MATH 117–Methods of Analysis, Winter 2017
Final Exam Study Guide

GENERAL INFORMATION AND FINAL EXAM RULES

- The exam will have a duration of 3 hours. No extra time will be given. Failing to submit your solutions within this time will result in your exam not being graded.
- The Final Exam is comprehensive. The material corresponds to sections 4–5, 7–12, 14–15, 17–18.

THINGS YOU NEED TO KNOW

We have covered a variety of topics in this class. Even though you need to be very familiar with all the definitions, axioms, and theorems presented in class, remember that the most important thing overall is to understand the MEANING of the statements. Below you will find a (possibly incomplete) list of the main concepts that we have covered:

- Induction principle.
- Absolute value and triangular inequality.
- Bounded above and bounded below.
- Supremum, infimum, maximum, and minimum of a subset $S \subseteq \mathbb{R}$.
- Epsilon definition of sup S and inf S:
  (a) $s_0 = \sup S$ if and only if s_0 is an upper bound for S, and for every $\epsilon > 0$ there exists $s \in S$ such that $s_0 - \epsilon < s \leq s_0$.
  (b) $s_1 = \inf S$ if and only if s_1 is a lower bound for S, and for every $\epsilon > 0$ there exists $s \in S$ such that $s_1 \leq s < s_1 + \epsilon$.
- **Completeness axiom**: Every nonempty and bounded above subset of $\mathbb{R}$ has a least upper bound.
- **Archimedean property:** For every $a > 0$ there exists $n \in \mathbb{N}$ such that $n > a$.
- **Density of $\mathbb{Q}$:** Every nonempty open interval contains a rational number.
- **Epsilon definition of limit of a sequence:** We say that a sequence (s_n) converges to $s \in \mathbb{R}$ if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|s_n - s| < \epsilon$ whenever $n > N$. We say that (s_n) diverges to $+\infty$ if for every $M > 0$ there exists $N \in \mathbb{N}$ such that $s_n > M$ whenever $n > N$.
- Properties of limits of sequences.
- Convergent sequences are bounded.
- **Monotone Convergence Theorem:** If a sequence is monotone and bounded then it converges. Moreover, if (s_n) is increasing (decreasing) and bounded above (below), then (s_n) converges to $\sup s_n$ ($\inf s_n$).

- $\limsup s_n$, and $\liminf s_n$.

- A sequence is convergent if and only if it is a Cauchy sequence.

- Subsequences.

- If (s_n) converges to s then every subsequence (s_{n_k}) of (s_n) converges to s.

- (s_n) converges to s if and only if the subsequences (s_{2k}) and (s_{2k-1}) both converge to s.

- Every sequence has a monotone subsequence.

- (s_n) has a subsequence converging to $t \in \mathbb{R}$ if and only if the set $\{n \in \mathbb{N} \colon |s_n - t| < \epsilon\}$ is infinite for all $\epsilon > 0$.

- **Bolzano–Weierstrass Theorem:** Every bounded sequence has a convergent subsequence.

- If S denotes the set of subsequential limits of a sequence (s_n) then:

  (a) $\limsup s_n$ and $\liminf s_n$ are elements of S.

  (b) $\limsup s_n = \sup S$ and $\liminf s_n = \inf S$.

- A series $\sum_{k=1}^{\infty} a_k$ converges if and only if the sequence of partial sums $s_n = \sum_{k=1}^{n} a_k$ converges.

- **Cauchy criterion for series:** The series $\sum_{k=1}^{\infty} a_k$ converges if and only if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\left| \sum_{k=n}^{m} a_k \right| < \epsilon, \quad \text{whenever} \quad m > n > N.$$

- **Divergence test:** If $\sum_{k=1}^{\infty} a_k$ converges then $\lim a_k = 0$.

- **Comparison test:**

  (a) If $\sum_{k=1}^{\infty} a_k$ converges and $|b_k| \leq a_k$ for all $k \in \mathbb{N}$, then $\sum_{k=1}^{\infty} b_k$ converges.

  (b) If $\sum_{k=1}^{\infty} a_k = +\infty$, and $b_k \geq a_k \geq 0$ for all $k \in \mathbb{N}$, then $\sum_{k=1}^{\infty} b_k = +\infty$.

- If $\sum_{k=1}^{\infty} |a_k|$ converges then $\sum_{k=1}^{\infty} a_k$ converges.

- **Root test:** Let $\alpha = \limsup |a_n|^{1/n}$.
  (a) If $\alpha < 1$, then $\sum a_n$ converges.
  (b) If $\alpha > 1$, then $\sum a_n$ diverges.
  (c) If $\alpha = 1$ the test is inconclusive.
- **Ratio test:** Assume that $a_n \neq 0$ for all $n \in \mathbb{N}$.
  (a) If $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum |a_n|$ converges.
  (b) If $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum a_n$ diverges.
  (c) Otherwise, the test is inconclusive.
- $\sum \frac{1}{n^p}$ converges if and only if $p > 1$.
- **Alternating Series Theorem:** If (a_n) is a nonincreasing sequence with $a_n \geq 0$ and $\lim a_n = 0$, then the alternating series $\sum (-1)^{n+1} a_n$ converges.
- **Definition of continuity:** A function $f\colon S \subset \mathbb{R} \to \mathbb{R}$ is said to be continuous at $x_0 \in S$ if for every sequence $(x_n) \subset S$ with $\lim x_n = x_0$ we have

$$\lim f(x_n) = f(\lim x_n) = f(x_0).$$

- **Epsilon–delta definition of continuity:** A function $f\colon S \subset \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in S$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(x_0)| < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta.$$

- A function f is called continuous if f is continuous at each $x \in \mathrm{Dom}(f)$.
- **Intermediate Value Theorem:** Suppose that f is continuous on a closed interval $[a, b]$ with $a < b$, and that $f(a) < f(b)$. Then for every y in the open interval $(f(a), f(b))$, there exists at least one $x \in (a, b)$ such that $f(x) = y$.

SUGGESTED PROBLEMS

- 4.1, 4.5–4.8, 4.10, 4.14, 4.15.
- 8.1–8.4, 8.6, 8.9, 8.10.
- 9.9–9.11.
- 10.7, 10.9–10.12.
- 11.2, 11.3, 11.5, 11.7, 11.8, 11.11.
- 12.1, 12.6, 12.7, 12.10.
- 14.1–14.3, 14.6, 14.7, 14.9, 14.12.
- 15.1, 15.8.
- 17.9, 17.10, 17.12, 17.13.
- 18.1, 18.4–18.10.