of an affine patch. A set $\Omega \subset S(V)$ with non-empty interior is called *properly convex* if its closure is a convex subset of some affine patch. Each properly convex set Ω comes equipped with a group $\mathrm{SL}(\Omega)$ consisting of elements of $\mathrm{SL}^\pm(V)$ that preserve Ω. There is a similar definition for properly convex subsets of $\mathbb{RP}^n$ and we will allow ourselves to discuss properly convex geometry in whichever setting is more convenient.

An important example of a properly convex set is *hyperbolic n-space*, which can be constructed as follows. Let q be the quadratic form on V given by the matrix

$$(1.1) \qquad J_n = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}.$$

This form has signature $(n, 1)$, and let $\mathcal{C}_q$ be a component of the cone $\{v \in V \mid q(v) < 0\}$. The image of $\mathcal{C}_q$ in $S(V)$ gives a model of hyperbolic space called the *Klein model* of hyperbolic space which we denote $\mathbb{H}^n$. In this setting, $\mathrm{SL}(\mathbb{H}^n)$ is equal to the group $O(J_n)^+$ of elements of $\mathrm{SL}^\pm(V)$ that preserve both J_n and $\mathcal{C}_q$.

To each properly convex $\Omega \subset S(V)$ it is possible to construct a *dual convex set* $\Omega^* \subset S(V^*)$ defined by

$$\Omega^* = \{[\phi] \in S(V^*) \mid \phi(v) > 0 \ \forall [v] \in \overline{\Omega}\}$$

It is a standard fact that Ω^* is a properly convex subset of $S(V)$. For each $\gamma \in \mathrm{SL}(\Omega)$ there is a corresponding $\gamma^* \in \mathrm{SL}(\Omega^*)$ given by $\gamma^*([\phi]) = [\phi \circ \overline{\gamma}]$, where $\overline{\gamma}$ is any element of $\mathrm{GL}(V)$ in the projective class of γ. This map induces an isomorphism between $\mathrm{SL}(\Omega)$ and $\mathrm{SL}(\Omega^*)$. By choosing a basis for V and the corresponding dual basis for V^*, it is possible to identify $\mathrm{SL}(V^*)$ and $\mathrm{SL}(V)$ and in these coordinates the isomorphism between $\mathrm{SL}(\Omega)$ and $\mathrm{SL}(\Omega^*)$ is given by $\gamma \mapsto (\gamma^{-1})^t$.

It follows that if Ω/Γ is a properly convex manifold that there is a corresponding *dual group* $\Gamma^* \subset \mathrm{SL}(\Omega^*)$ and a corresponding *dual properly convex manifold* Ω^*/Γ^*. The manifolds Ω/Γ are diffeomorphic, but are in general not projectively equivalent.

Furthermore, if Ω is properly convex and $\Gamma \subset \mathrm{SL}(\Omega)$ is discrete then Ω/Γ is a *properly convex orbifold*. If Γ is torsion-free then this orbifold is a manifold. By Selberg's lemma, every properly convex orbifold is finitely covered by a properly convex manifold, and for the remainder of the paper we will almost exclusively be dealing with manifolds. An important example is when $\Omega = \mathbb{H}^n$ and $\Gamma \subset \mathrm{SL}(\mathbb{H}^n)$ is a discrete, torsion-free group. In this case Ω/Γ is a *complete hyperbolic manifold*.

If N is an orientable manifold then a *properly convex structure* on N is a pair $(\Omega/\Gamma, f)$ where Ω/Γ is a properly convex manifold and $f : N \to \Omega/\Gamma$ is a diffeomorphism. The map f induces an isomorphism $f_* : \pi_1 N \to \Gamma$. Since $\Gamma \subset \mathrm{SL}^\pm(V)$ we can regard f_* as a representation from $\pi_1 N$ into the Lie group $\mathrm{SL}^\pm(V)$
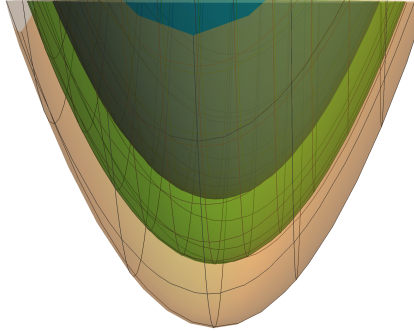
FIGURE 2. The domain Ω_0 and its foliation by horospheres

which we call the *holonomy* of the structure $(\Omega/\Gamma, f)$. Since N is orientable it is easy to show that the holonomy always has image on $\mathrm{SL}(V)$. Observe that by definition, the holonomy is an isomorphism between $\pi_1 N$ and Γ, and it follows immediately that the holonomy representation is injective.

Given a properly convex structure $(\Omega/\Gamma, f)$ on N and an element $g \in \mathrm{SL}^{\pm}(V)$ it is easy to check that $g : \Omega \to g(\Omega)$ induces a diffeomorphism $\overline{g} : \Omega/\Gamma \to g(\Omega)/g\Gamma g^{-1}$ and that $(g(\Omega)/g\Gamma g^{-1}, f \circ \overline{g})$ is also a properly convex structure on N. Furthermore, the holonomy of this new structure is obtained by post-composing f_* by conjugation in $\mathrm{SL}^{\pm}(V)$ by g. Two properly convex structures $(\Omega/\Gamma, f)$ and $(\Omega'/\Gamma', f')$ on N are *equivalent* if there is $g \in G$ such that $\Omega'/\Gamma' = \Omega/g\Gamma g^{-1}$, and f' is isotopic to $\overline{g} \circ f$.

1.1. Generalized cusps.

A generalized cusp is a certain type of properly convex manifold that generalizes a cusp in a finite volume hyperbolic manifold. Specifically, a properly convex manifold $C \cong \Omega/\Gamma$ is a *generalized cusp* if Γ is a virtually abelian and $C \cong \partial C \times (0, \infty)$ with ∂C a compact strictly convex submanifold of C. Such manifolds were recently classified by the first author, D. Cooper, and A. Leitner [1]. One consequence of this classification is that in dimension n there are $n + 1$ different *types* of generalized cusps. For the purposes of this work only two of these types (type 0 and type 1). We will also restrict to cusps with the property that ∂C is diffeomorphic to an $(n-1)$-torus will be relevant, and we now briefly describe these types of cusps.

Let

$$\Omega_0 = \left\{ [x_1 \cdots : x_{n+1}] \in P(V) \mid x_1 x_{n+1} > \frac{1}{2} \left(x_2^2 + \ldots + x_n^2 \right) \right\}.$$

It is not difficult to see that Ω_0 is projectively equivalent to the Klein model for hyperbolic space. Let P_0 be the collection (of equivalence classes) of matrices with block form

$$(1.2) \qquad \begin{pmatrix} 1 & v & \frac{1}{2}|v|^2 \\ 0 & I_{n-1} & v^t \\ 0 & 0 & 1 \end{pmatrix},$$

where v is a (row) vector in $\mathbb{R}^{n-1}$, I_{n-1} is the identity matrix and the zeros are blocks of the appropriate size to make (1.2) a $(n+1) \times (n+1)$ matrix. A simple computation shows that the elements of P_0 preserve Ω_0 (they are just the parabolic isometries of $\mathbb{H}^n$ that fix $\infty = [1 : 0 \ldots : 0]$). There is a foliation of Ω_0 by strictly convex hypersurfaces of the form

$$\mathcal{H}_c = \left\{ [x_1 : \ldots : x_n : 1] \mid x_1^2 - \frac{1}{2}(x_2 + \ldots + x_n^2) = c \right\},$$

for $c > 0$ whose leaves are preserved setwise by P_0. In terms of hyperbolic geometry the $\mathcal{H}_c$ are *horospheres* centered at ∞ and the convex hull of a leaf is a *horoball* centered at ∞. The group P_0 is isomorphic to $\mathbb{R}^{n-1}$
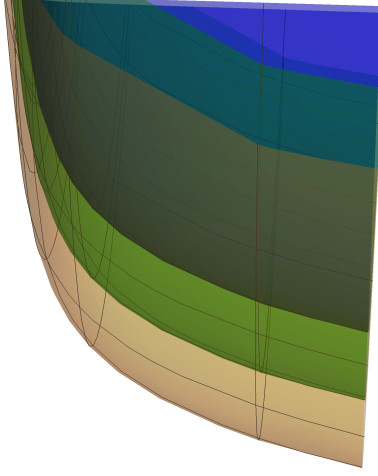
FIGURE 3. The domain Ω_1 and its foliation by horospheres

and so if $\Gamma \subset P_0$ is a lattice then Γ is isomorphic to $\mathbb{Z}^{n-1}$ and the quotient Ω/Γ is a *generalized (torus) cusp of type 0.*

Next, let

$$\Omega_1 = \left\{ [x_1 : \ldots : x_{n+1}] \mid x_1 x_{n+1} > -\log|x_2| + \frac{1}{2}(x_3^2 + \ldots + x_n^2),\ x_2 x_{n+1} > 0 \right\}$$

and let P_1 be the collection (of equivalence classes) of matrices of block form

$$(1.3) \qquad \begin{pmatrix} 1 & 0 & v & -u + \frac{1}{2}|v|^2 \\ 0 & e^u & 0 & 0 \\ 0 & 0 & I_{n-2} & v^t \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $u \in \mathbb{R}$, $v \in \mathbb{R}^{n-2}$, I_{n-2} is the identity matrix and the zeros are the appropriate size to make (1.3) an $(n+1) \times (n+1)$ matrix. Again, it is easy to check that P_1 preserves Ω_1. Again, there is a foliation of Ω_1 by strictly convex hypersurfaces of the form

$$\mathcal{H}_c = \left\{ [x_1 : \ldots : x_n : 1] \mid x_1 + \log x_2 - \frac{1}{2}(x_3^2 + \ldots + x_n^2) = c,\ x_2 > 0 \right\}$$

for $c > 0$ that is preserved by P_1. Again, each leaf is a P_1 orbit, we call the leaves of this foliation *horospheres* and call the convex hulls of a leaves *horoballs*. Again $P_1 \cong \mathbb{R}^{n-1}$ and if $\Gamma \subset P_1$ is a lattice then $\Gamma \cong \mathbb{Z}^{n-1}$ and Ω_1/Γ is a *generalized (torus) cusp of type 1.* For the remainder of this paper when we say generalized cusp that will mean a generalized torus cusp of type 0 or type 1.

Generalized cusps of a fixed type are closed under two important operations: taking finite sheeted covers and duality. If Ω/Γ is a generalized cusp then taking a finite sheeted cover corresponds to choosing a finite index subgroup $\Gamma' \subset \Gamma$. The group Γ' is also a lattice in P_0 or P_1 and hence Ω/Γ' is a generalized cusp. The fact that generalized cusps are closed under duality follows immediately from the observation that the group P_0^t (resp. P_1^t) obtained by taking the transpose of the elements of P_0 (resp. P_1) is conjugate to P_0 (resp. P_1).

One distinction between these two types of cusps that will be important for our purposes in Section 4 is that the group P_0 is Zariski closed, but the group P_1 is not. The Zariski closure, $\overline{P_1}$ of P_1 is n-dimensional

and consists of matrices of the form

$$(1.4) \qquad \begin{pmatrix} 1 & 0 & v & w \\ 0 & u & 0 & 0 \\ 0 & 0 & I_{n-2} & v^t \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $u \neq 0$, $w \in \mathbb{R}$, and $v \in \mathbb{R}^{n-2}$. Furthermore, we have the following lemma describing the generic orbits of $\overline{P_1}$ whose proof is a straightforward computation.

Lemma 1.1. *If $x \notin \ker(e_2^*) \cup \ker(e_{n+1}^*)$ then $\overline{P_1} \cdot x$ is open in $\mathbb{RP}^n$.*

Given a properly convex manifolds $M = \Omega/\Gamma$ we say that *M is a manifold with generalized cusp ends* if M can be written as $M = K \sqcup \mathcal{C}$ where K is a non-empty compact submanifold and $\mathcal{C} = \sqcup_i C_i$ and each C_i is projectively equivalent to a generalized cusp.

1.2. **Properties of the holonomy.** In this section we discuss some important properties of the holonomy representation of a convex projective structure, particularly in the presence of generalized cusps.

Given a properly convex manifolds $M = \Omega/\Gamma$ we say that *M is a manifold with generalized cusp ends* if M can be written as $M = K \sqcup \mathcal{C}$ where K is a non-empty compact submanifold and $\mathcal{C} = \sqcup_i C_i$ and each C_i is projectively equivalent to a generalized cusp.

A representation $\rho : \Gamma \to \mathrm{GL}(V)$ is called *strongly irreducible* if its restriction to any finite index subgroup is irreducible. The main result of this section is the following:

Theorem 1.2. *Let $(\Omega/\Gamma, f)$ be a convex projective structure on M and let ρ be its holonomy. If Ω/Γ is a manifold with generalized cusp ends then ρ is strongly irreducible.*

Before proceeding, we need a few lemmas. If P is a subset contained in some affine patch in $S(V)$ then let $\mathcal{CH}(P)$ denote the convex hull of P (note that since P is contained in an affine patch that this is well defined).

Lemma 1.3. *Suppose that $M = \Omega/\Gamma$ is a properly convex manifold with generalized cusp ends then for any $p \in \overline{\Omega}$, $\mathcal{CH}(\Gamma \cdot p)$ has non-empty interior.*

Proof. Let $p \in \overline{\Omega}$ and let Λ be the fundamental group of one of the generalized cusps of M. By taking a conjugate of Λ in Γ if necessary it is possible to ensure that Λ does not preserve any proper projective subspace containing p. It follows that $\mathcal{CH}(\Lambda \cdot p)$ contains a horoball and thus $\mathcal{CH}(\Gamma \cdot p)$ has non-empty interior. $\qquad \square$

The following Lemma is the basis for the proof of Theorem 1.2. The lemma and its proof are inspired by a similar result of J. Vey [25, Prop. 4].

Lemma 1.4. *Suppose that $\Omega \subset P(V)$ is properly convex and that $\Gamma \subset \mathrm{SL}(\Omega)$ is a group with the property that for every $p \in \overline{\Omega}$, $\mathcal{CH}(\Gamma \cdot p)$ has non-empty interior. If L is a Γ-invariant subspace of V and $P(L) \cap \overline{\Omega} \neq \emptyset$ then $L = V$.*

Proof. Let $L \subset V$ be a Γ-invariant subspace such that $P(L) \cap \overline{\Omega} \neq \emptyset$, and let p be a point in the intersection. Since $p \in \overline{\Omega}$ it follows that $\mathcal{CH}(\Gamma \cdot p)$ has non empty interior. Furthermore, since $p \in L$ and L is both Γ-invariant and convex it follows that $\mathcal{CH}(\Gamma \cdot p) \subset P(L)$. Since $\mathcal{CH}(\Gamma \cdot p)$ has non-empty interior so does $P(L)$. It follows that $L = V$. $\qquad \square$

We can now prove Theorem 1.2

Proof of Theorem 1.2. Suppose that $L \subset V$ is a Γ-invariant subspace. First assume that $P(L) \cap \overline{\Omega} \neq \emptyset$. Combining Lemmas 1.3 and 1.4 it follows that $L = V$. On the other hand, suppose that $L \cap \overline{\Omega} = \emptyset$ then L corresponds to a non-trivial subspace $L^* \subset V^*$ such that $P(L^*) \cap \overline{\Omega^*} \neq \emptyset$. Since Ω^*/Γ^* is also a manifold with generalized cusp ends we can apply the same argument as before to show that $L^* = V^*$, and so $L = 0$. It follows that Γ acts irreducibly on V.

Finally, if Γ' is a finite index subgroup of Γ then Ω/Γ' is also a properly convex manifold with generalized cusp ends and so by the argument above Γ' acts irreducibly on V.

$\square$

1.3. **Bending.** We now describe a construction that allows one to start with a (special) hyperbolic manifold and produce a family of inequivalent convex projective structures.

Suppose that $M = \mathbb{H}^n/\Gamma$ is a complete, finite-volume hyperbolic manifold, and suppose that M contains an embedded totally geodesic hypersurface, Σ. There is an embedding of $\mathrm{SO}(J_{n-1})$ into $\mathrm{SO}(J_n)$ via the embedding

$$\mathrm{SO}(J_{n-1}, \mathbb{R}) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{SO}(J_{n-1}, \mathbb{R}) \end{pmatrix}.$$

Under this embedding, the image of $\mathrm{SO}(J_{n-1})$ stabilizes a copy of $\mathbb{H}^{n-1}$ in $\mathbb{H}^n$ and $\Sigma \cong \mathbb{H}^{n-1}/\Lambda$, where Λ is a subgroup of $\mathrm{SO}(J_{n-1}) \cap \Gamma$. For each $t \in \mathbb{R}$, the element

$$B_t = \begin{pmatrix} e^{-nt} & \\ & e^t I_n \end{pmatrix}$$

centralizes the $\mathrm{SO}(J_{n-1})$ and hence centralizes Λ.

Let $N = M$ and let $id : N \to M$ be the identity, then (M, id) is a convex projective structure on N. Let $\rho : \pi_1 N \to \mathrm{SL}(V)$ be the holonomy of this structure. Concretely, ρ is the just the inclusion of $\pi_1 N \cong \Gamma$ into $\mathrm{SL}(V)$. We now define a family, $\rho_t : \pi_1 N \to \mathrm{SL}(V)$, of representations such that $\rho_0 = \rho$. The construction depends on whether or not Σ is separating.

If Σ is separating then Γ splits as an amalgamated product $\Gamma_1 *_\Lambda \Gamma_2$, where the Γ_i are the fundamental groups of the components of $M\backslash\Sigma$. Then ρ_t is defined by the property that $\rho_t(\gamma) = \rho(\gamma)$ if $\gamma \in \Gamma_1$ and $\rho_t(\gamma) = B_t\rho_0(\gamma)B_t^{-1}$ if $\gamma \in \Gamma_2$. Since B_t centralized Λ this gives a well defined representation $\rho_t : \pi_1 N \to \mathrm{SL}(V)$.

In the separating case, $\Gamma = \Gamma' *_s$ is an HNN extension where Γ' is the fundamental group of $M\backslash\Sigma$. In this case ρ_t is defined by the property that $\rho_t(\gamma) = \rho(\gamma)$ if $\gamma \in \Gamma'$ and $\rho_t(s) = B_t\rho(s)$. Again it is easy to see that since B_t centralizes Λ that this gives a well defined representation $\rho_t : \pi_1 N \to \mathrm{SL}(V)$.

In either case we say that the family of ρ_t is *obtained by bending M along Σ*. From the construction, it is not obvious that the representations ρ_t are the holonomy of a convex projective structure. However, the following theorem guarantees that this is the case

Theorem 1.5 (See [17, 22]). *For each $t \in \mathbb{R}$ the representation ρ_t obtained by bending M along Σ is the holonomy of a properly convex projective structure on N.*

The following theorem from [2] addresses which types of cusps arise when one bends a hyperbolic manifold along a totally geodesic hypersurface.

Theorem 1.6 (Cor. 5.10 of [2]). *Let M be a finite volume hyperbolic manifold and let Σ be an embedded totally geodesic hypersurface. If M' is the properly convex manifold obtained by bending M along Σ then each end of M is a generalized cusp of type 0 or type 1.*

1.4. **Zariski closures and limit sets.** We close this section by describing some properties of the Zariski closure of the groups obtained by bending. Before proceeding we introduce some terminology and notation. Let $g \in \mathrm{SL}(V)$ then g is *proximal* if g has a unique (counted with multiplicity) eigenvalue of maximum modulus. It follows that this eigenvalue must be real and that g is proximal if and only if g has a unique attracting fixed point for its action on $P(V)$. If G is a subgroup of $\mathrm{SL}(V)$ then G is *proximal* if it contains a proximal element.

If $G \subset \mathrm{SL}(V)$ is a group then we define the *limit set* of G, denoted Λ_G as

$$\Lambda_G = \overline{\{x \in P(v) \mid x \text{ a fixed point of some proximal } g \in G\}}$$

By construction, this Λ_G is closed and if G is proximal then Λ_G is non-empty. In this generality the limit set was introduced by Goldscheid–Guivarc'h [14] and this construction reduces to the more familiar notion of limit set when G is a Kleinian group. The limit set has the following important properties.

Theorem 1.7 (Thm. 2.3 of [14])**.** *If G is proximal and acts irreducibly on V then Λ_G is the unique minimal non-empty closed G-invariant subset of $P(V)$.*

Next, let $M = \mathbb{H}^n/\Gamma$ be a finite volume (non-compact) hyperbolic manifold containing an embedded totally geodesic hypersurface Σ, let $\Gamma_t = \rho_t(\Gamma)$ be the group obtained by bending M along Σ, and let G_t be the Zariski closure of Γ_t. The following lemma summarizes some properties of G_t and its relation to Λ_G.

Lemma 1.8. *Let ρ_t be obtained by bending M along Σ, let $\Gamma_t = \rho_t(\Gamma)$ and let G_t be the Zariski closure of Γ_t then*

- *The identity component, G_0, of G_t is semisimple, proximal, and acts irreducibly on V*
- *If $x \in \Lambda_{G_0}$ then $\Lambda_{G_0} = G_0 \cdot x$.*

Proof. The group G_0 is a finite index subgroup of G_t and contains the group $G_0 \cap \Gamma_t$ which has finite index in Γ_t. By Theorem 1.2 it follows that $G_0 \cap \Gamma_t$ and hence G_0 acts irreducibly on V. The group $\rho_0(\pi_1\Sigma)$ is easily seen to contain a proximal element and by construction $\rho_t(\pi_1\Sigma) = \rho_0(\pi_1\Sigma)$. It follows that Γ_t is proximal and therefore so is G_0. Since G_0 acts irreducibly on V it is a reductive group. Furthermore, since it is proximal it is easy to see that G_0 must have trivial center and it follows that G_0 is semisimple.

Next, let $G_0 = KAN$ be an Iwasawa decomposition of G_0. Since G is proximal, N has a unique global fixed point $x_N \in P(V)$, which is a weight vector for the highest weight of G with respect to this decomposition. Since A normalizes N it follows that A also preserves x_N, and so $G_0 \cdot x_N = K \cdot x_N$ is a closed orbit, (since K is compact). Furthermore, it is easy to see that $x_N \in \Lambda_{G_0}$ and so $G \cdot x_N$ is a closed G_0-invariant subset of Λ_{G_0}. Therefore, by Theorem 1.7, $G_0 \cdot x_N = \Lambda_{G_0}$. Finally, an orbit is the orbit of any of its points and so it follows that if $x \in \Lambda_{G_0}$ then $\Lambda_{G_0} = G_0 \cdot x$. $\square$

2. Arithmetic lattices

Up until now we have been implicitly working over the real numbers. In this section we will have to work with other fields and rings and we would like this to be explicit in our notation. For this reason when we discuss groups of matrices we will need to explicitly specify where the entries lie. Henceforth, we will denote $\mathrm{SO}(J_n)$ as $\mathrm{SO}(n, 1)$.

Let F be a number field and recall that F is *totally real* if every embedding $\sigma : F \to \mathbb{C}$ has the property that $\sigma(F) \subset \mathbb{R} \subset \mathbb{C}$. By choosing one of these embeddings we will regard F as a subfield of $\mathbb{R}$. If $\alpha \neq 0$ is an element of a totally real field then define $s(\alpha)$ to be the number of non-identity embeddings $\sigma : F \to \mathbb{R}$ for which $\sigma(\alpha) > 0$.

2.1. **Lattices in** $\mathrm{SO}(n, 1)$**.** There are multiple constructions that give rise to different classes of arithmetic lattices in $\mathrm{SO}(n, 1)$. We now explain the simplest of these constructions and the only one that will be relevant for our purposes.

Let F be a totally real number field, let $\mathcal{O}_F$ be its ring of integers and suppose we have chosen $\alpha_1, \ldots, \alpha_n$ be positive elements of F such that $s(\alpha_i) = 0$ (i.e. the α_i are negative under all other embeddings of F). Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and define $J^{\vec{\alpha}} = diag(\alpha_1, \ldots, \alpha_n, -1)$. Next, let $\mathbb{X} \in \{\mathbb{R}, F, \mathcal{O}_F\}$ and define the groups $\mathrm{SO}(J^{\vec{\alpha}}, \mathbb{X}) = \{A \in \mathrm{SL}(n + 1, \mathbb{X}) \mid A^t J^{\vec{\alpha}} A = J^{\vec{\alpha}}\}$. It is well known (see [26, §6.4]) that $\mathrm{SO}(J^{\vec{\alpha}}, \mathcal{O}_F)$ is a lattice in $\mathrm{SO}(J^{\vec{\alpha}}, \mathbb{R})$. Furthermore, the forms $J^{\vec{\alpha}}$ and J_n are $\mathbb{R}$-equivalent and so by a standard argument $\mathrm{SO}(J^{\vec{\alpha}}, \mathcal{O}_F)$ is commensurable with a lattice in $\mathrm{SO}(n, 1)$, and hence we can regard $\mathbb{H}^n/\mathrm{SO}(J^{\vec{\alpha}}, \mathcal{O}_F)$ as a hyperbolic orbifold. The lattices constructed in this fashion are cocompact if and only if $F \neq \mathbb{Q}$. A lattice in $\mathrm{SO}(n, 1)$ that is commensurable with $\mathrm{SO}(J^{\vec{\alpha}}, \mathcal{O}_F)$ for some choice of F and $\vec{\alpha}$ is called an *arithmetic lattice of orthogonal type*

If $\tilde{\Gamma} = \mathrm{SO}(J^{\vec{\alpha}}, \mathcal{O}_F)$ constructed above, then $O = \mathbb{H}^n/\tilde{\Gamma}$ will contain several immersed totally geodesic hypersurfaces, and we now describe one of them and show how it can be promoted to an embedded totally

geodesic hypersuface with nice intersection properties in a finite sheeted manifold cover of O. Specifically, let $\vec{\alpha}_1 = (\alpha_2 \ldots, \alpha_n)$, then $\tilde{\Gamma}_1 = \mathrm{SO}(J^{\vec{\alpha}_1}, \mathcal{O}_F)$ embeds reducibly in $\mathrm{SO}(J^{\vec{\alpha}}, \mathcal{O}_F)$ via

$$\mathrm{SO}(J^{\vec{\alpha}_i}, \mathcal{O}_F) \hookrightarrow \begin{pmatrix} 1 & \\ & \mathrm{SO}(J^{\vec{\alpha}_i}, \mathcal{O}_F) \end{pmatrix}$$

Furthermore, $\tilde{\Gamma}_1$ is (commensurable with) a lattice in $\mathrm{SO}(n - 1, 1)$. The obvious embedding of $\tilde{\Gamma}_1$ into $\tilde{\Gamma}$ induces an immersion of $\mathbb{H}^{n-1}/\tilde{\Gamma}_1$ in $\mathbb{H}^n/\tilde{\Gamma}$. By combining results of Bergeron [8], and Selberg's Lemma we can find finite index subgroups Γ (resp. Γ_1) so that $M = \mathbb{H}^n/\Gamma$ (resp. $M_1 = \mathbb{H}^{n-1}/\Gamma_1$) is a manifold and M_1 is an embedded totally geodesic hypersurface in M. Furthermore, if M is noncompact, then by using the argument from [3, Thm 7.1] it is possible pass to a further finite cover of M where all the cusps have torus cross sections and the intersection of M_1 with one of the cusp cross sections has a single connected component. Shortly we will bend M along M_1 in order to produce thin subgroups in lattices in $\mathrm{SL}(n+1, \mathbb{R})$.

2.2. **Lattices in** $\mathrm{SL}(n + 1, \mathbb{R})$. Next, we describe the lattices in $\mathrm{SL}(n + 1, \mathbb{R})$ in which we will construct thin subgroups. The construction is similar to the one in the previous section, and can be thought of as its "unitary" analogue.

Again, let F be a totally real number field, let $\mathcal{O}_F$ be its ring of integers, and suppose we have chosen $\alpha_1, \ldots, \alpha_n$ to be positive elements of F such that $s(\alpha_i) = 0$. Next, let L be a real quadratic extension of F and let $\mathcal{O}_L$ be the ring of integers of this number field. L is a quadratic extension of F and so there is a unique non-trivial Galois automorphism of L over F that we denote $\tau : L \to L$.

If M is a matrix with entries in L then the *conjugate transpose of M (over L)*, denote M^* is the matrix obtained by taking the conjugate of M and applying τ to its entries. A matrix M is called τ-*Hermitian* if it has entries in L and is equal to its conjugate transpose. Observe that the matrix $J^{\vec{\alpha}}$ is diagonal with entries in F, and so $J^{\vec{\alpha}}$ is τ-Hermitian. Furthermore, it is a standard result (see [26, §6.8], for example) that $\mathrm{SU}(J^{\vec{\alpha}}, \mathcal{O}_L, \tau) := \{A \in \mathrm{SL}(n + 1, \mathcal{O}_L) \mid A^* J^{\vec{\alpha}} A = J^{\vec{\alpha}}\}$ is an arithmetic lattice in $\mathrm{SL}(n + 1, \mathbb{R})$ that is cocompact if and only if $F \neq \mathbb{Q}$.

3. The construction

In this section we describe the the construction of the thin groups in Theorem 0.1. Recall that F is a totally real number field, $\alpha_1, \ldots, \alpha_n$ are positive elements of F such that $s(\alpha_i) = 0$.

Next, we construct a certain real quadratic extension of L. In order to proceed with the construction, we require the following:

Lemma 3.1. *Let F be any totally real field and $N > 0$, then F contains infinitely many units u with the properties that:*

(1) *At the identity embedding of F, $u > N$*
(2) *At all the other embeddings $\sigma : F \to \mathbb{R}$ one has $0 < \sigma(u) < 1$.*

Proof. Suppose that $[F : \mathbb{Q}] = k + 1$ and let $v_1, \ldots, v_k$ be generators of the unit group as determined by Dirichlet's Unit Theorem.

There is an embedding of F into $\mathbb{R}^{k+1}$ given by $x \mapsto (\sigma_1(x), \ldots, \sigma_{k+1}(x))$, where the σ_i are all the embeddings of F into $\mathbb{R}$. By replacing each v_i with its square we can suppose that the image of each of the v_j's has all its coordinates positive. This will replace $\mathcal{O}_L^\times$ with a subgroup of finite index in $\mathcal{O}_F^\times$.

Taking logarithms gives a map from the positive orthant of $\mathbb{R}^{k+1}$ to $\mathbb{R}^{k+1}$ so that each v_j lies in the hyperplane where the sum of the coordinates is equal to zero. Dirichlet's Unit Theorem implies that the images of the set $\{v_1, \ldots, v_k\}$ form a basis for this hyperplane, so there is a linear combination of their images which yield the vector $\vec{a} = (1, -1/k, -1/k, \ldots, -1/k)$, hence there is a rational linear combination giving a vector very close to $\vec{a}$. By scaling, one obtains an *integer* linear combination with the property that the last k coordinates are negative and the first coordinate is positive. After possibly taking further powers (to arrange $u > N$) and exponentiating one obtains a unit with the required properties. $\square$

Remark 3.2. *Notice that once a unit u satisfies the above conditions, so do all its powers.*

Next, let u be one of the units guaranteed by Lemma 3.1 for $N > 2$. Note that by construction, $u^2 - 4$ is not a square. Let s be a root of the polynomial $p_u(x) = x^2 - ux + 1$ and let $L = F(s)$. By construction, this is a real quadratic extension of F and L has exactly 2 real places. Let $\tau : L \to L$ be the unique non-trivial Galois automorphism of L over F. By construction, $s \in \mathcal{O}_L$ and since $\tau(s)$ is the other root of $p_u(x)$, a simple computation shows that $\tau(s) = 1/s$, and so $s \in \mathcal{O}_L^\times$. With this in mind, we henceforth call elements $u \in L$ such that $\tau(u) = 1/u$ τ-unitary or just unitary if τ is clear from context. Note, that τ-unitary elements in $\mathcal{O}_L$ are all units.

Every power of s (and indeed $-s$) is also unitary. Furthermore, we note that these are the only possible unitary elements of $\mathcal{O}_L^\times$. The reason is this: notice that the rank of the unit group of $\mathcal{O}_F$ is $[F : \mathbb{Q}] - 1$. Also, $F(s)$ has two real embeddings, (coming from s and $1/s$) and all the other embeddings lie on the unit circle (in other words, s is a so-called *Salem number*) since we required the other embeddings of u were less than 2 in absolute value. So by Dirichlet's theorem, the unit group of $\mathcal{O}_L$ has rank

$$2 + (2[F : \mathbb{Q}] - 2)/2 - 1 = [F : \mathbb{Q}],$$

which is 1 larger than the rank of $\mathcal{O}_F^\times$. Since τ induces an automorphism of the unit group that fixes $\mathcal{O}_F^\times$, the possibilities for are all accounted for by s and its powers.

From the discussion of the previous section we can find torsion-free subgroups Γ (resp. Γ_1) commensurable with $\mathrm{SO}(J^{\vec{\alpha}}, \mathcal{O}_F)$ (resp. $\mathrm{SO}(J^{\vec{\alpha}_i}, \mathcal{O}_F)$) such that $M_1 := \mathbb{H}^{n-1}/\Gamma_1$ is an embedded submanifold of $M := \mathbb{H}^n/\Gamma$. As previously mentioned, we can regard (M, id) as a complete hyperbolic (and hence convex projective) structure on M whose holonomy ρ is the inclusion of Γ into $\mathrm{SL}(n + 1, \mathbb{R})$. Since M contains an embedded totally geodesic hypersurface, M_1, it is possible to bend M along M_1 to produce a family of representations $\rho_t : \Gamma \to \mathrm{SL}(n + 1, \mathbb{R})$. We now show that for various special values of the parameter t, the group $\rho_t(\Gamma)$ will be a thin group inside a lattice in $\mathrm{SL}(n + 1, \mathbb{R})$. These special values turn out to be logarithms of unitary elements of $\mathcal{O}_L$.

The main goal of the remainder of this section is to prove the following theorem

Theorem 3.3. *If $u \in \mathcal{O}_L$ is unitary and $t = \log |u|$ then $\rho_t(\Gamma) \subset \mathrm{SU}(J^{\vec{\alpha}}, \mathcal{O}_L, \tau)$.*

In order to prove Theorem 3.3 we need a preliminary lemma. Recall that in Section 1.3 we defined for each $t \in \mathbb{R}$ the matrix

$$B_t = \begin{pmatrix} e^{-nt} & \\ & e^t I_n \end{pmatrix}$$

Lemma 3.4. *If $u \in \mathcal{O}_L$ is unitary and $t = \log |u|$*
- $B_t \in \mathrm{SU}(J^{\vec{\alpha}}, \mathcal{O}_L, \tau)$.
- B_t *centralizes* Γ_1.

Proof. If $u \in \mathcal{O}_L$ is unitary then so is $-u$, and so without loss of generality we assume that $u > 0$. Since u is unitary we have

$$B_t^* J^{\vec{\alpha}} B_t = \begin{pmatrix} u^{-n} & 0 \\ 0 & u I_n \end{pmatrix} \begin{pmatrix} \alpha_1 & 0 \\ 0 & J^{\vec{\alpha}_1} \end{pmatrix} \begin{pmatrix} u^n & 0 \\ 0 & u^{-1} I_n \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & J^{\vec{\alpha}_1} \end{pmatrix} = J^{\vec{\alpha}},$$

which proves that $B_t \in \mathrm{SU}(J^{\vec{\alpha}}, \mathcal{O}_L, \tau)$.

For the second point, let $\{e_1, \ldots, e_{n+1}\}$ be the standard basis for $\mathbb{R}^{n+1}$ end let $\{e_1^*, \ldots, e_{n+1}^*\}$ be the corresponding dual basis. For each t, B_t acts trivially on subspaces $\langle e_1 \rangle$ and $\ker(e_1^*)$. By construction Γ_1 preserves both of these subspaces, and so B_t centralizes Γ_1. $\square$

Proof of Theorem 3.3. First, observe that $\Gamma \subset \mathrm{SO}(J^{\vec{\alpha}}, \mathcal{O}_F) \subset \mathrm{SU}(J^{\vec{\alpha}}, \mathcal{O}_L, \tau)$ for any $L = F(s)$. There are now two cases. If $M \backslash M_1$ is separating then as describe in Section 1.3 Γ splits as an amalgamated product $G_1 *_{\Gamma_1} G_2$, and ρ_t is defined by the property that $\rho_t(\gamma) = \rho_0(\gamma)$ if $\gamma \in G_1$ and $\rho_t(\gamma) = B_t \rho_0(\gamma) B_t^{-1}$ if $\gamma \in G_2$. By the previous observation $\rho_0(\gamma) \in \mathrm{SU}(J^{\vec{\alpha}}, \mathcal{O}_L, \tau)$ for any $\gamma \in \Gamma$ and by Lemma 3.4 $B_t \in \mathrm{SU}(J^{\vec{\alpha}}, \mathcal{O}_L, \tau)$. It follows that $\rho_t(\Gamma) \leq \mathrm{SU}(J^{\vec{\alpha}}, \mathcal{O}_L, \tau)$.

The separating case is similar. In this case, $\Gamma = \Gamma' *_s$ is an HNN extension where $\Gamma' = \pi_1(M \backslash M_1)$ and ρ_t is defined by the property that $\rho_t(\gamma) = \rho_0(\gamma)$ if $\gamma \in \Gamma'$ and $\rho_t(s) = B_t \rho_0(s)$. Using a similar argument as before it follows that $\rho_t(\Gamma) \leq \mathrm{SU}(J^{\vec{\alpha}}, \mathcal{O}_L, \tau)$. $\square$

4. Certifying thinness

The goal of this section is to certify the thinness of the examples produced in the previous section. Before proceeding we recall some notation. Γ and Γ_1 are finite index subgroups of $\mathrm{SO}(J^{\vec{\alpha}}, \mathcal{O}_F)$ and $\mathrm{SO}(J^{\vec{\alpha}_1}, \mathcal{O}_F)$ such that $M = \mathbb{H}^n/\Gamma$ is a manifold and $M_1 = \mathbb{H}^{n-1}/\Gamma_1$ is an embedded totally geodesic submanifold. Furthermore, if M is non-compact then all of the cusp cross sections of M are tori and the intersection of M_1 with one of these cross sections is connected. Let ρ_t be obtained by bending M along M_1, let $\Gamma_t = \rho_t(\Gamma)$. By Theorems 1.3 and 1.6 there is a properly convex set Ω_t such that $M_t := \Omega_t/\Gamma_t$ is a properly convex manifold that is diffeomorphic to M. Furthermore, if M is non-compact then M_t has generalized cusp ends

The main theorem is a corollary of the following result.

Proposition 4.1. *Suppose that ρ_t is obtained by bending M along M_1 then*

 (1) *For every t, ρ_t is injective,*
 (2) *If $u \in \mathcal{O}_L$ is unitary and $t = \log|u|$ then $\rho_t(\Gamma)$ has infinite index in $\mathrm{SU}(J^{\vec{\alpha}}, \mathcal{O}_L, \tau)$, and*
 (3) *For any $t \neq 0$, $\rho_t(\Gamma)$ is Zariski dense in $\mathrm{SL}(n+1, \mathbb{R})$*

In particular, $\mathrm{SU}(J^{\vec{\alpha}}, \mathcal{O}_L, \tau)$ contains a thin group isomorphic to $\pi_1 M$.

Proof. The first two points are simple. For (1) observe that by Theorem 1.3, ρ_t is the holonomy of a convex projective structure on M.

Let $\Gamma_t = \rho_t(\Gamma)$. For (2), we can use the fact that the manifold $\mathbb{H}^n/\Gamma$ contains an embedded hypersurface, as we observed earlier. It follows from [23] implies that the group Γ virtually surjects $\mathbb{Z}$. Since $\mathrm{SU}(J^{\vec{\alpha}}, \mathcal{O}_L, \tau)$ is a lattice in a high rank Lie group, it follows that it has property (T) (see [26, Prop. 13.4.1]). Furthermore, any finite index subgroup of $\mathrm{SU}(J^{\vec{\alpha}}, \mathcal{O}_L, \tau)$ will also have property (T) has and thus will have finite abelianization (see [26, Cor. 13.1.5]). Since the groups Γ_t are all abstractly isomorphic it follows that Γ_t is not a lattice, this implies (2).

The third point breaks into two cases depending on whether or not Γ is a cocompact lattice in $\mathrm{SO}(n, 1)$. We treat the cocompact case first. By Theorem 1.5, it follows that Γ_t acts cocompactly on a properly convex set Ω_t. Since Γ is a cocompact lattice in $\mathrm{SO}(n, 1)$, the group Γ is word hyperbolic and it follows from work of Benoist [6] that for each t the domain Ω_t is strictly convex. Hence Ω_t cannot be written as a non-trivial product of properly convex sets. Applying [6, Thm 1.1] it follows that Γ_t is either Zariski dense or Ω_t is the projectivization of an irreducible symmetric convex cone. Suppose we are in the latter case. Irreducible symmetric convex cones are classified by Koecher (see [7, Fact 1.3] for a precise statement) and since Ω_t is strictly convex it follows that $\Omega_t \cong \mathbb{H}^n$. It follows that Γ_t is conjugate to a lattice in $\mathrm{SO}(n, 1)$, which by Mostow rigidity must be Γ. However, bending in this context never produces conjugate representations, since any such conjugacy would centralize the subgroup corresponding to the complement of the bending hypersurface. However this subgroup is nonelementary and this is a contradiction. Therefore, Γ_t is Zariski dense if $t \neq 0$, which concludes the cocompact case.

The non-cocompact case is an immediate corollary of the following Proposition whose proof occupies the remainder of this section. $\square$

Proposition 4.2. *If M is non-compact, ρ_t is obtained by bending M along M_1, and $\Gamma_t = \rho_t(\Gamma)$ then Γ_t is Zariski dense.*

The strategy for proving Proposition 4.2 is to apply the following two results from [5].

Theorem 4.3 (Lem. 3.9 of [5]). *Suppose that $G \subset \mathrm{SL}(V)$ is a connected, semisimple, proximal Lie subgroup acting irreducibly on V. If G acts transitively on $P(V)$ then either $V = \mathbb{R}^n$ and $G = \mathrm{SL}(n, \mathbb{R})$ or $V = \mathbb{R}^{2n}$ and $G = \mathrm{Sp}(2n, \mathbb{R})$.*

The next Theorem allows us to rule out the second possibility in our case of interest.

Theorem 4.4 (Cor. 3.5 of [5]). *If $\Gamma \subset \mathrm{SL}(V)$ acts strongly irreducibly on V and preserves an open properly convex subset then Γ does not preserve a symplectic form.*

Proof of Proposition 4.2. Let G_t be the Zariski closure of Γ_t and let G_0 be the identity component of G_t. We now show that $G_0 = \mathrm{SL}(n+1, \mathbb{R})$. By applying Lemma 1.8 we see that G_0 satisfies all of the hypotheses of Theorem 4.3 except for transitivity.

Since the intersection of M_1 with one of the cusps of M is connected we can apply [3, Thm. 6.1] to conclude that M_t has at least one type 1 cusp. It follows that (after possibly conjugating) G_0 contains the Zariski closure of P_1. Since Γ_t acts irreducibly on V it is not the case that Λ_{G_0} is contained in $\ker(e_2^*) \cup \ker(e_{n+1}^*)$, therefore we can choose a point $x \in \Lambda_{G_0}$ such that $\overline{P_1} \cdot x$ is open in $P(V)$. It follows that $G_0 \cdot x$ has non-empty interior and is hence open. Finally, by Lemma 1.8, $G_0 \cdot x = \Lambda_{G_0}$, which is closed, hence G_0 acts transitively on $P(V)$.

Finally, by Theorem 4.4, Γ_t does not preserve a symplectic form and hence neither does G_0. Applying Theorem 4.3 it follows that $G_0 = \mathrm{SL}(n+1, \mathbb{R})$. $\qquad\square$

We can now prove the Theorem 0.1.

Proof of Theorem 0.1. Since Γ is an arithmetic group of orthogonal type in $\mathrm{SO}(n,1)$ there is a totally real number field F with ring of integers $\mathcal{O}_F$ as well as $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$ such that Γ is commensurable with $\mathrm{SO}(J^{\vec{\alpha}}, \mathcal{O}_F)$. The group Γ is cocompact if and only if $F \neq \mathbb{Q}$.

By using standard separability arguments, we can pass to a finite index subgroup Γ' such that $M = \mathbb{H}^n / \Gamma'$ contains an embedded totally geodesic hypersurface M_1 with the property that if M is non-compact it has only torus cusps and such that M_1 has connected intersection with at least one of the cusps.

Let ρ_t be obtained by bending M along M_1. Let $v \in \mathcal{O}_F^\times$ be an element guaranteed by Lemma 3.1 and let $L = F(s)$, where s is a root of $p_v(x)$, and let τ be the non-trivial Galois automorphism of L over F. Next, let $u = s^n$ be a τ-unit in $\mathcal{O}_F^\times$. If $t = \log|u|$ then by Theorem 3.3 it follows that $\rho_t(\Gamma') \subset \mathrm{SU}(J^{\vec{\alpha}}, \mathcal{O}_L, \tau)$. Furthermore, by Theorem 4.1, $\rho_t(\Gamma')$ is a thin subgroup of $\mathrm{SU}(J^{\vec{\alpha}}, \mathcal{O}_L, \tau)$. Again, $\mathrm{SU}(J^{\vec{\alpha}}, \mathcal{O}_L, \tau)$ is cocompact if and only if $F \neq \mathbb{Q}$ and by varying v and $\vec{\alpha}$ it is possible to produce infinitely many non-commensurable lattices. $\qquad\square$

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E-mail address: `ballas@math.fsu.edu`

E-mail address: `long@math.ucsb.edu`

Department of Mathematics, Florida State University, Tallahassee, FL 32306, USA

Department of Mathematics, University of California Santa Barbara, Santa Barbara, CA 93106, USA