1 Midterm Response

Q4: Blue  Find the slope\(^1\) of the tangent line to the curve \(y = \frac{1}{\sqrt{x}}\) at \((4, \frac{1}{2})\).

Incorrect Solutions

* Many students wrote the difference quotient in terms of \(x\) and tried to split it:

\[
\lim_{h \to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h \to 0} \frac{1}{x+h} - \frac{1}{x} \quad \text{NO}
\]

What is your next move then? It is better to leave the numerator together. Other students tried immediately to conjugate

\[
\lim_{h \to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} \cdot \frac{\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}} = \lim_{h \to 0} \frac{1}{x+h+x+h} \quad \text{NO}
\]

This is possible but the algebra gets messy and this conclusion is certainly wrong. You get

\[
\frac{\sqrt{x} - \sqrt{x+h}}{h}\sqrt{x} \cdot \sqrt{x+h}
\]

What should you conjugate by?

\[
\frac{\sqrt{x} - \sqrt{x+h}}{h}\sqrt{x} \cdot \sqrt{x+h} \cdot \frac{\sqrt{x} \cdot \sqrt{x+h}}{\sqrt{x} \cdot \sqrt{x+h}} \quad \text{NO}
\]

Instead you should conjugate by the numerator.

\[
\frac{\sqrt{x} - \sqrt{x+h}}{h}\sqrt{x} \cdot \sqrt{x+h} \cdot \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}} \quad \text{YES}
\]

After this, the solution is very similar to the previous page with \(x = 2\).

* One incorrect solution had 4 different mistakes:

  • \(\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}} = h \left( \frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}} \right)\)

  • \(\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}} = (x+h)^{-2} - x^{-2}\) Instead, \(\frac{1}{\sqrt{x+h}} = (x+h)^{-1/2}\) and \(\frac{1}{\sqrt{x}} = x^{-1/2}\).

  • \(h[(x+h)^{-2} - x^{-2}] = h^2(x+h-x)\)

  • \(h^2(x+h-x) = h^4\)

Please don't make these mistakes again.

\(^1\)The original question asks you to find the tangent line.
Correct Solutions

* The first step is to use the difference quotient.

\[ m = \lim_{h \to 0} \frac{\frac{1}{\sqrt{4+h}} - \frac{1}{2}}{h} \]

The numerator is the sum of two fractions and we should add them. The common denominator is \(2\sqrt{4+h}\).

\[ \lim_{h \to 0} \frac{\frac{1}{\sqrt{4+h}} \cdot \frac{2}{2} - \frac{1}{2} \cdot \frac{\sqrt{4+h}}{\sqrt{4+h}}}{h} = \frac{2 - \sqrt{4+h}}{2h\sqrt{4+h}} \]

Conjugate by the numerator

\[ \lim_{h \to 0} \frac{2 - \sqrt{4+h}}{2h\sqrt{4+h}} \cdot \frac{2 + \sqrt{4+h}}{2 + \sqrt{4+h}} = \lim_{h \to 0} \frac{-h}{2h\sqrt{4+h} \cdot (2 + \sqrt{4+h})} = \lim_{h \to 0} \frac{-1}{2\sqrt{4+h} \cdot (2 + \sqrt{4+h})} \]

Once we set \(h = 0\)

\[ \frac{-1}{2\sqrt{4+0} \cdot (2 + \sqrt{4+0})} = \frac{-1}{16} \]

* Another correct solution uses another form of the derivative definition

\[ \lim_{x \to 4} \frac{\frac{1}{\sqrt{x}} - \frac{1}{2}}{x - 4} = \lim_{x \to 4} \frac{2 - \sqrt{x}}{2\sqrt{x}(x - 4)} \]

However instead of conjugating by the denominator, like this

\[ \lim_{x \to 4} \frac{2 - \sqrt{x}}{2\sqrt{x}(x - 4)} \cdot \frac{2\sqrt{x}(x + 4)}{2\sqrt{x}(x + 4)} \]

you should conjugate the top

\[ \lim_{x \to 4} \frac{2 - \sqrt{x}}{2\sqrt{x}(x - 4)} \cdot \frac{2 + \sqrt{x}}{2 + \sqrt{x}} = \lim_{x \to 4} \frac{4 - x}{2\sqrt{x}(x - 4)(2 + \sqrt{x})} \]

Here are some errors in this vein,

\[ \frac{4 - x}{2\sqrt{x}(x - 4)(2 + \sqrt{x})} = \frac{1}{2\sqrt{x}(2 + \sqrt{x})} \]

\[ \frac{2 - \sqrt{x}}{2\sqrt{x}(x - 4)} \cdot \frac{2 + \sqrt{x}}{2 + \sqrt{x}} = \frac{4 - x}{2\sqrt{x}(x - 4)} \]

The fraction \(\frac{4-x}{x-4}\) does cancels to \(-1\). And you get

\[ \lim_{x \to 4} \frac{-1}{2\sqrt{x}(2 + \sqrt{x})} = \frac{-1}{16} \]