MATH 4B–Differential Equations, Fall 2016
Final Exam Study Guide

GENERAL INFORMATION AND FINAL EXAM RULES

- The exam will have a duration of 3 hours. No extra time will be given. Failing to submit your solutions within 3 hours will result in your exam not being graded.
- The Final Exam is comprehensive. The sections are 1.1–1.3, 2.1–2.6, 3.1–3.7, 6.1–6.4, 7.1–7.9.
- 35% of the questions will be from Chapters 1 and 2, 35% from Chapters 3 and 6, and 30% from Chapter 7.
- You can bring ONE index card of dimensions up to $5'' \times 6''$. This index card should be handwritten and can be filled on both sides. However, note cards of higher dimensions than the ones mentioned above or typewritten WILL NOT be allowed.
- Calculators WILL NOT be needed, nor allowed for this exam.
- Last but not least, CHEATING WILL NOT BE TOLERATED.

SKILL'S LIST

- Verify that a given function is a solution for an Initial Value Problem (IVP).
- Sketch the direction (slope) fields of an ODE. Find equilibrium solutions for Autonomous ODEs, and determine whether equilibrium solutions are semistable or not.
- Solve a separable ODE and corresponding IVP.
- Find the general solution to a first order linear ODE.
- Determine if an ODE is exact.
- Solve an exact ODE.
- Read information from a word problem, and establish the corresponding ODE modeling the situation in the following cases:
  1. Free falling object.
  2. Population growth and decay.
  3. Tank model.
  4. Newton's law of cooling.
  5. Springs.
- Determine whether two functions $y_1(t),\ y_2(t)$ form a fundamental set of solutions for a second order linear ODE.
- Find the general solution for a second order homogeneous ODE with constant coefficients.
- Given a solution $y_1(t)$ for a second order linear ODE, use reduction of order to find a second (independent) solution of the form $y_2(t) = u(t)y_1(t)$.
- Find a particular solution for a second order ODE with constant coefficients using the method of undetermined coefficients, and/or variation of parameters.
- Find the general solution to a second order ODE with constant coefficients.
- Solve Initial Value Problems associated to a second order ODE with constant coefficients.

- Compute the Laplace transform of a given function.
- Compute inverse Laplace transforms to rational functions and piecewise continuous functions.
- Use Laplace transforms to solve second order IVP.
- Compute the determinant of a matrix (2×2 or 3×3 would suffice).
- Find the eigenvalues and eigenvectors of a matrix.
- Find the inverse of a matrix.
- Find the canonical Jordan form of a 2×2 matrix.
- Find the general solution of a homogeneous system of linear ODEs with constant coefficients.
- Find the fundamental matrix of a system of linear ODEs with constant coefficients.
- Find the solution to IVPs of the form:

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

- Find the general solution to a 2×2 system of linear ODEs with constant matrix $\mathbf{A}$. Must be familiar with:

  1. Jordan form method,
  2. Undetermined coefficients,
  3. Variation of parameters, and
  4. Laplace transform.

SUMMARY

- An Ordinary Differential Equation (ODE) is an equation whose variable is a differentiable function and that involves the function and its derivatives:

$$F(x, y, y', y'', \ldots, y^{(n)}, \ldots) = 0.$$

- The order of an ODE is the order of the highest derivative that appears in the equation. For instance, $y' + y = x$ is first order while $x^3y'' - 10x^7y' = y^4$ is second order.
- A first order ODE of the form

$$y' = f(y)$$

is called **autonomous**. Notice that autonomous equations are always separable, however the integration involved may be very complicated (if not impossible). One can in any case study solutions by looking at the direction (slope) field for the differential equation.

- A constant solution to an Autonomous ODE is called an **equilibrium solution**. These are the zeroes of the function $f(y)$. An equilibrium solution is called **semistable** if as t approaches infinity solutions on one side of the equilibrium solution approach the equilibrium solution while solutions on the other side become further and further away from the equilibrium solution; it is called **asymptotically stable** if for large values of t solutions always approach the equilibrium solution, in population dynamics this would say that no matter the initial size of a population it will (after a long time) stabilize to the equilibrium size; it is called **unstable** if for large values of t every solution stays away from the equilibrium.
- There are special kinds of ODEs that we know how to solve, these are:

  1. **Separable:** First order ODEs of the form

  $$y' = F(t)G(y).$$

  These can be solved by dividing both sides by $G(y)$ and then integrating with respect to t:

  $$\int \frac{y'(t)}{G(y(t))}\, dt = \int F(t)\, dt,$$

  the left hand side is actually the integral with respect to y since if $y = y(t)$ then $dy = y'(t)dt$, and so the equation becomes

  $$\int \frac{1}{G(y)}\, dy = \int F(t)\, dt.$$

2. **Linear:** First order ODEs of the form

$$y' + a(t)y = b(t). \tag{1}$$

These can be solved by multiplying by the integrating factor

$$\boxed{\boxed{\mu(t) = e^{\int a(t)\ dt}}}$$

Notice that $\mu'(t) = a(t)\mu(t)$ and so multiplying (1) by $\mu(t)$ we obtain

$$\begin{aligned} \mu(t)(y' + a(t)y) &= \mu(t)b(t) \\ \mu(t)y' + \mu'(t)y &= \mu(t)b(t) \\ \frac{d}{dt}\left(\mu(t)y\right) &= \mu(t)b(t). \end{aligned}$$

Thus,

$$\boxed{\boxed{y = \frac{1}{\mu(t)}\left(\int \mu(t)b(t)\ dt\right)}}$$

3. **Exact:** First order ODEs

$$M(x,y) + N(x,y)y' = 0 \tag{2}$$

that can be written as

$$\frac{d}{dx}\left(\Psi(x, y(x))\right) = 0 \tag{3}$$

for some differentiable function $\Psi(x,y)$. If it happens that $M, N, \frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$ are all continuous in some rectangle then such function Ψ exists if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

If an equation is exact, then the solutions are given by

$$\Psi(x,y) = c$$

where c is a constant. To find $\Psi(x,y)$ we notice that for (2) to be equal to (3) we require

$$\frac{\partial \Psi}{\partial x} = M, \quad \text{and} \quad \frac{\partial \Psi}{\partial y} = N.$$

Therefore

$$\Psi(x,y) = \int M(x,y)\ dx + C(y)$$

for some function $C(y)$, depending only on y, and that we determine from the condition $\dfrac{\partial \Psi}{\partial y} = N$.

- Some ODEs may not fall directly into any of the groups above, but after a modification or substitution they can be solved by the same methods.

  1. **Bernoulli equations:** ODEs of the from

  $$y' + a(t)y = b(t)y^n.$$

  These equations can be transform into linear equations by using the substitution $u = y^{1-n}$.

  2. **Integrating factors:** Assume that

  $$M(x,y) + N(x,y)y' = 0 \tag{4}$$

  satisfies one of the following conditions:

     – The quotient

     $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = P(x)$$

     is a function depending only on the variable x.

     – The quotient

     $$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = Q(y)$$

     is a function depending only on the variable y.

  Then (4) can be transformed into an exact equation by multiplying by $\mu(x) = \exp(\int P(x)\ dx)$ in the first case, or by $\mu(y) = \exp(\int Q(y)\ dy)$ in the second case.

- ODEs are used to model real-life situations, some common applications are:

  1. **Free falling objects:** Let $v(t)$ represent the velocity of an object at the time t. If the object falls freely (assuming there is no air resistance) then

  $$m\frac{dv}{dt} = m(9.8) \quad \text{or} \quad \frac{dv}{dt} = 9.8,$$

where $9.8\text{m}/s^2$ is the gravity constant, t is measured in seconds (s), and $v(t)$ in meters per second (m/s). If there is air resistance proportional to the velocity, say $\gamma v(t)$ for some constant γ, then we obtain the ODE

$$\frac{dv}{dt} = 9.8 - \frac{\gamma}{m}v,$$

which is separable. If instead, the object is thrown upwards then the gravity will act as a force of resistance, and under the assumption of an air resistance of the form $\gamma v(t)$, the associated ODE is

$$\frac{dv}{dt} = -9.8 - \frac{\gamma}{m}v.$$

2. **Exponential growth and decay:** $P(t)$ represents the size of some population at the time t. If we assume that the rate of growth of $P(t)$ is proportional to $P(t)$ then we obtain the separable ODE

$$\frac{dP}{dt} = kP.$$

3. **Predator/ Prey systems:** Again $P(t)$ represents the size of some population at the time t and the rate of growth of $P(t)$ is proportional to $P(t)$. We also assume that due to the presence of some predator there are d deaths per unit of time (here d is a constant). Then we obtain the ODE

$$\frac{dP}{dt} = kP - d,$$

which is again separable.

4. **Newton's law of cooling:** Suppose that an object is initially in a room with temperature T_0, and after a while is taken to the outside where the ambient temperature is T_a. Newton's Law of Cooling says that the rate of change of the temperature of the object as the time passes is proportional to the difference between the ambient temperature and the temperature of the object, as an IVP this can be written as:

$$\frac{dT}{dt} = -k(T - T_a), \quad T(0) = T_0,$$

for some constant $k > 0$. This ODE is separable.

5. **Tank model:** Suppose that a tank contains V_0 gallons of brine with y_0 pounds of salt in it. Suppose that brine with a concentration of s_{in} pounds of salt/gallon is poured into the tank at a rate of ν_{in} gallons/minute, and that the well mixed liquid comes out of the tank at a rate of ν_{out} gallons/minute. If $y(t)$ represents the amount of salt (in pounds) in the tank at the time t (in minutes), then

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$

Since rate in (out)=number of pounds of salt poured in (coming out) per minute, then assuming that the mixture is homogenous we obtain

$$\begin{aligned} \text{rate in} &= s_{in} \times \nu_{in} \\ \text{rate out} &= \frac{y(t)}{V(t)} \times \nu_{out}. \end{aligned}$$

Notice that the volume of liquid in the tank at the time t is given by

$$V(t) = V_0 + \nu_{in} t - \nu_{out} t.$$

To determine the amount of salt in the tank at the time t we have to solve the IVP

$$\frac{dy}{dt} = s_{in} \times \nu_{in} - \frac{y}{V(t)} \times \nu_{out}, \quad y(0) = y_0.$$

This is a linear ODE.

- A second order linear ODE is a differential equation of the form

$$a(t)y'' + b(t)y' + c(t)y = d(t) \tag{*}$$

- The associated homogeneous equation is

$$a(t)y'' + b(t)y' + c(t)y = 0 \tag{$\star$}$$

- Two solutions $y_1(t)$ and $y_2(t)$ for ($\star$) are said to form a fundamental set of solutions if they are linearly independent, i.e., if the Wronskian

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t) \neq 0.$$

- If y_1 and y_2 form a fundamental set of solutions for ($\star$), then every other solution is of the form
$$\boxed{\boxed{Y_H(t) = c_1y_1(t) + c_2y_2(t)}}$$
for some constants c_1 and c_2. We refer to $Y_H(t)$ as the general solution to the homogeneous equation ($\star$).

- Suppose that you know a particular solution $Y_P(t)$ for (*), then if $y(t)$ is any other solution one can easily show that $y(t) - Y_P(t)$ is a solution for ($\star$) and so
$$\boxed{\boxed{y(t) = Y_H(t) + Y_P(t)}}$$
Thus, the general solution for (*) consists of the general solution for the associated homogeneous plus a particular solution.

- When the coefficients of ($\star$) are constant then it is easy to find a fundamental set of solutions. Consider the equation
$$ay'' + by' + cy = 0. \tag{✠}$$
When looking for solutions of the form $y = e^{rt}$ one finds out that r must satisfy the **characteristic equation**
$$ar^2 + br + c = 0.$$
Of course, the solutions to the characteristic equation vary depending on the values of the constants $a,\ b,\ c$ and we have the following cases:

  1. Two real solutions: This happens when $b^2 - 4ac > 0$, in which case the solutions are
$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \text{ and } r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$
In this case the solutions $y_1 = e^{r_1t}$ and $y_2 = e^{r_2t}$ form a fundamental set of solutions and therefore the general solution for (✠) is
$$\boxed{\boxed{y(t) = c_1e^{r_1t} + c_2e^{r_2t}}}$$

2. Two complex solutions: This happens when $b^2 - 4ac < 0$, in which case the solutions are

$$r_1 = \frac{-b}{2a} + i\frac{\sqrt{4ac - b^2}}{2a}, \text{ and } r_2 = \frac{-b}{2a} - i\frac{\sqrt{4ac - b^2}}{2a}.$$

In this case the solutions $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ are complex, which may or may not be appropriate depending on what the solutions of the equation should represent. Fortunately, one is able to prove that if $y(t) = A(t) + iB(t)$ is a solution for (✠) then so are the real part $A(t)$ and the imaginary part $B(t)$. Using the more convenient notation

$$r_1 = \lambda + i\omega, \text{ where } \lambda = \frac{-b}{2a} \text{ and } \omega = \frac{\sqrt{4ac - b^2}}{2a},$$

one obtains the solution

$$y(t) = e^{\lambda t + i\omega}t = e^{\lambda t}e^{i\omega t} = e^{\lambda t}(\cos\omega t + i\sin\omega t).$$

The real and imaginary parts of $y(t)$ are the solutions

$$y_1(t) = e^{\lambda t}\cos\omega t \text{ and } y_2(t) = e^{\lambda t}\sin\omega t.$$

After computing $W(y_1, y_1)$ one sees that y_1 and y_2 form a fundamental set of solutions for (✠) and therefore the general solution is

$$\boxed{y(t) = c_1 e^{\lambda t}\cos\omega t + c_2 e^{\lambda t}\sin\omega t}$$

3. Repeated roots: This happens when $b^2 = 4ac$, in which case we obtain only one solution to the characteristic equation

$$r = \lambda = \frac{-b}{2a}.$$

In this case one finds out (using the technique of reduction of order) that a fundamental set of solutions is given by $y_1(t) = e^{\lambda t}$ and $y_2(t) = te^{\lambda t}$ and so the general solution for (✠) is

$$\boxed{y(t) = c_1 e^{\lambda t} + c_2 te^{\lambda t}}$$

- We are now interested in finding particular solutions to second order ODEs whose associated homogeneous equation has constant coefficients, i.e., equations of the form

$$ay'' + by' + cy = d(t). \qquad (\maltese\maltese)$$

1. **Undetermined Coefficients:** The idea is to look for a particular solution $Y_P(t)$ that looks like $d(t)$. We follow the following table:

$d(t)$	$Y_P(t)$
$P_n(t)$	$t^s Q_n(t)$
$P_n(t)e^{\alpha t}$	$t^s Q_n(t)e^{\alpha t}$
$P_n(t)e^{\alpha t}\sin\beta t$	$t^s e^{\alpha t}\left[Q_n(t)\cos\beta t + R_n(t)\sin\beta t\right]$
$P_n(t)e^{\alpha t}\cos\beta t$	$t^s e^{\alpha t}\left[Q_n(t)\cos\beta t + R_n(t)\sin\beta t\right]$

where

$$\begin{aligned} P_n(t) &= A_n t^n + \cdots + A_1 t + A_0 \\ Q_n(t) &= B_n t^n + \cdots + B_1 t + B_0 \\ R_n(t) &= C_n t^n + \cdots + C_1 t + C_0. \end{aligned}$$

2. **Variation of Parameters:** If it happens that the function $d(t)$ is nothing like the functions in the table above, then we can try to look for a particular solution of the form

$$Y_P(t) = u_1(t)y_1(t) + u_2(t)y_2(t).$$

After one extra assumption one is able to deduce formulas for u_1 and u_2 in terms of $y_1,\ y_2$, and d. These formulas are:

$$\boxed{u_1(t) = -\int \frac{y_2(t)d(t)}{aW(y_1,y_2)(t)}dt} \quad \text{and} \quad \boxed{u_2(t) = \int \frac{y_1(t)d(t)}{aW(y_1,y_2)(t)}dt}$$

NOTE: These integrals may be very difficult to compute.

- Suppose that a spring of length ℓ in vertical position is stretched to a length $\ell + L$, i.e., L units beyond its natural length, by a mass m attached to its end. Then Hooke's Law states that there is a constant k (independent of L and m) such that

$$mg = kL.$$

Suppose that the spring is stretched further a distance $u(t)$, then one obtains the second order ODE:

$$\begin{aligned} mu''(t) &= \sum \text{forces acting on } m \\ &= (\text{weight}) + (\text{force due to the spring}) + (\text{damping force}) + (\text{total external force}) \\ &= mg - k(L + u(t)) - \gamma u'(t) + F(t) \\ &= (mg - kL) - ku(t) - \gamma u'(t) + F(t) \\ &= -ku(t) - \gamma u'(t) + F(t). \end{aligned}$$

Here we are using the fact that $mg - kL = 0$, and we are assuming that for small $u(t)$ the damping force (due to air resistance or viscosity) is proportional to the velocity $u'(t)$. The constant γ is called damping constant. Then we obtain a second order ODE with constant coefficients, which will have unique solution once we specify initial position and velocity, i.e., we have the following IVP:

$$\boxed{mu''(t) + \gamma u'(t) + ku(t) = F(t), \quad u(0) = u_0, \quad u'(0) = u_0'}$$

- Recall that if a function $f(t)$ is continuous in every interval of the form $[0, b]$ for $b > 0$, then one can define the infinite integral

$$\int_0^\infty f(t)\ dt := \lim_{b\to\infty} \int_0^b f(t)\ dt.$$

If such limit exists (it is a number) then we say that the integral converges, otherwise (the limit is $\pm\infty$ or does not exist) we say that the integral diverges.

- Assume that a function $f(t)$ is continuous in every interval of the form $[0, b]$ for $b > 0$, and assume further that there are constants $K,\ M, \alpha$ such that

$$|f(t)| \leq Ke^{\alpha t}, \quad \text{for all } t \geq M.$$

Then the integral

$$\mathcal{L}\{f(t)\} := F(s) = \int_0^\infty e^{-st} f(t)\ dt$$

converges for all $s > \alpha$. We refer to the function $\mathcal{L}\{f(t)\}$ as the **Laplace Transform** of $f(t)$.

- It follows from the additive properties of the integral that the Laplace transform is linear, i.e.,
$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\},$$
for any constants a, and b.

- If $\mathcal{L}\{f(t)\} = F(s)$ then we write $\mathcal{L}^{-1}\{F(s)\} = f(t)$. $\mathcal{L}^{-1}$ is also linear and is called the **Inverse Laplace Transform**.

- For a real constant c we define the step function
$$u_c(t) = \begin{cases} 1 & \text{if } \; t \geq c \\ 0 & \text{otherwise.} \end{cases}$$

- The following list of common Laplace transforms will be provided to you in the exam:

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
t^n, n positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
$\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
$\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \lvert a\rvert$
$\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \lvert a\rvert$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$t^n e^{at}$, n positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$e^{ct}f(t)$	$F(s-c)$

- Laplace transforms provide a valuable tool when solving IVP with non homogeneous linear ODEs involving piecewise defined functions. For second order linear ODEs the advantage becomes evident from the relations

$$\boxed{\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\} - y(0)} \quad \boxed{\mathcal{L}\{y''(t)\} = s^2\mathcal{L}\{y(t)\} - sy(0) - y'(0)}$$

- The algorithm to solve an IVP using Laplace transform is very simple:
  1. Apply Laplace transform to both sides of the equation.
  2. Use the relations for $\mathcal{L}\{y''\}$ and $\mathcal{L}\{y'\}$ boxed above.
  3. Use the given initial conditions $y(0) = y_0$ and $y'(0) = y_0'$.
  4. Solve for $\mathcal{L}\{y\}$.
  5. Apply $\mathcal{L}^{-1}$ to obtain y. This step is critical since computing $\mathcal{L}^{-1}$ may be challenging, the idea is to rewrite the function of s as a linear combination of functions appearing in the big table of Laplace transforms.

- A system of first order linear ODEs is an equation of the form:

$$\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}(t) + \mathbf{g}(t),$$

  where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{P}(t) = \begin{pmatrix} p_{1,1}(t) & p_{1,2}(t) & \cdots & p_{1,n}(t) \\ p_{2,1}(t) & p_{2,2}(t) & \cdots & p_{2,n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n,1}(t) & p_{n,2}(t) & \cdots & p_{n,n}(t) \end{pmatrix}, \quad \mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix}.$$

- Though the theory we will describe in what follows applies to matrices of any size, we will focus on the 2×2 and 3×3 cases. We will also assume that the matrix $\mathbf{P}(t)$ is constant and we will denote it by $\mathbf{A}$.

- A matrix $\mathbf{A}$ has an inverse if and only if $\det(\mathbf{A}) \neq 0$.

- Assume that the matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible. Then its inverse if given by

$$\boxed{\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

- Recall that the (column) vectors $X^{(1)},\ X^{(2)}, \dots,\ X^{(n)}$ are linearly independent if and only if

$$\det\begin{pmatrix} | & | & & | \\ X^{(1)} & X^{(2)} & \cdots & X^{(n)} \\ | & | & & | \end{pmatrix} \neq 0.$$

- If $X^{(1)}(t),\ X^{(2)}(t), \dots,\ X^{(n)}(t)$ are linearly independent solutions for the homogeneous system

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$

then the **general solution** is of the form

$$\boxed{\mathbf{x}(t) = c_1 X^{(1)}(t) + c_2 X^{(2)}(t) + \cdots + c_n X^{(n)}(t)}$$

In this case the matrix

$$\boxed{\mathbf{\Psi}(t) = \begin{pmatrix} | & | & & | \\ X^{(1)} & X^{(2)} & \cdots & X^{(n)} \\ | & | & & | \end{pmatrix}}$$

is called a **fundamental matrix** for the system. Thus the general solution can be written as

$$\boxed{\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{c}, \quad \text{where} \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}}$$

- To solve the IVP

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0,$$

we look for the the vector $\mathbf{c}$ such that

$$\mathbf{x}(t_0) = \mathbf{\Psi}(t_0)\mathbf{c} = \mathbf{x}_0.$$

Since the matrix $\mathbf{\Psi}(t)$ is invertible then

$$\boxed{\mathbf{c} = (\mathbf{\Psi}(t_0))^{-1}\mathbf{x}_0}$$

- The **eigenvalues** of a square matrix are the values λ satisfying the equation:

$$\boxed{\boxed{\det(\mathbf{A} - \lambda\mathbf{I}) = 0}}$$

- If $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ then the system of equations corresponding to the matrix $\mathbf{A} - \lambda\mathbf{I}$ admits more than one solution and so it is possible to find a nonzero vector ξ such that

$$\boxed{\boxed{(\mathbf{A} - \lambda\mathbf{I})\xi = 0}}$$

  Such vector is called an **eigenvector** associated to the eigenvalue λ.

- If $\mathbf{A}$ is square matrix whose entries are complex numbers, then $\mathbf{A}$ can be written as

$$\mathbf{A} = \mathbf{T}\mathbf{J}\mathbf{T}^{-1}$$

  for some invertible matrix $\mathbf{T}$. The matrix $\mathbf{J}$ is upper triangular and carries all the relevant information from $\mathbf{A}$. $\mathbf{J}$ is called the canonical Jordan from of $\mathbf{A}$. In the case when $\mathbf{A}$ is size 2×2, we have the following possibilities:

  1. $\mathbf{A}$ has two different eigenvalues λ_1 and λ_2. In this case

$$\mathbf{J} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \text{and} \quad \mathbf{T} = \begin{pmatrix} | & | \\ T^{(1)} & T^{(2)} \\ | & | \end{pmatrix},$$

     where $T^{(1)}$ is an eigenvector for the eigenvalue λ_1, and $T^{(2)}$ is an eigenvector for the eigenvalue λ_2.

  2. $\mathbf{A}$ has only one eigenvalue λ, but it has two linearly independent eigenvectors $T^{(1)}$ and $T^{(2)}$. In this case

$$\mathbf{J} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad \text{and} \quad \mathbf{T} = \begin{pmatrix} | & | \\ T^{(1)} & T^{(2)} \\ | & | \end{pmatrix}.$$

  3. $\mathbf{A}$ has only one eigenvalue λ and one eigenvector $T^{(1)}$ (up to multiples). In this case

$$\mathbf{J} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad \text{and} \quad \mathbf{T} = \begin{pmatrix} | & | \\ T^{(1)} & T^{(2)} \\ | & | \end{pmatrix},$$

     where the vector $T^{(2)}$ is solution to

$$(\mathbf{A} - \lambda\mathbf{I})T^{(2)} = T^{(1)}.$$

- To solve an homogeneous system of linear ODEs with constant coefficients

$$\mathbf{x} = \mathbf{A}\mathbf{x}$$

  we follow the following steps:

  1. Compute the canonical Jordan form of the matrix $\mathbf{A}$.
  2. Use the substitution $\mathbf{x} = \mathbf{T}\mathbf{y}$. In this case we get $\mathbf{x}' = \mathbf{T}\mathbf{y}'$ and therefore we obtain the new system
$$\mathbf{T}\mathbf{y}' = \mathbf{A}\mathbf{T}\mathbf{y}.$$
  3. Multiplying both sides by $\mathbf{T}^{-1}$ the system becomes
$$\mathbf{y}' = \mathbf{J}\mathbf{y}.$$
     Reading out the two equations we obtain two very simple first order linear ODEs, solving first the equation for $y_2(t)$ and then the equation for $y_1(t)$ we obtain $\mathbf{y}$.
  4. We find $\mathbf{x}$ from our substitution $\mathbf{x} = \mathbf{T}\mathbf{y}$.

- The output of the algorithm above will depend on the shape of $\mathbf{J}$.

  1. If $\mathbf{J} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and $\mathbf{T} = \begin{pmatrix} T^{(1)} & T^{(2)} \end{pmatrix}$ then
$$\boxed{\mathbf{x}(t) = c_1 T^{(1)} e^{\lambda_1 t} + c_2 T^{(2)} e^{\lambda_2 t}}$$
     The same answer if there is only one eigenvalue (i.e., $\lambda = \lambda_1 = \lambda_2$), but two linearly independent eigenvectors $T^{(1)}$ and $T^{(2)}$.
  2. If $\mathbf{J} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, $T^{(1)}$ is an eigenvector and $T^{(2)}$ satisfies
$$(\mathbf{A} - \lambda \mathbf{I}) T^{(2)} = T^{(1)},$$
     then the solution is
$$\boxed{\mathbf{x}(t) = c_1 T^{(1)} e^{\lambda t} + c_2 \left(T^{(1)} t e^{\lambda t} + T^{(2)} e^{\lambda t} \right)}$$

- It could happen that the entries of a matrix $\mathbf{A}$ are all real numbers, but $\mathbf{A}$ has complex eigenvalues. In this case, we can take the real and imaginary parts of the general solution to obtain a fundamental set of real solutions. We show this with an example:

Example 1. Find the general solution to the system

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x}.$$

First, we need to find the eigenvalues:

$$\det \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 + 1 = 0 \;\Rightarrow\; \lambda = \pm i.$$

Eigenvector for $\lambda_1 = i$:

$$\left(\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \Longleftrightarrow \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \longrightarrow \begin{pmatrix} -i & 1 \\ 0 & 0 \end{pmatrix},$$

Thus, $-i\xi_1 + \xi_2 = 0$ and therefore the solutions are of the form

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ i\xi_1 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

We choose the eigenvector $T^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix}$. Similarly for $\lambda_2 = -i$ we obtain the eigenvector $T^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

The Jordan form is $\mathbf{J} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, and $\mathbf{T} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$.

The general solution is

$$\begin{aligned}
\mathbf{x}(t) &= c_1 \begin{pmatrix} 1 \\ i \end{pmatrix} e^{it} + c_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-it} \\
&= c_1 \begin{pmatrix} 1 \\ i \end{pmatrix} (\cos t + i \sin t) + c_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} (\cos t - i \sin t) \\
&= \begin{pmatrix} c_1 \cos t + ic_1 \sin t + c_2 \cos t - ic_2 \sin t \\ -c_1 \sin t + ic_1 \cos t - c_2 \sin t - ic_2 \cos t \end{pmatrix} \\
&= \begin{pmatrix} c_1 \cos t + c_2 \cos t \\ -c_1 \sin t - c_2 \sin t \end{pmatrix} + i \begin{pmatrix} c_1 \sin t - c_2 \sin t \\ c_1 \cos t - c_2 \cos t \end{pmatrix} \\
&= (c_1 + c_2) \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i(c_1 - c_2) \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.
\end{aligned}$$

Thus, the general solution can be written as

$$\mathbf{x}(t) = A\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix} + B\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}$$

for constants A and B.

- As in the case of second order ODEs, the general solution to the system of linear ODEs

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{g}(t)$$

can be written as $\mathbf{x}(t) = \mathbf{x}_H(t) + \mathbf{x}_P(t)$, where $\mathbf{x}_H$ is the general solution to the associated homogeneous system and $\mathbf{x}_P$ is a particular solution.

- To solve a non-homogeneous system of linear ODEs of the form

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{g}(t)$$

we follow the following steps:

1. Compute the canonical Jordan form of the matrix $\mathbf{A}$ (this includes $\mathbf{J}$, $\mathbf{T}$, and $\mathbf{T}^{-1}$).
2. Use the substitution $\mathbf{x} = \mathbf{T}\mathbf{y}$. In this case we get $\mathbf{x}' = \mathbf{T}\mathbf{y}'$ and therefore we obtain the new system
$$\mathbf{T}\mathbf{y}' = \mathbf{A}\mathbf{T}\mathbf{y} + \mathbf{g}(t).$$
3. Multiplying both sides by $\mathbf{T}^{-1}$ the system becomes
$$\mathbf{y}' = \mathbf{J}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}(t).$$
Reading out the two equations we obtain two first order linear ODEs, solving first the equation for $y_2(t)$ and then the equation for $y_1(t)$ we obtain $\mathbf{y}$.
4. We find $\mathbf{x}$ from our substitution $\mathbf{x} = \mathbf{T}\mathbf{y}$.

- Even though for 2×2 systems you can just solve the equation by finding the Jordan form of the matrix $\mathbf{A}$ and using the substitution $\mathbf{x} = \mathbf{T}\mathbf{y}$ as explained above, for systems of more variables it is sometimes quicker to find a particular solution by using either undetermined coefficients or variation of parameters:

1. **Undetermined Coefficients:** The idea is to look for a particular solution $\mathbf{x}_P(t)$ that looks like $\mathbf{g}(t)$. We follow the table:

$\mathbf{g}(t)$	$\mathbf{x}_P(t)$
$P_n(t)$	$t^s Q_n(t)$
$P_n(t)e^{\alpha t}$	$t^s Q_n(t)e^{\alpha t}$
$P_n(t)e^{\alpha t}\sin\beta t$	$t^s e^{\alpha t}\left[Q_n(t)\cos\beta t + R_n(t)\sin\beta t\right]$
$P_n(t)e^{\alpha t}\cos\beta t$	$t^s e^{\alpha t}\left[Q_n(t)\cos\beta t + R_n(t)\sin\beta t\right]$

where

$$\begin{aligned} P_n(t) &= \mathbf{a}_n t^n + \cdots + \mathbf{a}_1 t + \mathbf{a}_0 \\ Q_n(t) &= \mathbf{b}_n t^n + \cdots + \mathbf{b}_1 t + \mathbf{b}_0 \\ R_n(t) &= \mathbf{c}_n t^n + \cdots + \mathbf{c}_1 t + \mathbf{c}_0. \end{aligned}$$

NOTE: Here $\mathbf{a_n}, \ldots, \mathbf{a}_0, \mathbf{b}_n, \ldots, \mathbf{b}_0, \mathbf{c}_n, \ldots, \mathbf{c}_0$ are constant vectors.

2. **Variation of Parameters:** We know that the solution to the associated homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ can be written as

$$\mathbf{x}_H(t) = \mathbf{\Psi}(t)\mathbf{c}.$$

The idea is to look for a particular solution of the form

$$\boxed{\boxed{\mathbf{x}_P(t) = \mathbf{\Psi}(t)\mathbf{u}(t)}}$$

for some vector

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix}.$$

Computing derivatives we obtain

$$\mathbf{x}_P'(t) = \mathbf{\Psi}'(t)\mathbf{u}(t) + \mathbf{\Psi}(t)\mathbf{u}'(t).$$

Replacing into the equation we get

$$\mathbf{x}_P'(t) = \mathbf{\Psi}'(t)\mathbf{u}(t) + \mathbf{\Psi}(t)\mathbf{u}'(t) = \mathbf{A}\mathbf{\Psi}(t)\mathbf{u}(t) + \mathbf{g}(t).$$

The second equality can be rewritten as

$$\underbrace{\left(\mathbf{\Psi}'(t)\mathbf{u}(t) - \mathbf{A}\mathbf{\Psi}(t)\mathbf{u}(t)\right)}_{=0} + \mathbf{\Psi}(t)\mathbf{u}'(t) = \mathbf{g}(t)$$

The first term vanishes because $\mathbf{\Psi}'(t)\mathbf{c} = \mathbf{A}\mathbf{\Psi}(t)\mathbf{c}$ for every vector $\mathbf{c}$, including $\mathbf{u}(t)$. Therefore we must have

$$\mathbf{\Psi}(t)\mathbf{u}'(t) = \mathbf{g}(t), \quad \text{or} \quad \mathbf{u}'(t) = (\mathbf{\Psi}(t))^{-1}\mathbf{g}(t).$$

Thus

$$\boxed{\mathbf{u}(t) = \int_{t_0}^{t} (\mathbf{\Psi}(t))^{-1}\mathbf{g}(t)\, dt}$$

As for the case of second order ODEs with constant coefficients, these integrals may be very hard to compute.

- As in the case of second order ODEs with constant coefficients, it is possible to use Laplace transform to solve IVP of the form

  $$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{g}(t), \quad \mathbf{x}(0) = \mathbf{x}_0.$$

  Define

  $$\mathcal{L}(\mathbf{x}(t)) = \mathcal{L}\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} \mathcal{L}\{x_1(t)\} \\ \mathcal{L}\{x_2(t)\} \\ \vdots \\ \mathcal{L}\{x_n(t)\} \end{pmatrix} = \begin{pmatrix} X_1(s) \\ X_2(s) \\ \vdots \\ X_n(s) \end{pmatrix} = \mathbf{X}(s).$$

  Then it follows that

  $$\boxed{\mathcal{L}\{\mathbf{x}'(t)\} = s\mathcal{L}\{\mathbf{x}(t)\} - \mathbf{x}(0) = s\mathbf{X}(s) - \mathbf{x}_0}$$

  Applying Laplace Transform to the system we obtain

  $$s\mathbf{X}(s) - \mathbf{x}_0 = \mathbf{A}\mathbf{X}(s) + \mathbf{G}(s),$$

  where $\mathbf{G}(s) = \mathcal{L}\{\mathbf{g}(t)\}$. Solving for $\mathbf{X}(s)$ the equation becomes

  $$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}(\mathbf{G}(s) + \mathbf{x}_0).$$

  Therefore, the solution is given by

  $$\boxed{\mathbf{x}(t) = \mathcal{L}^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}(\mathbf{G}(s) + \mathbf{x}_0)\right\}}$$

  To compute the Laplace inverse one needs to compute $(s\mathbf{I} - \mathbf{A})^{-1}(\mathbf{G}(s) + \mathbf{x}_0)$ as a matrix of s and as usual apply partial fractions to write each entry of the matrix as combinations of the functions appearing in the table for Laplace transforms.