Anosov Closing Lemma

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Let $M$ be a smooth manifold, $U \subset M$ an open subset, $f : U \rightarrow M$ a $C^1$ diffeomorphism onto its image, and $\Lambda \subset U$ a compact $f$-invariant set, i.e., $f\Lambda \subset \Lambda$.

**Definition**

The set $\Lambda$ is called a **hyperbolic set for the map** $f$ if there exists a Riemannian metric in an open neighborhood $U$ of $\Lambda$ and $\lambda < 1 < \mu$ such that for any point $x \in \Lambda$ the sequence of differentials $(Df)_{f^n x} : T_{f^n x} M \rightarrow T_{f^{n+1} x} M$, $n \in \mathbb{Z}$, admits a $(\lambda, \mu)$-splitting, i.e., there exist decompositions $T_{f^n x} M = E^s_n \oplus E^u_n$ such that $(Df)_{f^n x} E^s_n = E^s_{n+1}$ and

$$
\|(Df)_{f^n x}\big|_{E^s_n}\| \leq \lambda, \quad \|(Df)_{f^n x}^{-1}\big|_{E^u_{n+1}}\| \leq \mu^{-1}.
$$

Example. The Arnold’s cat map on 2-torus: $f = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} : \mathbb{T}^2 \rightarrow \mathbb{T}^2$, $\lambda = \frac{3-\sqrt{5}}{2}$, $\mu = \frac{3+\sqrt{5}}{2}$. 
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**Definition**

We call a sequence $x_0, x_1, \ldots, x_{m-1}, x_m = x_0$ of points a **periodic** $\epsilon$-**orbit** if $\text{dist}(fx_k, x_{k+1}) < \epsilon$ for $k = 0, \ldots, m-1$.

**Theorem**

Let $\Lambda$ be a hyperbolic set for $f : U \to M$. Then there exists an open neighborhood $V \supset \Lambda$ and $C, \epsilon_0 > 0$ such that for $\epsilon < \epsilon_0$ and any periodic $\epsilon$-orbit $(x_0, \ldots, x_m) \subset V$ there is a point $y \in U$ such that $f^m y = y$ and $\text{dist}(f^k y, x_k) < C \epsilon$ for $k = 0, \ldots, m - 1$.