MATH 117–Methods of Analysis, Winter 2017
Midterm Study Guide

GENERAL INFORMATION AND MIDTERM RULES

- The exam will have a duration of 1 hour and 15 minutes. No extra time will be given. Failing to submit your solutions within this time will result in your exam not being graded.
- The material included for the first midterm corresponds to everything that will be covered by Thursday, February 9th. The sections are 1–5, and 7–10.
- You should bring a blue book to your exam, and use a separate page for each problem.

THINGS YOU NEED TO KNOW

You need to be very familiar with the following definitions/axioms/theorems and, if possible, remember some of the proofs involved. The most important thing overall is to understand the MEANING of the definitions, WHY the statements of theorems are expected to be true, and WHAT axioms imply.

- Induction principle.
- The Rational Zeros Theorem.
- Ordered fields: axioms and basic properties.
- Absolute value and triangular inequality.
- Maximum and Minimum of a set.
- Bounded above and bounded below.
- Least upper bound ($\sup S$), and greatest lower bound ($\inf S$).
- Epsilon definition of $\sup S$ and $\inf S$.
- Completeness axiom.
- Archimedean property.
- Unbounded above, and unbounded below.
- Sequences.
- Epsilon definition of limit of a sequence.
- Properties of limits of sequences.
- Convergent sequences are bounded.
- Bounded and monotone sequences converge.
- $\lim\ \sup s_n$, and $\lim\ \inf s_n$.
- Cauchy sequences.
- A sequence is convergent if and only if it is a Cauchy sequence.

SKILLS LIST

- Prove an equality or inequality using the induction principle.
- Use the rational zeros theorem to prove that certain algebraic numbers are irrational.
- Be able to differentiate between upper bound, supremum, and maximum of a set. Similarly, differentiate between lower bound, infimum, and minimum of a set. You should create a list of examples exemplifying their differences.
- Be able to prove using the epsilon definition that a number is the supremum/infimum of a set.
- Be able to use effectively the completeness axiom to show that a set has a supremum.
- Master the Archimedean property. You should, by know, be able to recognize when and how to use this important property. This includes proving that a number is a supremum or infimum, and proving limits.
- Be comfortable with the symbols $+\infty$ and $-\infty$. Understand what expressions as $\sup S = +\infty$ or $\lim\ s_n = -\infty$ mean.
- Need to know what the epsilon definition of convergence for a sequence means. Keep in mind that to prove new theorems we usually think graphically and then write formally. In the last step is when mastering the Archimedean property will become useful.
- Prove that certain number is the limit of a sequence. Tons of practice problems are required in order to prove that certain number is the limit of certain sequence. Archimedean property, the absolute value definition, and the triangular inequality are key here.
- Master the properties of limits and the basic examples. Make a list of examples showing the importance of working with convergent sequences, i.e., examples of divergent sequences that make the limit properties fail.
- Compute and prove limits of sequences defined recursively using the monotone convergence theorem.
- Compute $\limsup$ and $\liminf$ of sequences.
- Be able to prove that a sequence is a Cauchy sequence.

SOME PROOFS TO REMEMBER

- Let $S \subset \mathbb{R}$ be a nonempty and bounded above subset. Then $s_0 = \sup S$ if and only if for every $n \in \mathbb{N}$ there exists $s \in S$ such that
$$s_0 - \frac{1}{n} < s.$$
- $S \subset \mathbb{R}$ has a least upper bound if and only if the set
$$-S = \{-s \colon s \in S\}$$
has a greatest lower bound.
- If $S \subset \mathbb{Z}$ is bounded above then S has a largest element.
- The The Archimedean property: Given $a, b > 0$ there exists $n \in \mathbb{N}$ such that $an > b$.
- The Archimedean property is equivalent to the following statement: For every $a \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that $n > a$.
- The rational numbers are dense: Every nonempty open interval contains a rational number.
- A sequence $\{s_n\}$ is convergent if and only if both of the sequences $E_k = s_{2k}$ and $O_k = s_{2k-1}$ converge to the same limit.
- The sequence $\{(-1)^n\}_{n\in\mathbb{N}}$ is not convergent.
- $\lim \dfrac{1}{n^k} = 0$.
- All limit properties.
- If $\lim |s_n| = 0$ then $\lim s_n = 0$. Is the converse true?
- Let $P(n)$ and $Q(n)$ be polynomials. Consider the sequence $s_n = \dfrac{P(n)}{Q(n)}$. Prove that
$$\begin{aligned} \lim s_n &= \frac{\text{leading coefficient of } P(n)}{\text{leading coefficient of } Q(n)} && \text{if } \deg P(n) = \deg Q(n), \\ \lim s_n &= 0 && \text{if } \deg P(n) < \deg Q(n), \\ \lim |s_n| &= +\infty && \text{if } \deg P(n) > \deg Q(n). \end{aligned}$$

- Convergent sequences are bounded.
- Monotonic and bounded sequences are convergent.
- Convergent sequences are Cauchy.
- Cauchy sequences are convergent.
- For every $x \in \mathbb{R}$ there exists a sequence $\{r_n\}_{n\in\mathbb{N}}$ with $r_n \in \mathbb{Q}$ such that $\lim r_n = x$.