MATH 6B–Spring 2017
Vector Calculus II
Study Guide Final Exam
Chapters 8, 9, and Sections 11.1, 11.2, 11.7, 12.2, 12.3.

Before starting with the summary of the main concepts covered in the quarter, here there is a list of the skills you should have acquired by the end of the quarter and before the Final Exam (as you prepare for your exam, check the boxes corresponding to the skills you dominate):

- ☐ Recognize what circulation and (outward) flux of a vector field mean in terms of path and surface integrals.
- ☐ Identify when it is possible to use either the Stokes or Divergence theorems.
- ☐ Identify the n-th term of a sequence.
- ☐ Compute the limit of a sequence given by a formula.
- ☐ Compute the limit of a sequence using squeeze theorem.
- ☐ Check that a sequence is increasing (or non-decreasing) or decreasing (or non-increasing).
- ☐ Compute the limit of a sequence given recursively.
- ☐ Successfully identify a convergent geometric series and compute its sum.
- ☐ Identify when to use the various convergence tests for series.
- ☐ Find the interval of convergence of a power series.
- ☐ Find the Taylor or Maclaurin expansion of a function.
- ☐ Estimate the number of terms necessary in the Taylor series expansion of a function in order to obtain a value that is exactly the value of the function in the first k decimal places.
- ☐ Compute the Fourier coefficients of a periodic function of arbitrary period.
- ☐ Compute the Fourier integral expression of a function.
- ☐ Find the solution to the 1-dimensional wave equation satisfying given boundary conditions.

Part I: Integration Theorems of Vector Calculus (about 15% of Final Exam)

Before we start with some problems, let's make sure we recall the fundamental integration theorems of vector calculus:

Theorem 1 (Green's Theorem)**.** *Let $D \subset \mathbb{R}^2$ be a "nice" region of the plane, and let ∂D be its positively oriented boundary. If $F = P\mathbf{i} + Q\mathbf{j}$ is a C^1–vector field on D then*

$$\int_{\partial D} F \cdot ds = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Theorem 2 (Divergence Theorem)**.** *Let $W \subset \mathbb{R}^3$ be a "nice" region, and let ∂W be its positively oriented boundary (outward normal vector). If F is a C^1–vector field on W then*

$$\iint_{\partial W} F \cdot d\mathbf{S} = \iiint_W div(F) dV.$$

Theorem 3 (Stokes' Theorem)**.** *Let $S \subset \mathbb{R}^3$ be a surface with a parametrization $r(u,v) \colon D \to \mathbb{R}^3$ defined on a "nice" region $D \subseteq \mathbb{R}^2$, and let ∂S be the boundary of S with the orientation induced by the parametrization $r(u,v)$. If F is a C^1–vector field on S then*

$$\int_{\partial S} F \cdot ds = \iint_S curl(F) \cdot d\mathbf{S}.$$

It is also important to recall the following two applications of the integration theorems:

Theorem 4 (Divergence Theorem on the Plane)**.** *Let $D \subset \mathbb{R}^2$ be a "nice" region of the plane, and let ∂D be its positively oriented boundary. If F is a C^1–vector field on D then*

$$\int_{\partial D} F \cdot \mathbf{n} \; ds = \iint_D div(F) dA.$$

Theorem 5 (Gauss' Law)**.** *Consider the electrostatic vector field*

$$E(x,y,z) = \frac{Q}{4\pi\varepsilon_0} \frac{1}{(x^2+y^2+z^2)^{3/2}} (x,y,z)$$

produced by a charge Q located at the origin. Then the flux of E through every closed surface S enclosing the origin is

$$\iint_S E \cdot d\mathbf{S} = \frac{Q}{\varepsilon_0}.$$

The following are some recommendations that you should take into account when working with the integration theorems:

- Remember the definition of the surface integral of a function:

$$\iint_S f \; dS := \iint_D f(X(u,v), Y(u,v), Z(u,v)) \cdot \|N(u,v)\| dA$$

where $\mathbf{r}(u,v) = \langle X(u,v), Y(u,v), Z(u,v) \rangle$ is a parametrization of S with normal vector

$$N(u,v) = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}.$$

- The flux of a vector field F through a surface S is defined as the surface integral of the function $F \cdot \mathbf{n}$, where $\mathbf{n}$ denotes the **unit normal** to the surface. We write

$$\iint_S F \cdot d\mathbf{S} := \iint_S F \cdot \mathbf{n} dS = \iint_D F(X(u,v), Y(u,v), Z(u,v)) \cdot N(u,v) dA,$$

since

$$\mathbf{n} = \frac{N(u,v)}{\|N(u,v)\|}.$$

- Similarly, the flux of a vector field in $\mathbb{R}^2$ through a curve $C \subset \mathbb{R}^2$ is given by

$$\int_C F \cdot \mathbf{n} \, ds.$$

- If S_1 and S_2 are two surfaces with oriented boundaries $\partial S_1 = \partial S_2$ for which Stokes' theorem applies, then

$$\iint_{S_1} curl(F) \cdot d\mathbf{S} = \iint_{S_2} curl(F) \cdot d\mathbf{S}$$

for any C^1–vector field on $S_1 \cup S_2$. **Note:** This can be easily obtained by applying Stokes' theorem to S_1 and then to S_2 and compare.

- Assume that a 3D region W is enclosed by two surfaces S_1 and S_2, and assume that F is a C^1–vector field on W with $\text{div}(F) = 0$, then

$$\iint_{S_1} F \cdot d\mathbf{S} + \iint_{S_2} F \cdot d\mathbf{S} = \iiint_W div(F) \; dV = 0.$$

- Assume that S is a surface contained in a plane with unit normal vector $\mathbf{n}$, and assume that F is a C^1–vector field on S with

$$\text{curl}(F) \cdot \mathbf{n} = c$$

for some constant $c \in \mathbb{R}$. Then

$$\int_{\partial S} F \cdot ds = \iint_S \text{curl}(F) \cdot d\mathbf{S} = \iint_S \text{curl}(F) \cdot \mathbf{n} dS = c \iint_S dS = c \times (\text{area of } S).$$

Moreover, if F is C^1–vector field on the whole $\mathbb{R}^3$ and $\tilde{S}$ is any other surface with $\partial \tilde{S} = \partial S$, then

$$\int_{\partial S} F \cdot ds = \int_{\partial \tilde{S}} F \cdot ds = c \times (\text{area of } S).$$

Part II: Convergence Sequences and Series (about 55% of Final Exam)

An **infinite sequence** is an infinite list of numbers. It can be regarded as a function $f \colon (n_0, \infty) \cap \mathbb{Z} \to \mathbb{R}$ where $n_0 \in \mathbb{Z} \cup \{-\infty\}$. The number $f(n)$ is called the n-th term of the sequence.

- If $f\colon (n_0, \infty) \cap \mathbb{Z} \to \mathbb{R}$ is a sequence, the corresponding list is

$$f(n_0+1),\ f(n_0+2),\ f(n_0+3), \ldots$$

Since the notation is very heavy, we often denote sequences by lower case letters with subindexes. For instance, we could write:

  - $a_n = f(n)$, for $n > n_0$.
  - $\{a_n\}_{n=n_0+1}^{\infty}$, or simply $\{a_n\}$ if there is no need to remember n_0.
  - $\{a_n\}_{-\infty}^{\infty}$ if $n_0 = -\infty$.

Thus the sequence becomes the list

$$a_{n_0+1},\ a_{n_0+2},\ a_{n_0+3}, \ldots$$

- Sequences can often be defined recursively, i.e., one defines some few terms, and the following terms are defined using a formula involving only earlier terms in the list. For instance, the **Fibonacci** sequence is defined by

$$\begin{aligned} a_0 &= 0, \\ a_1 &= 1, \\ a_n &= a_{n-1} + a_{n-2}, \text{ for } n \geq 2. \end{aligned}$$

The first few terms in this sequence are:

$$\begin{aligned} a_0 &= 0, \\ a_1 &= 1, \\ a_2 &= 1 + 0 = 1, \\ a_3 &= 1 + 1 = 2, \\ a_4 &= 2 + 1 = 3, \\ a_5 &= 3 + 2 = 5, \\ &\vdots \end{aligned}$$

- If the values of a sequence $\{a_n\}$ approach a single value as n grows, we say that the sequence $\{a_n\}$ is **convergent**. Otherwise, i.e., if the values approach $\pm\infty$ or if they do not approach a single value, we say that the sequence is **divergent**. If the sequence $\{a_n\}$ converges to the value L, we write $\lim\limits_{n\to\infty} a_n = L$. Formally,

Definition 6. Let $\{a_n\}$ be an infinite sequence. We say that $\lim\limits_{n\to\infty} a_n = L$ if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for all $n \geq N$.

- Limits of sequences satisfy the usual limit rules that limits of functions satisfy, i.e., limits of sums are sums of limits, limits of products are products of limits, limits of constants are the same constants, etc.

- The following three theorems give us the tools to compute limits of sequences.

  Theorem 7 (Even and odd terms)**.** *The sequence $\{a_n\}$ converges to L if and only if both of the sequences $\{a_{2k}\}$ and $\{a_{2k+1}\}$ converge to L.*

  Theorem 8 (Replace by a function)**.** *If $a_n = f(n)$ for a function $f\colon \mathbb{R} \to \mathbb{R}$, and $\lim_{x\to\infty} f(x) = L$, then*

  $$\lim_{n\to\infty} a_n = L.$$

  Theorem 9 (Squeeze Theorem)**.** *If $a_n \le b_n \le c_n$ for all $n \ge N$ for some $N \in \mathbb{N}$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then*

  $$\lim_{n\to\infty} b_n = L.$$

- There is another theorem that is fundamental to understand sequences, it is practically used to find limits of sequences defined recursively, and theoretically to prove many theorems about series.

  Definition 10. A sequence $\{a_n\}$ is said to be non–decreasing (resp. non-increasing) if $a_n \le a_{n+1}$ (resp. $a_n \ge a_{n+1}$) for all $n \in \mathbb{N}$. Such sequence is called monotonic.

- To prove that a sequence is non–decreasing or non–increasing, it is enough to check the sign of $a_{n+1} - a_n$.
- If replacing a_n by a function makes sense (no factorials, no negative numbers to power n, etc.,) then determining if a_n is increasing or decreasing can be done by analyzing the sign of the first derivative.

  Example 11. Prove that the sequence $a_n = \dfrac{\sqrt{n^2-1}}{3n^2-n+2}$ is decreasing for $n \ge 2$.

  Indeed, if $f(x) = \dfrac{\sqrt{x^2-1}}{3x^2-x+2}$ then

  $$\begin{aligned} f'(x) &= \frac{\frac{x}{\sqrt{x^2-1}}(3x^2-x+2) - \sqrt{x^2-1}(6x-1)}{(3x^2-x+2)^2} \\ &= \frac{\frac{x(3x^2-x+2)-(x^2-1)(6x-1)}{\sqrt{x^2-1}}}{(3x^2-x+2)^2} \\ &= \frac{x(3x^2-x+2)-(x^2-1)(6x-1)}{\sqrt{x^2-1}(3x^2-x+2)^2} \\ &= \frac{(3x^3-x^2+2x)-(6x^3-x^2-6x+1)}{\sqrt{x^2-1}(3x^2-x+2)^2} \\ &= \frac{-3x^3+8x-1}{\sqrt{x^2-1}(3x^2-x+2)^2} \\ &\le 0 \text{ for (at least) } x \ge 2. \end{aligned}$$

  Then the sequence a_n is decreasing for $n \ge 2$.

  Definition 12. A sequence $\{a_n\}$ is said to be bounded from above (resp. from below) if there is a constant M (resp. m) such that $a_n \le M$ (resp. $m \le a_n$) for all n.

Theorem 13 (Monotonic Convergence Theorem)**.** *If a sequence $\{a_n\}$ is non–decreasing and bounded from above then it is convergent. If a sequence $\{a_n\}$ is non–decreasing and bounded from below then it is convergent.*

- There is an extra thing that might help to prove that a sequence is divergent, the idea is to have some partial criteria similar to the Squeeze Theorem but to conclude a sequence is divergent.

  Theorem 14. *Assume that $b_n \leq a_n$ for all $n \geq N$ (i.e., the inequality happens from the N-terms on). If $\lim_{n\to\infty} b_n = \infty$ then $\lim_{n\to\infty} a_n = \infty$.*

  Example 15. Prove that the sequence $a_n = \dfrac{n^n}{n!}$ is divergent.

  Indeed,

$$\begin{aligned} a_n &= \frac{n}{1} \cdot \frac{n}{2} \cdot \frac{n}{3} \cdots \frac{n}{n} \\ &= n \left[\frac{n}{2} \cdot \frac{n}{3} \cdots \frac{n}{n} \right] \\ &\geq n. \end{aligned}$$

  Since $\lim_{n\to\infty} n = \infty$ then $\lim_{n\to\infty} a_n = \infty$ by Theorem 14 with $b_n = n$.

A **series of numbers** or **numerical series** is the formal sum of the elements of a sequence. The series corresponding to the sequence $\{a_n\}_{n=1}^{\infty}$ is denoted by

$$\sum_{n=1}^{\infty} a_n.$$

- The sequence of *partial sums* of $\sum_{n=1}^{\infty} a_n$ is the sequence $\{S_k\}$ defined by

$$S_k = a_1 + a_2 + \cdots + a_{k-1} + a_k.$$

- If the sequence $\{S_k\}$ is convergent we say that the series is *convergent*.
- If the sequence $\{S_k\}$ is divergent we say that the series is *divergent*.
- A series of the form $\displaystyle\sum_{n=1}^{\infty} cr^{n-1}$ is called *geometric* (c and r are constants). A geometric series with $|r| < 1$ is convergent and

$$\sum_{n=1}^{\infty} cr^{n-1} := \lim_{k\to\infty} S_k = \frac{c}{1-r}.$$

  For any other value of r this kind of series is divergent.

- A series $\displaystyle\sum_{n=1}^{\infty} a_n$ is called *telescoping series* (we also called these series collapsing series) if there is a sequence $\{c_n\}$ such that $a_n = c_n - c_{n+1}$ for all $n \in \mathbb{N}$. In this case

$$\begin{aligned} S_k = a_1 + a_2 + \cdots + a_k &= (c_1 - c_2) + (c_2 - c_3) + \cdots + (c_{k-1} - c_k) + (c_k - c_{k+1}) \\ &= c_1 - c_{k+1}. \end{aligned}$$

A telescoping series is convergent if and only if the sequence $\{c_n\}$ is convergent. In this case we have

$$\sum_{n=1}^{\infty} a_n := \lim_{k\to\infty} S_k = c_1 - \lim_{k\to\infty} c_{k+1}.$$

- *Divergence Test (DT):* If $\lim_{n\to\infty} a_n \neq 0$ (either the limit exists and is a non-zero number, or is infinity, or does not exist) then the series $\sum_{n=1}^{\infty} a_n$ is divergent. If the limit is zero we can not say anything about the convergence of the series.

- *Integral Test (IT):* Assume that $f(x)$ is a positive, continuous, and non-increasing function in the interval $[k, \infty)$ with $\lim_{x\to\infty} f(x) = 0$. Then

$$\sum_{n=k}^{\infty} f(n) \text{ converges } \iff \int_k^{\infty} f(x)\, dx \text{ converges.}$$

- *p-series Test:* The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges otherwise.

- *Comparison Test (CT):* Assume $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$. Then

  1. If $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{n=1}^{\infty} a_n$.
  2. If $\sum_{n=1}^{\infty} a_n$ diverges, so does $\sum_{n=1}^{\infty} b_n$.

- *Limit Comparison Test (LCT):* Assume $a_n \geq 0$ and $b_n > 0$ for all $n \in \mathbb{N}$. Let

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L.$$

If $L \neq 0,\ \infty$, then $\sum_{n=1}^{\infty} a_n$ converges $\iff \sum_{n=1}^{\infty} b_n$ converges.

- *(Absolute) Ratio Test (RT):* Let

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$$

  1. If $\rho < 1$ then the series $\sum_{n=1}^{\infty} a_n$ converges (absolutely).
  2. If $\rho > 1$ or ∞ then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
  3. If $\rho = 1$ the test is inconclusive.

- *Absolute Convergence Test (ACT):* If the series $\sum_{n=0}^{\infty} |a_n|$ converges, then the series $\sum_{n=0}^{\infty} a_n$ converges. In this case we say that the series converges absolutelly.
- A series of the form $a_1 - a_2 + a_3 - a_4 + \cdots$ is called *alternating*.
- *Alternating Series Test (AST):* Assume that $\{a_n\}$ is a decreasing sequence of positive numbers with $\lim_{n\to\infty} a_n = 0$. Then the series

$$\sum_{n=1}^{\infty}(-1)^n a_n$$

  is convergent.
- If $\lim_{n\to\infty} a_n \neq 0$, then the alternating series $\sum(-1)^n a_n$ diverges by the divergence test.
- A convergent series that is not absolutely convergent is called *conditionally convergent*.

Part II: Power Series

A *power series* centered at $a \in \mathbb{R}$ is a series of functions of the form

$$\sum_{n=0}^{\infty} a_n (x-a)^n.$$

It is a series of functions because for each value of the variable x we get a numerical series. The set of values $x \in \mathbb{R}$ for which a power series is a convergent numerical series is called *convergence set*.

- The convergence set of a power series is given by the *absolute ratio test*. Indeed, if we want the series $\sum a_n(x-a)^n$ to be convergent we would like to have

$$\rho_x := \underbrace{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}_{\rho} |x-a| < 1.$$

  this inequality gives us an interval that is contained in the convergence set. To determine the convergence set, we need to check by hand the extremes of such interval. The possibilities are:

  1. $\rho = 0 \Rightarrow \rho_x = 0 < 1$ for all values of x and therefore the series is always convergent.
  2. $\rho = \infty \Rightarrow \rho_x = \infty > 1$ for all values of $x \neq a$ and therefore the series is divergent for $x \neq a$.
  3. $\rho \neq 0, \infty \Rightarrow \rho_x < 1$ only if $|x-a| < \dfrac{1}{\rho}$. The number $R = 1/\rho$ is called the *radius of convergence*, then set of values for which the power series converges is one of the intervals:

$$(a-R, a+R)\,,\ [a-R, a+R)\,,\ (a-R, a+R]\,,\ \text{or}\ [a-R, a+R]\,.$$

     We refer to such interval as the *interval of convergence*.
- When $\rho = 0$, the radius of convergence is $R = \infty$ and the interval of convergence is $(-\infty, \infty)$.
- When $\rho = \infty$, the radius of convergence is $R = 0$, and the interval of convergence is just $\{a\}$.

- If a power series if convergent at $x \in \mathbb{R}$, we denote its sum (at x) by $S(x)$. $S(x)$ is a function defined on the convergence set of the power series.

- Consider the series
$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$$
This is a geometric series with $r = x$ for each value of x. It is therefore convergent only if $|x| < 1$. The sum is
$$S(x) = \frac{1}{1-x}, \quad -1 < x < 1.$$

- Conversely, if a function $f(x)$ is the sum of a power series in some interval I then we say that $f(x)$ has a power series expansion in I.

- *Taylor Polynomials:* If a function $f(x)$ has derivatives $f'(a),\ f''(a), \ldots, f^{(n+1)}(a)$ at $x = a$, then the degree n Taylor Polynomial of $f(x)$ is
$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(x)}{n!}.$$
We should think of $P_n(x)$ as the degree n approximation of f for values of x very close to a.

- When $a = 0$, $P_n(x)$ is called the Maclaurin Polynomial.

- The ERROR term in the Taylor approximation is
$$R_n(x) = f(x) - P_n(x).$$

- Suppose that f has derivatives $f',\ f'', \cdots,\ f^{(n)},\ f^{(n+1)}$ in an interval containing $x = a$. If $|f^{(n+1)}(c)| \leq M$ for all c in the interval and some constant M, then
$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$
for every x in the interval. **Note:** For most applications one uses the interval $[a, x]$ if $a < x$, or the interval $[x, a]$ if $x < a$.

- Suppose that f has derivatives of all orders at $x = a$, then the Taylor series of f centered at a is the series
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$
When $a = 0$ the Taylor series is called *Maclaurin Series.*

- The equality
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$
holds for every x such that $\lim_{n\to\infty} R_n(x) = 0$. This means that for all such x the Taylor series of f converges to $f(x)$.

- *Uniqueness:* Suppose that a function $f(x)$ has a power series expansion centered at $a \in \mathbb{R}$ in the interval $I = (a - R, a + R)$. That is, for every $x \in I$ we can write

$$f(x) = \sum_{n=0}^{\infty} a_n (x - a)^n.$$

Then

$$a_n = \frac{f^{(n)}(a)}{n!}.$$

This means that the only possible power series expansion of f is its Taylor series.

Part IV: Fourier Series, Fourier Integrals, and Wave Equation (about 30% of Final Exam).

- A *periodic function* is a function $f(x)$ defined for all real x, except possibly at some points, such that there is a number $p \in \mathbb{R}$ such that

$$f(x + p) = f(x).$$

The smallest of such numbers p is called *the fundamental period* of f.

- Examples of periodic functions include $\sin x$, $\cos x$ of period 2π, and $\tan x$ of period π, which is defined everywhere except at $x = \pm\pi/2, \ \pm 3\pi/2, \ldots$.

Theorem 16 (Fourier Series Expansion)**.** *Suppose that $f(x)$ is a periodic function of period $2L$, that is piecewise continuous in the interval $[-L, L]$. Furthermore, let f have a right-hand and a left-hand derivative at each point of the interval. Then $f(x)$ has a series expansion of the form*

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} x\right) + b_n \sin\left(\frac{n\pi}{L} x\right)$$

at every point where the function is continuous. At the points where f is discontinuous the series converges to the average of the left-hand and right-hand limits of f.

- The expansion above is called the Fourier Series of $f(x)$, its coefficients are given by the formulas

$$\begin{aligned}
a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x)\ dx \\
a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi}{L} x\right)\ dx, \quad n = 1, 2, \ldots \\
b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L} x\right)\ dx, \quad n = 1, 2, \ldots
\end{aligned}$$

- Even if a function is not periodic, it is possible to see it as the limit of periodic functions. If each of these periodic functions has a Fourier series expansion, then it is sensible to think that the limit of such Fourier expansions will represent the function we started with. This is how we obtain the following theorem:

Theorem 17 (Fourier Integral)**.** *Assume that $f(x)$ is piecewise continuous in every interval, has right-hand and left-hand derivatives at every point, and is absolutely integrable. Then the equality*

$$f(x) = \int_0^\infty A(w)\cos wx + B(w)\sin wx \; dw$$

holds for every x at which f is continuous, where the coefficients are given by the formulas:

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv \; dv,$$
$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv \; dv.$$

- There is a **Partial Differential Equation** (henceforth PDE) modeling small transverse vibrations of an elastic string that has been stretched to length L on the x-axis, and fastened at the ends $x = 0$ and $x = L$. For the modeling process we make the following assumptions:
  - The mass of the string per unit length is constant ρ. The string is perfectly elastic and does not offer any resistance to bending.
  - The tension caused by stretching the string before fastening it is so large that the action of the gravitational force can be neglected.
  - The string moves only vertically.

  Then if $u(x,t)$ denotes the position of the string at the time t, one deduces that $u(x,t)$ must satisfy the following PDE
  $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$
  where $c^2 = \dfrac{T}{\rho}$. This is called the **1-dimensional wave equation**.

- Under the considerations above we have the **boundary conditions**:
  $$u(0,t) = 0, \quad u(L,t) = 0, \quad \text{for all } t \geq 0.$$

- The solution to the wave equation will depend not only on the boundary conditions but on some initial data (**initial conditions**), corresponding to the initial position and velocity of the string:
  $$u(x,0) = f(x), \quad \text{and} \quad u_t(x,0) = g(x), \quad x \in [0,L].$$

- Using the method of **separating variables**, that is, assuming that there is a solution of the form $u(x,t) = F(x)G(t)$, one can reduce the problem to solving two ODEs. The solution obtained this way looks like
  $$\boxed{u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{n\pi}{L}ct\right) + B_n \sin\left(\frac{n\pi}{L}ct\right)\right]\sin\left(\frac{n\pi}{L}x\right)},$$

where the coefficients A_n and $\frac{n\pi}{L}cB_n$ are the "Fourier coefficients" of the odd extensions of $f(x)$ and $g(x)$ respectively, i.e.,

$$\boxed{A_n = \frac{2}{L}\int_0^L f(v)\sin\left(\frac{n\pi}{L}v\right)\,dv} \qquad \boxed{\frac{n\pi}{L}cB_n = \frac{2}{L}\int_0^L g(v)\sin\left(\frac{n\pi}{L}v\right)\,dv}$$

- Assuming that the initial conditions $f(x)$ and $g(x)$ are piecewise continuous and have right-hand and left-hand derivatives in $[0, L]$, one can prove that the series solution to the wave equation converges to

$$u(x,t) = \frac{1}{2}[f_{odd}(x+ct) + f_{odd}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g_{odd}(s)\,ds,$$

where $f_{odd}(x)$ and $g_{odd}(x)$ are the odd extensions of $f(x)$ and $g(x)$ to the interval $[-L, L]$, i.e.,

$$f_{odd}(x) = \begin{cases} f(x) & \text{if} \quad 0 < x < L \\ -f(-x) & \text{if} \quad -L < x < 0 \end{cases}, \quad g_{odd}(x) = \begin{cases} g(x) & \text{if} \quad 0 < x < L \\ -g(-x) & \text{if} \quad -L < x < 0 \end{cases}$$

- In particular, if there is no initial velocity ($g(x) = 0$), then the solution becomes

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}ct\right)\sin\left(\frac{n\pi}{L}x\right)} \quad \text{with} \quad \boxed{A_n = \frac{2}{L}\int_0^L f(v)\sin\left(\frac{n\pi}{L}v\right)\,dv},$$

or

$$\boxed{u(x,t) = \frac{1}{2}\left[f_{odd}(x+ct) + f_{odd}(x-ct)\right]}$$

EXTRA PROBLEMS

You should definitely review your WebWork problems before the exam, and the problems from chapters 11 and 12 below. If you want to practice even more, here there are some suggested problems from the book:

- **8.2.** 5, 6, 7, 12, 14, 17.
- **8.3.** 1, 4, 5, 7, 8, 12.
- **9.1.** 4, 7–22, 42.
- **9.2.** 1–6, 13–16, 17–20.
- **9.3.** 3–14.
- **9.4.** 3–24.
- **9.5.** 1–16, 25–39.
- **9.6.** 1–28.
- **9.7.** 35, 36.
- **9.8.** 1–6, 11–14, 29–50.
- **11.2.** 8, 9, 11, 12, 14, 16, 28.
- **11.7.** 7, 8, 9, 16, 17.
- **12.3.** 5, 6, 7.