Simplification of metric spaces
Metric graph approximations

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04/27/2019

\begin{itemize}
\item This is a joint work with Facundo Mémoli
\item https://arxiv.org/abs/1809.05566. This work was partially supported by grants NSF AF 1526513, NSF DMS 1723003, NSF CCF 1740761.
\end{itemize}
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What can we say about the approximation if we put an upper bound the first Betti number of the approximating graphs?

Given a compact geodesic space $X$, we define the sequence $(\delta_n^X)_{n \geq 0}$ as follows:

$$\delta_n^X := \inf \{ d_{GH}(X, G) : G \text{ a finite metric graph, } \beta_1(G) \leq n \}.$$
Approximation by the Reeb graph

Given a function \( f : X \to \mathbb{R} \), the Reeb graph \( X_f \) is the quotient space \( X/\sim \) where \( x \sim y \) if there is a continuous path between \( x \) and \( y \) on which \( f \) is constant. This is a graph under certain conditions and it can be given a length structure pulled back by \( f \). If \( f = d(p, \cdot) \) for some \( p \) in \( X \), then we denote \( X_f \) by \( X_p \). It is known that \( \beta_1(X_p) \leq \beta_1(X) \).
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**Theorem**

Let $X$ be a compact geodesic space such that $\beta = \beta_1(X)$ is finite and $p$ be a point in $X$. Then,
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1. For \( n \geq \beta \),

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\frac{d_{GH}(X, X_p)}{16n + 13} \leq \delta_n^X \leq d_{GH}(X, X_p).
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ii) Let \( a_1^X \geq a_2^X \geq \ldots \) be the lengths of the intervals in the first persistent barcode of the open Vietoris-Rips filtration of \( X \). For \( n < \beta \),

\[
\frac{d_{GH}(X, X_p)}{16 \beta + 13} \leq \delta_n^X \leq d_{GH}(X, X_p) + (6 \beta + 6) a_{n+1}^X.
\]