MATH 8–Transition to Higher Mathematics, Spring 2017
Week 1 Summary

The following is a summary of the concepts and definitions that we covered during the first week.

1. **First rule of proof writing:** Never lie!

2. **Basic mathematical law:** Every statement is either TRUE or FALSE, but not both.

3. Statements can be connected to form more complicated statements. We use the symbols $\wedge$, $\vee$, $\Rightarrow$, and $\Leftrightarrow$ for the connectives "AND", "OR", "IMPLIES", and IF AND ONLY IF, respectively.

4. One can derive the validity (or falsity) of a composed statement from the validity (or falsity) of the parts. The associated truth tables are:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

5. We can negate statements, we use $\neg P$ for the negation of P. The associated truth table is:

P	$\neg P$
T	F
F	T

6. A TAUTOLOGY is a statement that is always true. A CONTRADICTION is a statement that is always false. The most common tautology is $P \vee \neg P$ which states that the statement P can be either truth or false. The most common contradiction is $P \wedge \neg P$ which states that the statement P is true and false at the same time.

7. Sometimes when proving theorems we have to replace our given statement for an equivalent statement that is easier to work with ($\Leftrightarrow$), or for a simpler statement that is a consequence of the givens ($\Rightarrow$). Some of those replacement rules are given by the following tautologies:

- $\neg\neg P \Leftrightarrow P$.
- $P \Rightarrow P \vee Q$.
- $P \wedge Q \Rightarrow P$.
- $(P \to Q) \Leftrightarrow \neg Q \vee P$.
- $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$.
- $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$.
- $[(P \vee Q) \wedge \neg P] \Rightarrow Q$.
- $(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$.

8. **Distributive Laws:** The following are tautologies:
   - $P \wedge (Q \vee R) \Leftrightarrow [(P \wedge Q) \vee (P \wedge R)]$.
   - $P \vee (Q \wedge R) \Leftrightarrow [(P \vee Q) \wedge (P \vee R)]$.

9. **Tautology Laws:**

$$\begin{aligned} P \wedge (\text{tautology}) &\Leftrightarrow P \\ P \vee (\text{tautology}) &\text{ is a tautology} \\ \neg(\text{tautology}) &\text{ is a contradiction} \end{aligned}$$

10. **Contradiction Laws:**

$$\begin{aligned} P \wedge (\text{contradiction}) &\text{ is a contradiction} \\ P \vee (\text{contradiction}) &\Leftrightarrow P \\ \neg(\text{contradiction}) &\text{ is a tautology} \end{aligned}$$

11. A **theorem** is a tautology. A **proof** for a theorem is a verification of the tautology.

12. Statements in theorems are usually of the form $P \Rightarrow Q$. Here P can be a composed statement called the **hypothesis** of the theorem, and the statement Q, which can also be composed, is called the **conclusion** of the theorem.

13. An argument is a chain of implications $P \Rightarrow P_1 \Rightarrow P_2 \Rightarrow \cdots \Rightarrow P_k \Rightarrow Q$. This is usually the way in which proofs are written.

14. The connective $\Rightarrow$ (or $\rightarrow$) is called a conditional connective. It is a master of disguise, and appears in mathematics under

$$P \Rightarrow Q \begin{cases} P \text{ implies } Q \\ \text{If } P, \text{ then } Q \\ P \text{ is a sufficient condition for } Q \\ Q \text{ is a necessary condition for } P \end{cases}$$

15. **An important tautology:** $[P \Leftrightarrow Q] \Leftrightarrow [(P \Rightarrow Q) \wedge (Q \Rightarrow P)]$.

16. The connective $\Leftrightarrow$ (or $\leftrightarrow$) is a biconditional connective, by the tautology above it expresses $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. It disguises as

$$P \Leftrightarrow Q \begin{cases} P \text{ is equivalent to } Q \\ P \text{ if and only if } Q \\ P \text{ iff } Q \\ P \text{ is a sufficient and necessary condition for } Q \end{cases}$$

17. There are two quantifiers that are used to express for "how many" elements of a set certain property holds. These are:

$$\forall := \text{ for all, and } \exists := \text{ exists.}$$

18. The statement $\forall\ x \in A$ reads "for every element x in the set A". The statement $\exists\ x \in A$ reads "there is an element x in the set A".

19. **Quantifier negation law:**

$$\neg(\forall\ x \colon P(x)) \Leftrightarrow \exists x \colon \neg P(x),$$
$$\neg(\exists\ x \colon P(x)) \Leftrightarrow \forall x \colon \neg P(x).$$

20. **Uniqueness**. To express that there is a unique element x satisfying a property $P(x)$ we write $\exists! x \colon P(x)$, in other words

$$\exists! x \colon P(x) \Leftrightarrow (\exists x \colon P(x)) \wedge (\forall y \neq x \colon \neg P(y)) .$$

21. A set is a collection of objects. The objects in a set are called elements. We write $x \in A$ to state that x is an element in the set A. In the same way, $x \notin A$ means that x is not an element of A, i.e., $\neg(x \in A)$.

22. We use $A \subseteq B$ to state that every element of the set A is also an element of the set B. We say that A is a subset of B. Thus,

$$(A \subseteq B) \Leftrightarrow (x \in A \Rightarrow x \in B).$$

23. Two sets A and B are equal if they have the same elements, i.e., each element of A is also an element of B and each element of B is also an element of A. Thus,

$$A = B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A).$$

24. Given two sets A and B, one can define the following operations:

    - **Union**
    $$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}.$$
    - **Intersection**
    $$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}.$$
    - **Difference**
    $$A \setminus B = \{x \mid (x \in A) \wedge (x \notin B)\}.$$

25. There is a set with no elements. It is called the **empty set** and we denote it by $\emptyset$.

26. Every set A has two distinguished subsets, namely $\emptyset$ and A itself.

27. The **power set** of a set A is the set of subsets of A. We denote the power set of A by $\mathscr{P}(A)$, i.e.,
$$\mathscr{P}(A) = \{x \mid x \subseteq A\}.$$

28. The **cartesian product** of two sets A and B is the set of pairs (a, b) with $a \in A$ and $b \in B$. Formally
$$A \times B := \{(a, b) \mid (a \in A) \wedge (b \in B)\}.$$

29. A **relation** from a set A to a set B is a subset $R \subset A \times B$.

30. The **domain** of a relation $R \subset A \times B$ is the set

$$\text{Dom } R := \{a \in A \mid \exists b \in B \text{ such that } (a, b) \in R\}.$$

31. The **range** of a relation $R \subset A \times B$ is the set

$$\text{Ran } R := \{b \in B \mid \exists a \in A \text{ such that } (a, b) \in R\}.$$

32. The **composition** of two relations $R \subset A \times B$ and $S \subset B \times C$ is the relation from A to C defined by

$$S \circ R := \{(a, c) \in A \times C \mid \exists b \in B \text{ such that } ((a, b) \in R) \wedge ((b, c) \in S)\}.$$

33. The **inverse** of a relation $R \subset A \times B$ is the relation from B to A defined by

$$R^{-1} := \{(b, a) \in B \times A \mid (a, b) \in R\}.$$

34. When we are asked to prove a statement, we usually refer to the hypothesis as **givens** and to the conclusion as the **goal**. Sometimes it is possible to change the goal to an equivalent form that makes the statement easier to prove or that indicates how to proceed with the proof. Here are some examples:

    - If the goal of the statement is of the form $P \to Q$, then we can take P as part of the givens and change the goal to Q.
    - If the goal is of the form $P \vee Q$ we can change it to the equivalent form $\neg P \to Q$, then assume $\neg P$ as one of the givens and prove Q. (This may be confusing but think that if you want to prove that either P or Q hold then you may as well prove that if P doesn't hold then necessarily Q should hold).
    - If the goal is of the form $P \leftrightarrow Q$ then you should prove two statements: $P \to Q$ and $Q \to P$, their proofs should be independent.
    - If your goal is to prove an inclusion of sets $A \subset B$ you should write this statement in logical terms, i.e.,
      $$A \subset B \Leftrightarrow (x \in A) \to (x \in B).$$
      Therefore you can take the statement $x \in A$ as part of your givens and change your goal to $x \in B$.
    - If your goal is to prove that a set A is empty, then a good strategy is to argue by contradiction, i.e., assume that there is an element $x \in A$ and take this as one of your givens, you should be able to build up to a statement that is both true and false.
    - If your goal is to prove that two sets A and B are equal, then you should prove two different statements: $A \subset B$ and $B \subset A$.
    - If the goal is of the form $\exists x P(x)$, then you should either exhibit x satisfying $P(x)$ or you should assume that no such x exists and build up to a contradiction.

- If the goal is of the form $\exists! x P(x)$, then you should prove two things: $\exists x P(x)$, and that if there is a second element y satisfying $P(y)$ then necessarily $x = y$.
- If the goal is a statement about the power set $\mathscr{P}(A)$ then it is convenient to interpret the statement using words, replacing "$x \in \mathscr{P}(A)$" by "x is a subset of A". For instance, if the goal is to prove that $\mathscr{P}(A) \cap \mathscr{P}(B) \neq \emptyset$, then you should prove that there is a subset of A that is also a subset of B.
- If the goal is to prove that $\mathscr{P}(A) \subset \mathscr{P}(B)$(i.e., that every subset of A is also a subset of B), then it is convenient to change the statement to:

$$\begin{aligned} \mathscr{P}(A) \subset \mathscr{P}(B) &\Leftrightarrow (C \in \mathscr{P}(A)) \to (C \in \mathscr{P}(B)) \\ &\Leftrightarrow (C \subset A) \to (C \subset B) \\ &\Leftrightarrow (x \in C \to x \in A) \to (x \in C \to x \in B) \end{aligned}$$

  and so take the statements "$x \in C \to x \in A$" and "$x \in C$" as part of your givens, and change the goal to $x \in B$.
- If $A \subset B$ and $B \subset C$ then $A \subset C$. We refer to this property of the inclusion ($\subset$) as **transitivity**. As a practice exercise prove the above statement. You should feel free to use the transitivity property in any of the problems.

SUGGESTED PROBLEMS

Many of the following exercises have already been included as homework problems, so you should be familiar with many of them. Nevertheless, it would be good practice to write new clean proofs.

1.2. 8, 9, 12.

1.4. 1, 5, 7.

1.5. 10.

2.3. 3, 5, 6, 8, 11.

3.1. 5–13.

3.2. 3, 4, 7, 8, 12.

3.3. 2, 3, 4, 5, 8, 9, 16, 18, 21.

3.4. 2, 3, 4, 5, 8, 9, 10, 13, 14, 16, 19, 20, 25, 26.

3.5. 4, 7, 8, 9, 12–15, 27.

3.6. 5(a), 6, 7, 8, 10.

4.1. 4, 5, 7, 10.

4.2. 1, 2, 4, 8, 9, 10.