MATH 8–Transition to Higher Mathematics, Spring 2017
Final Exam Study Guide

1. Write the definitions of contradiction and tautology.
2. Define the notion of subset.
3. Define the power set of a given set.
4. Name at least two elements of the power set of a nonempty set.
5. What does it mean for two sets to be equal?
6. Describe strategies to prove the following:
   (a) $A \subseteq B$.
   (b) $A = \varnothing$.
   (c) $A = B$.
   (d) The set A has one single element.
7. Prove that $\sqrt{2}$ is irrational.
8. What is a relation?
9. Define domain and range of a relation.
10. Given a relation R, what do we mean by R^{-1}?
11. Define the composition of two relations.
12. Describe a relation between the composition of two relations and their domains and ranges.
13. What does it mean for a relation to be reflexive, symmetric, antisymmetric, transitive? Give examples of all these notions.
14. Give examples of relations satisfying one and only one of the properties above.
15. What is an order relation?
16. Give examples of at least three different order relations on $\mathbb{Z}$.

17. Prove that every nonempty set admits an order relation.

18. Given an order relation R on a set A and $B \subset A$, define the notion of lower and upper bound for B with respect to R.

19. What do we mean by R-maximal and R-minimal elements of a subset of a set A with respect to a partial order R on A?

20. What do we mean by R-largest and R-smallest elements of a subset of a set A with respect to a partial order R on A?

21. When are the notions of largest and maximal elements equal? Give examples.

22. When are the notions of smallest and minimal elements equal? Give examples.

23. What is the smallest element of $\mathbb{N}$ with the order given by divisibility "$|$" (recall that $a|b$ if and only if b is a multiple of a)?

24. Describe all subsets of $\mathbb{N}$ for which 3 is a lower bound with respect to the order $|$.

25. Let p be a positive prime number. How many subsets of $\mathbb{N}$ are there with p as an upper bound with respect to the order $|$?

26. Let A be a set and $\mathcal{F} \subseteq \mathscr{P}(A)$ be a family of subsets of A. Prove that $\varnothing$ is a lower bound and that A is an upper bound for $\mathcal{F}$ with respect to the partial order $\subseteq$. What is its greatest lower bound? What is its least upper bound?

27. State and proof the well-ordering principle.

28. What is an equivalence relation?

29. Give examples of at least three different equivalence relations on $\mathbb{Z}$.

30. Denote by $m\mathbb{Z}$ the equivalence relation "congruence modulo m" on $\mathbb{Z}$. Prove that $\mathbb{Z}/m\mathbb{Z}$ has exactly m elements.

31. Let $[x] \in \mathbb{Z}/m\mathbb{Z}$ with $[x] \neq [0]$. Prove that if $n \in \{1, 2, \ldots, m-1, m\}$ is the smallest number with the property that $n[x] = [0]$ then $m|n$. What can you conclude if n is a prime number?

32. What is a function?

33. When can an order relation be a function?

34. When can an equivalence relation be a function?

35. Define the notions of one-to-one, onto, and bijective function.

36. Give an example of a function $f\colon \mathbb{N} \to \mathbb{N}$ that is one-to-one but not onto.

37. Give an example of a function $g\colon \mathbb{N} \to \mathbb{N}$ that is onto but not one-to-one.

38. Construct a bijection $f\colon (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$.

39. Given two numbers $a, b \in \mathbb{R}$ with $a < b$, construct a bijection between the interval $[a, b]$ and the interval $[0, 1]$.

40. A function $f\colon \mathbb{R} \to \mathbb{R}$ is increasing if for all $x, y \in \mathbb{R}$ we have $f(x) < f(y)$ whenever $x < y$. Prove that an increasing function is one-to-one.

41. Let $L\colon \mathbb{Z} \to \mathbb{Z}$ be a function such that $L(0) = 0$ and $L(n + m) = L(n) + L(m)$. Prove that $L(-m) = -L(m)$. Moreover, prove that $L(\lambda n) = \lambda L(n)$ for all $\lambda \in \mathbb{Z}$. Find a formula for $L(x)$.

42. Let $\mathscr{C} = \{f \mid f\colon \mathbb{R} \to \mathbb{R}\}$. Define at least two different order relations on $\mathscr{C}$.

43. Prove that every integer is either a prime number or a product of prime numbers.

44. Prove that there are infinitely many prime numbers.

45. Use induction to prove that there are exactly $n!$ bijections from $\mathcal{I}_n = \{1, 2, \ldots, n\}$ to itself.

46. Let $J_n = \{x_1, x_2, \ldots, x_n\}$ for every $n \in \mathbb{N}$.

    - Consider the subsets of $\mathscr{P}(J_{n+1})$:

      $$U = \{B \in \mathscr{P}(J_{n+1}) \mid x_{n+1} \in B\}, \quad V = \{B \in \mathscr{P}(J_{n+1}) \mid x_{n+1} \notin B\}.$$

      Prove that $\mathscr{P}(J_{n+1}) = U \cup V$ and that $U \cap V = \varnothing$.

    - Consider the function $\varphi\colon U \to \mathscr{P}(J_n)$ defined by

      $$\varphi(B) = B \setminus \{x_{n+1}\}.$$

      Prove that φ is bijective.

- Similarly, prove that the function $\psi\colon V \to \mathscr{P}(J_n)$ defined by

$$\psi(B) = B$$

is bijective.

47. Use the problem above to prove that if a set A has n elements then $\mathscr{P}(A)$ has 2^n elements. ***Hint:*** *Use induction.*

48. Prove that $\mathbb{Z}$ is countable

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51. Recall that

$$\mathbb{Q} = \left\{ \frac{n}{m} \ : \ n \in \mathbb{Z}, \ m \in \mathbb{N}, \ n \text{ and } m \text{ with no common factors} \right\}.$$

Prove that the function $f\colon \mathbb{Q} \to \mathbb{Z} \times \mathbb{N}$ defined by $f(n/m) = (n, m)$ is bijective and so $\mathbb{Q}$ is countable.

52. Prove that $\mathbb{R}$ is uncountable.