MATH BATTLE

Problem 1 A thin rectangular piece of tin is 26 inches long and 21 inches wide. A thin square piece of metal with side length of 4 inches is cut from each corner, and the tin is folded to form an open rectangular box. How many cubic inches are in the volume of the box?

Problem 2 What is the smallest positive integer greater than 1 such that division by each element of the set {4,5,6,9,10} gives a remainder of 1?

Problem 3 The numbers 1447, 1005, and 1231 have something in common. Each is a four-digit number beginning with 1 that has exactly two identical digits. How many such numbers are there?

Problem 4 John and Pete have three pieces of paper. Each of the boys picks one piece, tears it up, and puts the smaller pieces back. They repeat this process several times. John always tears a piece of paper into 3 smaller pieces while Pete always tears a piece of paper into 5 smaller pieces. After a few minutes can there be exactly 100 pieces of paper?

Problem 5 The increasing sequence 2, 3, 5, 6, 7, 10, 11, ... consists of all positive integers that are neither the square nor the cube of a positive integer. Find the 500th term of this sequence.

Problem 6 A biologist wants to calculate the number of fish in a lake. On May 1 she catches a random sample of 60 fish, tags them, and releases them. On September 1 she catches a random sample of 70 fish and finds that 3 of them are tagged. To calculate the number of fish in the lake on May 1, she assumes that 25% of these fish are no longer in the lake on September 1 (because of death and emigrations), that 40% of the fish were not in the lake May 1 (because of births and immigrations), and that the number of untagged fish and tagged fish in the September 1 sample are representative of the total population. What does the biologist calculate for the number of fish in the lake on May 1?

Problem 7 In a shooting match, eight clay targets are arranged in two hanging columns of three targets each and one column of two targets. A marksman is to break all the targets according to the following rules:
1) The marksman first chooses a column from which a target is to be broken.
2) The marksman must then break the lowest remaining target in the chosen column.

If the rules are followed, in how many different orders can the eight targets be broken?

Problem 8 Find \( ax^5 + by^5 \) if the real numbers \( a, b, x, \) and \( y \) satisfy the equations \( ax + by = 3, \ ax^2 + by^2 = 7, \ ax^3 + by^3 = 16, \ ax^4 + by^4 = 42. \)

Problem 9 Your cabin is two miles due north of a stream that runs east-west. Your grandmothers’s cabin is located 12 miles west and one mile north of your cabin. Every day, you go from your cabin to Grandma’s, but first visit the stream (to get fresh water for Grandma). What is the length of the route with minimum distance?

Problem 10 Someone observed that \( 6! = 8 \cdot 9 \cdot 10. \) Find the largest positive integer \( n \) for which \( n! \) can be expressed as the product of \( n-3 \) consecutive integers.

Problem 11 The rectangle \( ABCD \) has dimensions \( AB = 12\sqrt{13} \) and \( BC = 13\sqrt{3}. \) Diagonals \( AC \) and \( BD \) intersect at \( P. \) If triangle \( ABP \) is cut out and removed, edges \( AP \) and \( BP \) are joined and the figure is then creased along segments \( CP \) and \( DP, \) we obtain a triangular pyramid, all four of whose faces are isosceles triangles. Find the volume of this pyramid.