Geometry of Random Surfaces

Sahana Vasudevan

Massachusetts Institute of Technology

April 27, 2019
Random surfaces in moduli space

- $\mathcal{M}_g = \text{moduli space of compact Riemann surfaces of genus } g$
- Fenchel-Nielsen coordinates on $\mathcal{M}_g$: given by length of curves in a pair of pants decomposition $\ell_1, \ldots, \ell_{3g-3}$, and twist parameters $\tau_1, \ldots, \tau_{3g-3}$ that indicate how to glue along the boundaries of the pairs of pants

Question: If we pick a random surface from $\mathcal{M}_g$ according to the WP volume, what does it look like geometrically?

- shortest geodesic $\geq C$ with high probability asymptotically
- diameter $\leq C \log g$ with probability 1 asymptotically
- Cheeger constant $\geq C$ with probability 1 asymptotically

[Mirzakhani]
Random surfaces in moduli space

- $\mathcal{M}_g =$ moduli space of compact Riemann surfaces of genus $g$
- Fenchel-Nielsen coordinates on $\mathcal{M}_g$: given by length of curves in a pair of pants decomposition $\ell_1, ..., \ell_{3g-3}$, and twist parameters $\tau_1, ..., \tau_{3g-3}$ that indicate how to glue along the boundaries of the pairs of pants
- Weil-Petersson (WP) metric on $\mathcal{M}_g$: Kahler metric, volume form given by $d\ell_1 \wedge d\tau_1 \wedge ... \wedge d\ell_{3g-3} \wedge d\tau_{3g-3}$

Question: if we pick a random surface from $\mathcal{M}_g$ according to the WP volume, what does it look like geometrically?

- shortest geodesic $\geq C$ with high probability asymptotically
- diameter $\leq C \log g$ with probability 1 asymptotically
- Cheeger constant $\geq C$ with probability 1 asymptotically

[Mirzakhani]
Random surfaces in moduli space

- $\mathcal{M}_g = \text{moduli space of compact Riemann surfaces of genus } g$
- Fenchel-Nielsen coordinates on $\mathcal{M}_g$: given by length of curves in a pair of pants decomposition $\ell_1, ..., \ell_{3g-3}$, and twist parameters $\tau_1, ..., \tau_{3g-3}$ that indicate how to glue along the boundaries of the pairs of pants
- Weil-Petersson (WP) metric on $\mathcal{M}_g$: Kahler metric, volume form given by $d\ell_1 \wedge d\tau_1 \wedge ... \wedge d\ell_{3g-3} \wedge d\tau_{3g-3}$
- Question: if we pick a random surface from $\mathcal{M}_g$ according to the WP volume, what does it look like geometrically?
  - shortest geodesic?
  - diameter?
  - Cheeger constant?
Random surfaces in moduli space

- $\mathcal{M}_g =$ moduli space of compact Riemann surfaces of genus $g$
- Fenchel-Nielsen coordinates on $\mathcal{M}_g$: given by length of curves in a pair of pants decomposition $\ell_1, \ldots, \ell_{3g-3}$, and twist parameters $\tau_1, \ldots, \tau_{3g-3}$ that indicate how to glue along the boundaries of the pairs of pants
- Weil-Petersson (WP) metric on $\mathcal{M}_g$: Kahler metric, volume form given by $d\ell_1 \wedge d\tau_1 \wedge \ldots \wedge d\ell_{3g-3} \wedge d\tau_{3g-3}$
- Question: if we pick a random surface from $\mathcal{M}_g$ according to the WP volume, what does it look like geometrically?
  - shortest geodesic? $\geq C$ with high probability asymptotically
  - diameter? $\leq C \log g$ with probability 1 asymptotically
  - Cheeger constant? $\geq C$ with probability 1 asymptotically [Mirzakhani]
Random triangulated surfaces

- Triangulated surface: genus $g$ surface $S$ built out of $T$ equilateral triangles, comes with a canonical complex structure

Question: if we pick a random triangulated surface with genus $g$ and $T$ triangles, what does it look like geometrically?

- shortest geodesic $\geq C$ with probability 1 asymptotically
- diameter $\leq C \log g$ with probability 1 asymptotically
- Cheeger constant $\geq C$ with probability 1 asymptotically

Conjecture [Brooks-Makover, Mirzakhani, Guth-Parlier-Young]: discrete measure is a good asymptotic approximation for the WP volume on $M_g$
Random triangulated surfaces

- Triangulated surface: genus $g$ surface $S$ built out of $T$ equilateral triangles, comes with a canonical complex structure
- if $T \sim 4g$, then the flat metric on $S$ is roughly similar to the hyperbolic metric
Random triangulated surfaces

- Triangulated surface: genus $g$ surface $S$ built out of $T$ equilateral triangles, comes with a canonical complex structure
- if $T \sim 4g$, then the flat metric on $S$ is roughly similar to the hyperbolic metric
- Question: if we pick a random triangulated surface with genus $g$ and $T$ triangles, what does it look like geometrically?
  - shortest geodesic?
  - diameter?
  - Cheeger constant?
Random triangulated surfaces

- Triangulated surface: genus $g$ surface $S$ built out of $T$ equilateral triangles, comes with a canonical complex structure
- if $T \sim 4g$, then the flat metric on $S$ is roughly similar to the hyperbolic metric
- Question: if we pick a random triangulated surface with genus $g$ and $T$ triangles, what does it look like geometrically?
  - shortest geodesic? $\geq C$ with probability 1 asymptotically
  - diameter? $\leq C \log g$ with probability 1 asymptotically
  - Cheeger constant? $\geq C$ with probability 1 asymptotically
[Brooks-Makover]
Random triangulated surfaces

- Triangulated surface: genus $g$ surface $S$ built out of $T$ equilateral triangles, comes with a canonical complex structure.

- If $T \sim 4g$, then the flat metric on $S$ is roughly similar to the hyperbolic metric.

- Question: if we pick a random triangulated surface with genus $g$ and $T$ triangles, what does it look like geometrically?
  - Shortest geodesic? $\geq C$ with probability 1 asymptotically.
  - Diameter? $\leq C \log g$ with probability 1 asymptotically.
  - Cheeger constant? $\geq C$ with probability 1 asymptotically. [Brooks-Makover]

- Conjecture [Brooks-Makover, Mirzakhani, Guth-Parlier-Young]: discrete measure is a good asymptotic approximation for the WP volume on $\mathcal{M}_g$. 
References

R. Brooks, E. Makover
*Random constructions of Riemann surfaces*

L. Guth, H. Parlier, R. Young
*Pants decompositions of random surfaces*

M. Mirzakhani
*Growth of Weil-Petersson volumes and random hyperbolic surfaces of large genus*