122A–Introduction to Theory of Complex Variables, Summer A, 2017
A Comprehensive Study Guide

1. The field of complex numbers is the set

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\},$$

together with the operations

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2),$$
$$(x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2).$$

If $z = x + iy$, the inverse of z is the complex number

$$z^{-1} = \frac{1}{z} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}.$$

2. For a complex number $z = x + iy$, we define its conjugate as $\bar{z} = x - iy$. The real and imaginary parts are defined as

$$\mathbf{Re}z = x = \frac{z + \bar{z}}{2}$$
$$\mathbf{Im}z = y = \frac{z - \bar{z}}{2i}.$$

3. The absolute value (or modulus) of $z = x + iy$ is the "real" number

$$|z| = \sqrt{x^2 + y^2}.$$

Thus, we can write

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}.$$

4. We define the complex number

$$e^{i\theta} := \cos\theta + i\sin\theta, \quad \text{for} \quad \theta \in \mathbb{R}.$$

This definition is obtained after replacing $i\theta$ in the power series expansion of the exponential function

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

and taking real and imaginary parts.

5. If $z \in \mathbb{C} \setminus \{0\}$ then we can write z in polar coordinates as

$$z = |z|e^{i\theta}.$$

The angle θ is called an argument of z and is defined only modulo 2π as $e^{i(\theta+2\pi)} = e^{i\theta}$.

6. The principal argument of z is the only argument of z in the interval $(-\pi, \pi]$, and it is denoted by $\mathrm{Arg}(z)$.

7. The disk with center z_0 and radius R is the set

$$D(z_0, R) = \{z \in \mathbb{C} \mid |z - z_0| < R\}.$$

8. A set $S \subseteq \mathbb{C}$ is called open if for every $z_0 \in S$ there exists $\epsilon > 0$ such that $D(z_0, \epsilon) \subseteq S$.

9. A set $C \subseteq \mathbb{C}$ is called closed if its complement $S = \mathbb{C} \setminus C$ is open.

10. An open set D is called a region if it can not be written as the disjoint union of two nonempty disjoint open subsets.

11. We say that a sequence $\{z_n\} \subseteq \mathbb{C}$ converges to $z_0 \in \mathbb{C}$ if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|z_n - z_0| < \epsilon \quad \text{for all} \quad n \geq N.$$

Equivalently, if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $z_n \in D(z_0, \epsilon)$ for all $n \geq N$.

12. One can prove that a set $C \subseteq \mathbb{C}$ is closed if and only if for every convergent sequence $\{z_n\} \subseteq C$ one has that $\lim z_n \in C$.

13. An ϵ-neighborhood of a point $z_0 \in \mathbb{C}$ is simply a disk $D(z_0, \epsilon)$. A **neighborhood** of z_0 is any open subset containing an ϵ-neighborhood of z_0.

14. Let $f \colon D \to \mathbb{C}$ be a function defined on a region D. We say that $f(z)$ approaches L as z approaches z_0 and write

$$\lim_{z \to z_0} f(z) = L,$$

if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|z - z_0| < \delta \implies |f(z) - L| < \epsilon.$$

Equivalently, if for every ϵ-neighborhood $V = D(L, \epsilon)$ of L, there is a δ-neighborhood $U = D(z_0, \delta)$ such that

$$f(U) \subseteq V.$$

15. Let $f\colon D \to \mathbb{C}$ be a function defined on a region D. We say that f is **continuous** at $z_0 \in D$ if
$$\lim_{z \to z_0} f(z) = f(z_0).$$

16. For every $M > 0$, we define the M-neightborhood of ∞ to be the set
$$D(\infty, M) := \{z \in \mathbb{C} \mid |z| > M\}.$$
Then one can make sense of
$$\lim_{z \to \Gamma} f(z) = L$$
for every $\Gamma, L \in \mathbb{C} \cup \{\infty\}$.

17. Let $f\colon D \to \mathbb{C}$ be a function defined on a region D. We say that f is **differentiable** at $z_0 \in D$ if the limit
$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z)}{h} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$
exists, in which case we denote it by $f'(z_0)$.

18. Let $f\colon D \to \mathbb{C}$ be a function defined on a region D. If f is differentiable at $z_0 \in D$ then f is continuous at z_0.

19. Let $f\colon D \to \mathbb{C}$ be a function defined on a region D, and assume that f is differentiable at $z_0 \in D$. If we write $f = u + iv$, then the partial derivatives of u and v must satisfy the Cauchy–Riemann equations at $z_0 = x_0 + iy_0$, i.e.,
$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \quad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0).$$

20. Conversely, if $f = u+iv$ and u and v have continuous partial derivatives at $z_0 = x_0+iy_0$ that satisfy the Cauchy–Riemann equations, then f is is differentiable at z_0. Moreover,
$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0).$$

21. We say that a function $f\colon D \to \mathbb{C}$ defined on a region $D \subset \mathbb{C}$ is **analytic** at $z_0 \in D$ if f is differentiable in a neighborhood of z_0.

22. If $f\colon \mathbb{C} \to \mathbb{C}$ is differentiable at every $z \in \mathbb{C}$ then we say that f is an **entire** function.

23. Let $u\colon D \to \mathbb{R}$ be a real valued function defined on a region $D \subseteq \mathbb{C}$ and assume that $u_x = u_y = 0$ at every point of D (we say that u has vanishing partial derivatives on D), then u must be a constant function.

24. (Chain Rule) Suppose that f and g are complex valued functions such that g is differentiable at z_0 and f is differentiable at $g(z_0)$, then the composition $f \circ g$ is differentiable at z_0 and moreover
$$(f \circ g)'(z_0) = f'(g(z_0))g'(z_0).$$

25. Thus, sums, products, and compositions of differentiable (resp. analytic) functions are differentiable (resp. analytic). Quotients of analytic are analytic in every open set where both numerator and denominator are analytic and the denominator is non-zero.

26. Let f be a complex valued function. We say that g is an inverse of f in an open set $U \subseteq \mathbb{C}$ if for every $z \in U$ we have $f(g(z)) = z$. We say that g is an inverse of f at z_0 if g is an inverse of f in some neighborhood of z_0.

27. Let f be a complex valued function and suppose that g is an inverse of f at z_0. Assume furthermore that g is continuous at z_0 and that f is differentiable at $g(z_0)$ with $f'(g(z_0)) \neq 0$. Then g is differentiable at z_0 and
$$g'(z_0) = \frac{1}{f'(g(z_0))}.$$

28. If $f\colon D \to \mathbb{C}$ is a complex valued function defined on a region $D \subseteq \mathbb{C}$ such that $f'(z) = 0$ for all $z \in D$, then f must be constant. Indeed,
$$f' = u_x + iv_x = 0 \implies u_x = 0, \quad \text{and} \quad v_x = 0,$$
and since f is differentiable it should satisfy the Cauchy–Riemann equations implying that $v_y = 0$ and $u_y = 0$ at every point of D. Therefore, u and v must be constant because D is a region.

29. Using the Cauchy–Riemann equations one can also prove that there are no nonconstant analytic functions with constant real or imaginary parts, or with constant absolute value.

30. The image of a map $z\colon [a,b] \to \mathbb{C}$ is called a curve if z is nonconstant and continuous at all but possibly finitely many values of $t \in [a,b]$. A curve C with parametrization $z(t)$ is called smooth if additionally $z(t)$ is differentiable with derivative $\dot{z}(t) \neq 0$ at all but possibly finitely many values of t.

31. Let $u\colon [a,b] \to \mathbb{C}$ be a complex valued function of real parameter $t \in [a,b]$. We define the integral of u in the interval $[a,b]$ as

$$\int_a^b u(t)\ dt := \int_a^b \mathbf{Re}(u(t))\ dt + i \int_a^b \mathbf{Im}(u(t))\ dt.$$

32. Let C be a smooth curve with parametrization $z\colon [a,b] \to \mathbb{C}$. We define the line integral of a complex function along C as

$$\int_C f(z)dz := \int_a^b f(z(t))\dot{z}(t)\ dt.$$

Notice that for this definition to make sense, C must be contained in the domain of the function f.

33. The arc length of smooth curve C with parametrization $z\colon [a,b] \to \mathbb{C}$, is defined as

$$\text{arc-length}(C) := \int_C |dz| = \int_a^b |\dot{z}(t)|\ dt.$$

34. **The ML Formula:** Suppose that $|f(z)| \leq M$ for all $z \in C$, then

$$\left| \int_C f(z)\ dz \right| \leq \int_C |f(z)||dz| \leq M \int_C |dz| = ML,$$

where $L = \text{arc-length}(C)$.

35. **Fundamental Theorem of Calculus:** Let C be a smooth curve with parametrization $z\colon [a,b] \to \mathbb{C}$. Suppose that f has an antiderivative F on C, i.e., $F'(z) = f(z)$ for all $z \in C$. Then

$$\int_C f(z)\ dz = F(z(b)) - F(z(a)).$$

36. Let $z_0, z_1 \in \mathbb{C}$. Define C_{z_0,z_1} as the curve obtained by moving horizontally from z_0 to $\mathbf{Re}z_1$ and then vertically from $\mathbf{Re}z_1$ to $\mathbf{Im}z_1$. For a complex valued function f we define

$$\int_{z_0}^{z_1} f(z)\ dz := \int_{C_{z_0,z_1}} f(z)\ dz.$$

37. **The Rectangle Theorem I:** Let $f\colon D \to \mathbb{C}$ be a complex valued function defined on a region $D \subseteq \mathbb{C}$. Assume that f is analytic on D and let $R \subset D$ be a rectangle with boundary Γ, then
$$\int_\Gamma f(z)\ dz = 0.$$

38. **The Integral Theorem:** Let f be an entire function, then f has an antiderivative, i.e., there exists an entire function F such that $F'(z) = f(z)$ for all $z \in \mathbb{C}$.

39. The proof of the integral theorem consists in showing that the function
$$F(z) := \int_0^z f(w)\ dw$$
is differentiable with $F'(z) = f(z)$. The key thing to prove is that
$$\int_0^{z+h} f(w)\ dw = \int_0^z f(w)\ dw + \int_z^{z+h} f(w)\ dw,$$
which follows immediately from the rectangle theorem.

40. We say that a curve C with parametrization $z\colon [a,b] \to \mathbb{C}$ is closed if $z(a) = z(b)$.

41. **The Closed Curve Theorem:** If f is an entire function then
$$\int_C f(z)\ dz = 0$$
for every closed curve.

42. Notice that the closed curve theorem only requires for f to have an antiderivative along C and then the result follows from the Fundamental Theorem of Calculus. If the function is entire then the existence of the antiderivative is the statement of the Integral Theorem.

43. **The Rectangle Theorem II:** Let f be an entire function and $a \in \mathbb{C}$. Then the function
$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & \text{if } z \neq a, \\ f'(a) & \text{if } z = a, \end{cases}$$
is continuous for every $z \in \mathbb{C}$, differentiable for every $z \in \mathbb{C}\setminus\{a\}$, and if R is a rectangle with boundary Γ then
$$\int_\Gamma g(z)\ dz = 0.$$

44. **The Integral Theorem** and **The Closed Curve Theorem** hold for $g(z)$ as defined in the item above.

45. **Cauchy Integral Formula I:** Let f be an entire function and $a \in \mathbb{C}$. Let C_R be the circle centered at the origin and of radius R with parametrization $z(\theta) = Re^{i\theta}$, $0 \leq \theta \leq 2\pi$. Then if $R > |a|$ we have

$$f(a) = \frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{z-a}\, dz.$$

46. **Liouville's Theorem:** A bounded entire function is constant.

47. **Fundamental Theorem of Algebra:** If $P(z)$ is a nonconstant polynomial whose coefficients are complex numbers, then $P(z)$ has a root in $\mathbb{C}$.

48. **Power Series Expansion:** Suppose that f is analytic on a disk $D(\alpha, r)$, then there are constants C_k such that

$$f(z) = \sum_{n=0}^{\infty} C_k (z-\alpha)^k \quad \text{for all} \quad z \in D(\alpha, r).$$

49. If f is analytic on a disk $D(\alpha, r)$ then f is infinitely differentiable on $D(\alpha, r)$.

50. Suppose that f is analytic on a disk $D(\alpha, r)$, then the function defined by

$$g(z) = \begin{cases} \frac{f(z)-f(\alpha)}{z-\alpha} & \text{if } z \neq \alpha \\ f'(\alpha) & \text{if } z = \alpha \end{cases}$$

is analytic on $D(\alpha, r)$.

51. Suppose that f is analytic on a disk $D(\alpha, r)$ and suppose that $f(\alpha) = 0$, then

$$f(z) = (z-\alpha)g(z)$$

for some analytic function g on $D(\alpha, r)$.

52. **Cauchy Integral Formula II:** Suppose that f is analytic in $D(\alpha, r)$, and let $0 < \rho < r$, and let a be a complex number such that $|a - \alpha| < \rho$. Then

$$f(a) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{z-a} dz,$$

where C_ρ is the circle centered at α and with radius ρ, and with parametrization $z(\theta) = \alpha + \rho e^{i\theta}$, $0 \leq \theta \leq 2\pi$.

53. **Cauchy Integral Formula for Derivatives:** Suppose that f is analytic on a disk $D(\alpha, r)$, and let $0 < \rho < r$. Then

$$f^{(k)}(\alpha) = \frac{k!}{2\pi i} \int_{C_\rho} \frac{f(z)}{(z-\alpha)^{k+1}} dz,$$

where as before C_ρ represents the circle centered at α, with radius ρ, and with parametrization $z(\theta) = \alpha + \rho e^{i\theta}$, $0 \le \theta \le 2\pi$.

54. **Uniqueness Theorem:** Suppose that f is analytic on a region D and that there is a sequence $\{z_n\}$ converging to $z_0 \in D$. Then $f(z_n) = 0$ for all n implies that $f(z) = 0$ for all $z \in D$.

55. If f is entire and if $f(z) \to \infty$ as $z \to \infty$, then f is a polynomial.

56. **Maximum-Modulus Principle:** A nonconstant analytic function in a region D does not have any interior maximum points. More precisely, for each $z \in D$ and $\delta > 0$, there exists some $w \in D(z, \delta) \cap D$, such that $|f(w)| > |f(z)|$.

57. **Open Mapping Theorem:** The image of an open set under a nonconstant analytic mapping is an open set.

58. **Schwarz' Lemma:** Suppose that f is analytic in the unit disk $D(0,1)$, that $|f(z)| \le 1$ for all $z \in D(0,1)$, and that $f(0) = 0$. Then

    (a) $|f(z)| \le |z|$,

    (b) $|f'(0)| \le 1$,

    with equality in either of the above if and only if $f(z) = e^{i\theta} z$, i.e., f is a rotation.

59. **Morera's Theorem:** Let f be a continuous function on an open set D. If

$$\int_\Gamma f(z)\, dz = 0$$

whenever Γ is the boundary of a closed rectangle in D, then f is analytic on D.

60. **Useful theorems from topology:**

    - Image of a connected set by a continuous function is connected: If $U \subseteq \mathbb{C}$ is a connected subset and $f\colon D \to \mathbb{C}$ is a continuous functions whose domain contains U, then $f(U) \subseteq \mathbb{C}$ is connected.

- Image of a compact set by a continuous function is compact: Let $f\colon D \to \mathbb{C}$ be a continuous complex valued function. Let $K \subseteq D$ be a closed and bounded subset, then the image $f(K) \subset \mathbb{C}$ is also closed and bounded.
- Cantor's Theorem: Intersection of nested compact subsets is nonempty. More precisely, if

$$K_0 \supset K_1 \supset K_2 \supset \cdots$$

is a nested sequence of closed and bounded subsets of $\mathbb{C}$, then

$$\bigcap_{n \geq 0} K_n \neq \varnothing.$$

61. **Some examples to keep in mind:**

- The function $f(z) = \bar{z}$ is not differentiable at any point. Indeed, the Cauchy–Riemann equations are never satisfied.
- The function $f(z) = |z|^2$ is differentiable only at $z = 0$, which is the only point at which the Cauchy–Riemann equations are satisfied.
- Every polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ is analytic. To see this we only need to check that z^n is analytic as products and sums of analytic functions is analytic. This is easy to check as

$$\frac{d}{dz}(z^n) = nz^{n-1}.$$

- One can extend the exponential function to the whole complex plane by defining

$$e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

One can easily verify that e^z satisfy the Cauchy-Riemann equations and since

$$u(x,y) = e^x \cos y, \quad \text{and} \quad v(x,y) = e^x \sin y$$

have both continuous partial derivatives then e^z is differentiable at every $z \in \mathbb{C}$ and so entire. Moreover,

$$\frac{d}{dz}(e^z) = u_x(x,y) + iv_x(x,y) = e^x \cos y + ie^x \sin y = e^z.$$

- Once can easily verify:

$$|e^z| = e^x, \quad \text{and} \quad \overline{e^z} = e^{\bar{z}}.$$

- The principal branch of the complex logarithm is defined as

$$\log z = \ln |z| + i\text{Arg}(z).$$

One can easily verify that $\log z$ is continuous at every $z \in \mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leq 0\}$, and

$$e^{\log z} = z, \quad \text{for all} \quad z.$$

Moreover, since $\frac{d}{dz}(e^z) \neq 0$ then $\log z$ is differentiable at every point where it is continuous and

$$\frac{d}{dz}(\log z) = \frac{1}{z}.$$

- We can use the exponential function to extend the sine and cosine function to all the complex numbers as follows

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

It is clear that $\cos z$ and $\sin z$ are analytic and satisfy

$$\frac{d}{dz}(\sin z) = \cos z, \quad \frac{d}{dz}(\cos z) = -\sin z, \quad \text{and} \quad \cos^2 z + \sin^2 z = 1.$$

62. **Some remarks on power series:**

- A *power series* is a formal sum of the form

$$\sum_{n=0}^{\infty} C_n (z-a)^n, \quad C_n, \ a \in \mathbb{C}.$$

- For a fixed value of z, we say that the series $\sum C_n(z-a)^n$ converges if and only if the sequence of partial sums

$$S_k = C_0 + C_1(z-a) + C_2(z-a)^2 + \cdots + C_k(z-a)^k$$

converges. In such case we write

$$\sum_{n=0}^{\infty} C_n (z-a)^n = \lim_{k\to\infty} S_k.$$

- Every power series $\sum C_n(z-a)^n$ converges in some disk $D(a, R)$ for some $R \in [0, \infty]$, where $D(a, 0) = \{a\}$ and $D(a, \infty) = \mathbb{C}$. We refer to such value R as the radius of convergence.

- The radius of convergence of a power series $\sum C_n(z-a)^n$ can be computed as

$$R = \frac{1}{\limsup \sqrt[n]{C_n}}.$$

- If $\rho = \lim_{n\to\infty} \left|\frac{C_{n+1}}{C_n}\right|$ exists, then the radius of convergence of the power series $\sum C_n(z-a)^n$ can be computed as

$$R = \frac{1}{\rho} = \frac{1}{\lim_{n\to\infty} \left|\frac{C_{n+1}}{C_n}\right|}.$$

- The most remarkable example of a power series is the geometric series. Consider the power series

$$\sum_{n=0}^{\infty} z^n.$$

This power series converges on the disk $D(0,1)$ since $C_n = 1$ for all n and so

$$R = \frac{1}{\lim_{n\to\infty} \left|\frac{C_{n+1}}{C_n}\right|} = 1.$$

Most importantly, we can determine what the value of this power series is by looking at the sequence of partial sums and noticing that

$$S_k - zS_k = 1 - z^{k+1} \implies S_k = \frac{1-z^{k+1}}{1-z}.$$

Therefore, for $|z| < 1$ we have

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{1-z^{k+1}}{1-z} = \frac{1}{1-z}.$$

Thus we write

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \text{for all} \quad z \in D(0,1).$$

- If a power series $\sum C_n(z-a)^n$ converges to a function $f(z)$ on a disk $D(a,R)$, then $f(z)$ is analytic on $D(a,R)$ and

$$f'(z) = \sum_{n=1}^{\infty} nC_n(z-a)^{n-1}.$$

- If a power series $\sum C_n(z-a)^n$ converges to an analytic function $f(z)$ on a disk $D(a, R)$ with $R > 0$, then

$$C_n = \frac{f^n(a)}{n!}, \quad \text{for every} \quad n = 0, 1, \ldots$$

63. **Some suggested problems:**

(a) Prove that

$$\int_{C_R} \frac{1}{z}\, dz = 2\pi i.$$

(b) Prove that if C is any closed curve enclosing a region $D \subset \mathbb{C}$ with $0 \in D$, then

$$\int_{C} \frac{1}{z}\, dz = 2\pi i.$$

(c) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a continuous complex valued function. Suppose that

$$\lim_{z \to \infty} f(z) = 0.$$

Prove that f is bounded, i.e., there exists $M > 0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$.

(d) We say that a sequence of functions $\{S_k\}$ converges uniformly to a function S on a set K if and only if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ (depending only on ϵ) such that

$$|S_k(z) - S(z)| < \epsilon \;\text{ for all }\; k \geq N, \;\text{ and for all } z \in K.$$

Suppose that a series of continuous functions $\sum f_k(z)$ converges to uniformly to a function $f(z)$ on a curve C (i.e., the sequence of partial sums converges to f uniformly on C). Prove that the series of complex numbers

$$\sum_{k=0}^{\infty} \int_C f_k(z)\, dz$$

converges to $\int_C f(z)\, dz$.