Waist estimates
for the maps from the sphere

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Fibers of maps dropping dim

\[ f : \underbrace{S^n} \rightarrow \underbrace{Y^m}, \quad m < n \]

the round unit \( n \)-sphere \quad \( m \)-dim. top. space

Can one guarantee the existence of a fiber \( F = f^{-1}(y), \ y \in Y^m \), large in some sense?

- in terms of volume (Gromov’s waist)
- in terms of the diameter (Urysohn’s width):

\[ \text{UW}_m(X) = \inf_{f:X \rightarrow Y^m} \sup_{y \in Y^m} \text{diam}(f^{-1}(y)) \]

- in terms of the Urysohn width
A FIBER OF LARGE URYSOHN WIDTH

**Theorem (Gromov?)**

1. Let $n = (m + 1)(d + 1)$. Every map $f : S^n \to Y^m$ has a fiber $F = f^{-1}(y)$ of Urysohn $d$-width $UW_d(F) \geq UW_{n-1}(S^n) > \frac{\pi}{2}$.

2. Let $n = (m + 1)(d + 1) - 1$, $\varepsilon > 0$ (small). There is a map $f : S^n \to \Delta^m$, whose fibers all have $UW_d(F) < \varepsilon$.

**Theorem**

Assume that a generic PL-map $f : S^3 \to [0, 1]$ is such that all regular fibers $f^{-1}(y), y \in (0, 1)$, are PL-isomorphic to $S^2$. Then there is a fiber $F = f^{-1}(y)$ of Urysohn 1-width $UW_1(F) \geq \frac{1}{2}$.

Conjecture: if $f : S^n \to Y^m$ is generic and almost all fibers have "bounded topological complexity", then $\exists F = f^{-1}(y)$ with $UW_{n-m-1}(F) > \frac{1}{10^{10}}$. 