Directed filling functions and the groups $\Diamond_n$

Fedor Manin

University of Toronto

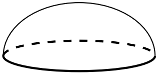
manin@math.toronto.edu

March 14, 2016

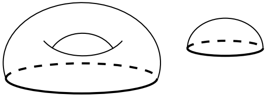
Filling functions in 1D

Let Γ be a finitely presented group and X its Cayley complex. Then we can construct...

...its Dehn function $\delta_\Gamma(k)$: how hard is it to fill a circle in X of length k with a disk?



...its homological filling function $\mathrm{FV}^1_\Gamma(k)$: how hard is it to fill a cellular 1-cycle in X of volume k with a 2-chain?

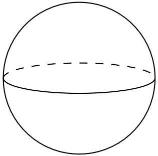


Notice that $\mathrm{FV}^1_\Gamma(k) \lesssim \delta_\Gamma(k)$; for some Γ, this inequality is strict (Abrams–Brady–Dani–Young 2013).

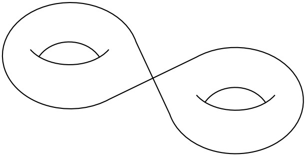
Higher-dimensional filling functions

Let Γ be a group of type $\mathcal{F}_{n+1}$; that is, there is a $K(\Gamma, 1)$ with finite $(n+1)$-skeleton. Let X be the universal cover of this $(n+1)$-skeleton. Then we can construct...

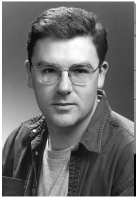
...the higher Dehn function $\delta^n_\Gamma(k)$: how hard is it to fill an n-sphere in X of volume k with a ball?



...the homological filling function $\mathrm{FV}^n_\Gamma(k)$: how hard is it to fill a cellular n-cycle in X of volume k with an $(n+1)$-chain?



What do we know about filling functions?



- Up to coarse equivalence, they depend only on the group Γ.
  (Alonso–Wang–Pride '99, Robert Young '11)
- When $n \geq 3$, $\mathrm{FV}^n_\Gamma(k) \lesssim \delta^n_\Gamma(k)$.
  (Brady–Bridson–Forester–Shankar '09)

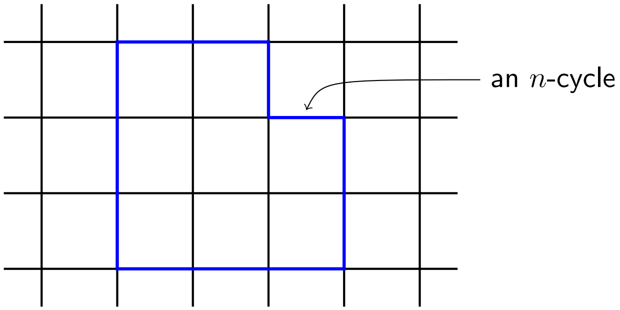
- This is not necessarily the case for $n = 2$.
  (Young '11)
- A group is hyperbolic if and only if all of its filling functions are linear.
  (Mineyev '00)

What functions can be filling functions?

- There are groups with higher-dimensional filling functions k^α for a dense set of α in $[1, \infty)$.
  (BBFS '09, Brady–Forester '10)
- Unlike the Dehn function, δ^2_Γ is computable for any group Γ of type $\mathcal{F}_3$.
  (Papasoglu '00)
- For any n, there are groups Γ such that the function $\mathrm{FV}^n_\Gamma(k)$ grows uncomputably fast. In particular, when $n \geq 3$, $\delta^n_\Gamma(k)$ can grow uncomputably fast as well.
  (Young '11)
- There are groups with 2-dimensional Dehn functions which grow as any tower of exponentials.
  (Barnard–Brady–Dani '12)

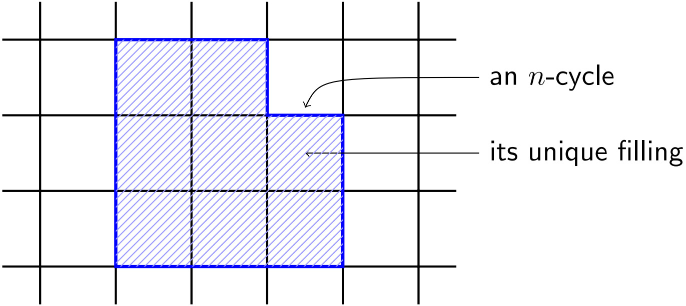
Counting fundamental domains

Suppose Γ is the fundamental group of an $(n+1)$-dimensional aspherical manifold M, with a cell structure which has one $(n+1)$-cell.



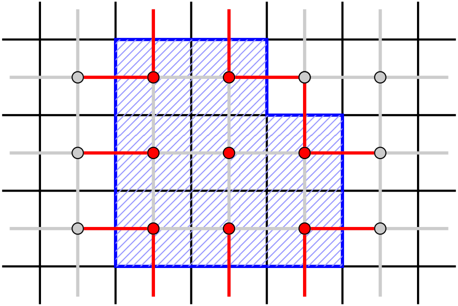
Counting fundamental domains

Suppose Γ is the fundamental group of an $(n+1)$-dimensional aspherical manifold M, with a cell structure which has one $(n+1)$-cell. Then FV^n_Γ counts the number of fundamental domains you can lasso with an n-cycle.



Counting fundamental domains

Suppose Γ is the fundamental group of an $(n+1)$-dimensional aspherical manifold M, with a cell structure which has one $(n+1)$-cell. Then FV^n_Γ counts the number of fundamental domains you can lasso with an n-cycle.



Notice FV^n_Γ is linear if and only if Γ is non-amenable!

Filling homology classes

Here are some features of the situation in the previous slide:

- Any n-cycle c in $\tilde{M}$ projects to the chain $0 \in C_n(M)$.
- Fillings are unique. Therefore, they project to a unique cycle in $C_{n+1}(M)$.
- In other words, every c is assigned a *filling homology class* $\mathrm{Fill}(c) \in H_{n+1}(M) \cong \mathbb{Z}$. The filling volume of c is $|\mathrm{Fill}(c)|$.

Notice that the only requirement for this is that the covering map $\pi : \tilde{M} \to M$ sends $c \mapsto \pi_\#(c) = 0 \in C_n(M)$. There is a well-defined filling homology class in $H_{n+1}(\Gamma)$ for any such c in any group Γ of type $\mathcal{F}_{n+1}$. However, we cannot necessarily compare filling classes to create a filling function.

Directed filling functions

As before, let X be an n-connected complex which Γ acts on, and let $w \in H^{n+1}(\Gamma)$ be a cohomology class. Then we can define a filling function

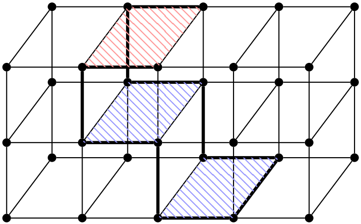
$$\mathrm{FV}^n_{\Gamma,w}(k) = \sup\{|\langle w, \mathrm{Fill}(c)\rangle| : c \in \tilde{X}, \mathrm{vol}\, c \le k, 0 = \pi_* c \in C_n(X/\Gamma)\}.$$

Notice that for any w,

$$\mathrm{FV}^n_{\Gamma,w}(k) \lesssim \mathrm{FV}^n_{\Gamma}(k).$$

Examples

- Let $\Gamma = \mathbb{Z}^d$, $d > n$, and fix a $0 \neq w \in H^{n+1}(\mathbb{Z}^d)$. Then $\langle w, \mathrm{Fill}\, c\rangle$ measures the signed area in a direction indicated by w. In particular, $\mathrm{FV}^n_{\Gamma,w}(k) \sim \mathrm{FV}^n_\Gamma(k) \sim k^{\frac{n+1}{n}}$.



horizontal filling size is 2-1=1.

- Let $\Gamma = BS(1,2) = \langle a, b \mid bab^{-1}a^{-2} = 1\rangle$. Then $\mathrm{FV}^1_\Gamma(k) \sim 2^k$, but $H^2(\Gamma) = 0$ and so there are no directed 1-dimensional filling functions.
- Other known examples of large filling functions also collapse (e.g. those from Young '11.)

Why are they useful?

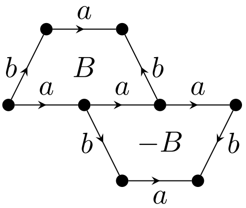
- Potentially a way to find easy lower bounds for other filling functions.
- Relates characteristic classes of certain bundles to their large-scale geometry.

In the remainder of the talk, I will demonstrate a sequence of groups with large directed filling functions.

The group $\diamondsuit_1$

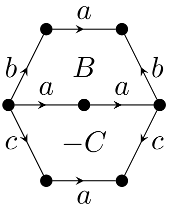
We would like to "fix" the fact that $BS(1,2)$ has $H_2 = 0$.

$BS(1,2) = \langle a, b \mid bab^{-1}a^{-2} \rangle$



The only 2-cell B has nonempty boundary.

$\diamondsuit_1 = \langle a, b, c \mid bab^{-1}a^{-2}, cac^{-1}a^{-2} \rangle$

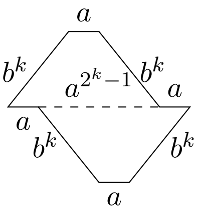


Here, $B - C$ is a cycle which generates $H_2(\diamondsuit_1)$.

The group $\diamondsuit_1$

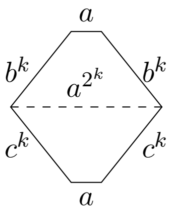
We would like to "fix" the fact that $BS(1,2)$ has $H_2 = 0$.

$BS(1,2) = \langle a, b \mid bab^{-1}a^{-2} \rangle$



A hard-to-fill cycle with homologically trivial filling.

$\diamondsuit_1 = \langle a, b, c \mid bab^{-1}a^{-2}, cac^{-1}a^{-2} \rangle$



A cycle whose filling class is $(2^k - 1)(B - C)$.

The group $\diamondsuit_n$

Define

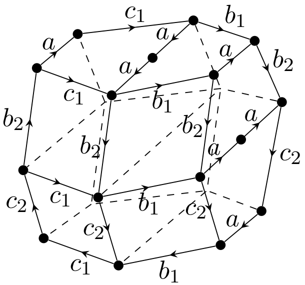
$$\diamondsuit_n = \left\langle b_1, c_1, \ldots, b_n, c_n, a \,\middle|\, \begin{array}{l} b_i^{-1} a b_i = c_i^{-1} a c_i = a^2 \\ [b_i, b_j] = [b_i, c_j] = [c_i, c_j] = 0 \text{ for } i \neq j \end{array} \right\rangle,$$

or inductively as a double HNN-extension of $\diamondsuit_{n-1}$ via the endomorphism

$$a \mapsto a^2, b_i \mapsto b_i, c_i \mapsto c_i.$$

By induction, $\diamondsuit_n$ has an $(n+1)$-dimensional classifying complex X_n. In fact,

$$H_{n+1}(X_n) \cong \mathbb{Z}.$$



A generator of $H_3(X_2)$.

Filling functions in $\diamondsuit_n$

Theorem

Let $w \neq 0 \in H^{n+1}(X_n)$*. Then*

$$\mathrm{FV}^n_{\diamondsuit_n,w}(k) \sim \mathrm{FV}^n_{\diamondsuit_n}(k) \sim \delta^n_{\diamondsuit_n}(k) \sim \exp(\sqrt[n]{k}).$$

Proof

Note first that

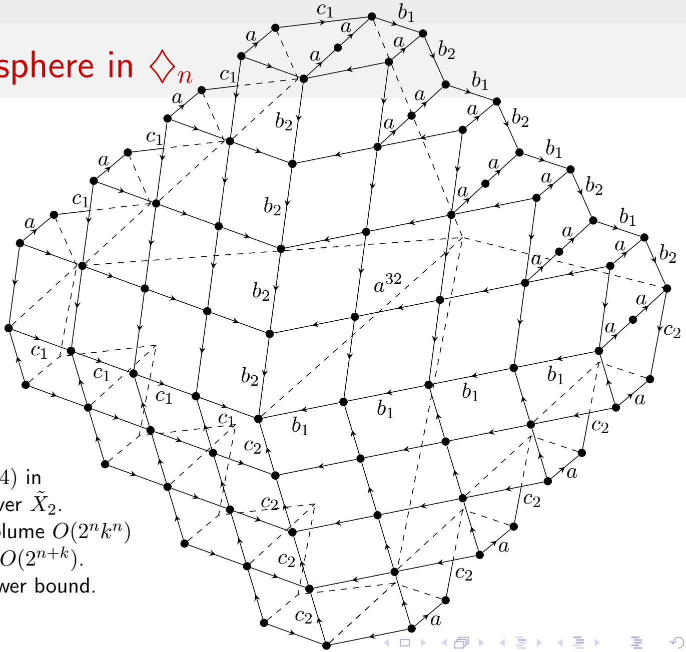
$$\mathrm{FV}^n_{\diamondsuit_n,w}(k) \lesssim \mathrm{FV}^n_{\diamondsuit_n}(k) \lesssim \delta^n_{\diamondsuit_n}(k).$$

So we need to show that

$$\mathrm{FV}^n_{\diamondsuit_n,w}(k) \gtrsim \exp(\sqrt[n]{k})$$

and

A hard-to-fill sphere in $\diamondsuit_n$



The chain $\tau_2(4)$ in
the universal cover $\tilde{X}_2$.
In general, $\tau_n(k)$ has volume $O(2^n k^n)$
and filling volume $O(2^{n+k})$.
This gives us the lower bound.

Finding an upper bound for $\delta^n_{\diamondsuit_n}(k)$

- We show by induction that if $\delta^{n-1}_{\diamondsuit_{n-1}}(k^{n-1}) = O(2^k)$, then $\delta^n_{\diamondsuit_n}(k^n) = O(2^k)$.
- Look at layers in the Bass–Serre tree.
- A refinement of the methods of BBFS '09.

Thank you!