RESEARCH STATEMENT

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I am an Algebraic Geometer. I am mainly interested in the study of the birational geometry of moduli spaces of sheaves via variation of stability conditions.

Complex Algebraic Geometry is concerned with the study of solution sets of homogeneous (complex) polynomial equations, each of which is called a projective algebraic variety or variety for short. Two varieties X and Y are said to be *birationally equivalent* if their fields of functions $\mathbb{C}(X)$ and $\mathbb{C}(Y)$ are isomorphic, or equivalently if there is an isomorphism $\phi\colon U \cong V$ between open sets $U \subseteq X$ and $V \subseteq Y$. My research focusses in identifying varieties that have the same fields of functions as varieties that parametrize compactified families of vector spaces over a fixed base, henceforth coherent sheaves. It turns out that in order to get a parameter space for coherent sheaves that is a variety we have to fix the topological type (Chern classes) and get rid of some coherent sheaves: those that violate a numerical condition called stability. The technique to get new varieties that are birationally equivalent to the parameter variety (moduli space) of stable sheaves is to vary the notion of stability not only numerically but also categorically, allowing not only sheaves but complexes of sheaves.

1. Background and past research

Let X be a smooth projective variety over $\mathbb{C}$ and choose an embedding of X into a projective space, i.e., choose a positive line bundle $H \in \operatorname{Amp}(X)$. We say that a coherent sheaf E is H-Gieseker semistable if for all subsheaves $A \hookrightarrow E$ one has

$$\frac{p_H(A)(n)}{r(A)} \leq \frac{p_H(E)(n)}{r(E)} \tag{1}$$

where $p_H(\,\cdot\,)(n)$ is the Hilbert polynomial with respect to the embedding given by H, which can be computed from the Riemann-Roch Theorem as

$$p_H(\,\cdot\,)(n) = \int_X e^{nH} \operatorname{ch}(\,\cdot\,) \operatorname{td}(X).$$

The resultant moduli space $M_H(v)$ is a projective variety and is our main object of study, mainly when X is a surface or a threefold.

One thing to observe is that H-Gieseker stability depends on the choice of the positive class H. In the 90's many papers were devoted to study how the moduli spaces $M_H(v)$ relate as H varies. It turns out that for a fixed Chern character v the ample cone has a wall and chamber decomposition such that the moduli spaces $M_H(v)$ are constant on each chamber. For a generic choice of v it was also known that the moduli spaces $M_H(v)$ would be birational for polarizations in different chambers. Thus, by varying the definition of stability one could get different birational models of $M_H(v)$.

Stability conditions were introduced by Bridgeland in [Bri07] in an effort to understand mathematically Douglas' notion of Π-stability for D-branes [Dou02]. A (numerical) pre-stability condition on a smooth variety X is a pair $\sigma = (Z_\sigma, \mathcal{A}_\sigma)$ consisting of a full abelian subcategory $\mathcal{A}_\sigma \subset D^b(\operatorname{Coh}(X))$ (which is the heart of a bounded t-structure), and a linear map $Z_\sigma\colon \mathrm{H}^{2*}_{alg}(X) \to \mathbb{C}$ (the central charge) such that

$$Z_\sigma(\operatorname{ch}(E)) \in \{re^{\phi\pi i} \mid r > 0,\ \phi \in (0,1]\} \text{ for all } E \in \mathcal{A}_\sigma,$$

i.e., Z_σ should satisfy the positivity property that for every $E \in \mathcal{A}_\sigma$, $\mathfrak{Im}(Z_\sigma(\mathrm{ch}(E))) \geq 0$, and $\mathfrak{Im}(Z_\sigma(\mathrm{ch}(E))) = 0$ only when $\mathfrak{Re}(Z_\sigma(\mathrm{ch}(E))) < 0$.

An object $E \in \mathcal{A}_\sigma$ is called σ-semistable if for every subobject $A \hookrightarrow E$ in $\mathcal{A}_\sigma$ one has

$$\mu_\sigma(A) \leq \mu_\sigma(E),$$

where $\mu_\sigma(E) = -\mathfrak{Re}(Z_\sigma(\mathrm{ch}(E)))/\mathfrak{Im}(Z_\sigma(\mathrm{ch}(E)))$. The pre-stability condition σ is a stability condition if every element in $\mathcal{A}_\sigma$ has a unique Harder-Narasimhan filtration with respect to μ_σ, and if there is a quadratic form Q_σ on $\mathrm{ch}(\mathcal{A}_\sigma) \subset \mathrm{H}^{2*}_{alg}(X)$ that is positive definite on stable objects.

Even though the definition of a stability condition may seem obscure, the defining conditions seek to generalize the notion of stable bundle on a curve to complexes on arbitrary varieties. On a smooth curve C there is essentially only one stability condition, namely

$$\sigma = (Z = -\mathrm{ch}_1 + \sqrt{-1}\ \mathrm{ch}_0,\ \mathrm{Coh}(C)).$$

On an arbitrary surface X tons of examples have been constructed by Bridgeland [Bri08], and Arcara-Bertram [AB13]. These examples have proven to be especially useful to study the birational geometry of $M_H(v)$ (see [ABCH13], [BM14], [BMW14], [Nue16], [LZ16]). For instance, we can consider the 1-parameter family of linear maps

$$Z_{\alpha_t}(\mathrm{ch}(E)) = -\int_X \mathrm{ch}(E)\alpha_t + \sqrt{-1}\int_X \mathrm{ch}(E)\alpha_t H,$$

where $\alpha_t = e^{tH}\mathrm{td}(X) - \Gamma_t$ and $\Gamma_t \in H^4(X) \otimes \mathbb{Q}$. As proven by Bertram in [Ber14], Z_{α_t} is the central charge of a stability condition $\sigma_{\alpha_t} = (Z_{\alpha_t}, \mathcal{A}_{\alpha_t})$ as long as $\chi(\mathcal{O}_S) - \Gamma_t < K^2/8$.

Also, if $v = \mathrm{ch}(E)$ is the Chern character of an H-Gieseker semistable sheaf, then after choosing $\Gamma_t = p_H(E)(t)/r(E)$, one obtains

$$\frac{\int_X \mathrm{ch}(A)\alpha_t H}{\mathrm{ch}_0(A)}\mu_{\sigma_{\alpha_t}}(A) = \frac{1}{\mathrm{ch}_0(A)}\int_X \mathrm{ch}(A)\alpha_t = \frac{p_H(A)(t)}{r(A)} - \frac{p_H(E)(t)}{r(E)}.$$

Thus the sign of $\mu_{\sigma_{\alpha_t}}(A)$ on subsheaves $A \hookrightarrow E$ for $t \gg 0$ determines the H-Gieseker semistability of E. Moreover, one can show the existence of $t_0 > 0$ such that the only σ_{α_t}-semistable objects for $t > t_0$ are the H-Gieseker semistable sheaves of type v. In this direction, together with Bertram and Wang, we obtained the following:

Theorem 1 ([BMW14])**.** *Let $X = \mathbb{P}^2$ and let H be the hyperplane class. If v is primitive then every birational model of $M_H(v)$ can be obtained as a component of the moduli space of σ_{α_t}-semistable objects of Chern character v for some $t > 0$.*

Stability conditions have proven to be a source of very interesting geometry. For instance, in [Mar17] I proved that if $\nu_{d-3}(\mathbb{P}^2) \subset \mathbb{P}^N$ is the image of the Veronese embedding of degree $d-3$ ($d > 5$), then one can obtain an explicit sequence of birational transformations of $\mathbb{P}^N$ flipping the first $\lfloor \frac{d-1}{2} \rfloor$ secant varieties of $\nu_{d-3}(\mathbb{P}^2)$ by restricting an MMP for a moduli space of torsion sheaves on $\mathbb{P}^2$ to the main component of the fixed locus of a particular involution. More precisely, let $v_d = (0, d, s_d)$ where s_d equals $-3d/2$ for d odd, and equals $-d/2$ for d even. The moduli space of stable torsion sheaves on $\mathbb{P}^2$ supported on a curve of degree d and with Chern character vector v_d can be thought as a compactified Jacobian via the support map.

Theorem 2 ([Mar17])**.** *By restricting an MMP for the Mori dream space $M_H(v_d)$ to the main component of the fixed locus of the -1 involution, we obtain a diagram*

$$\begin{tikzcd}
\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(d-3))^\vee) \arrow[d] \arrow[dr] & & M_0 \arrow[dl] \arrow[dr] & & M_1 \arrow[dl] \arrow[dr] & \cdots & M_{\lfloor \frac{d-1}{2} \rfloor} \arrow[dl] \\
pt & N_0 & & N_1 & & \cdots &
\end{tikzcd}$$

where M_0 is the blowup of $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(d-3))^\vee)$ along $\nu_{d-3}(\mathbb{P}^2)$ and $M_k \dashrightarrow M_{k+1}$ flips the $(k+1)$-secant variety of $\nu_{d-3}(\mathbb{P}^2)$.

Theorem 2 was proven studying the wall-crossing for the family of stability conditions σ_{α_t} with $\Gamma_t = t$. Given the stability condition σ_{α_t} we can define its dual $\sigma^D_{\alpha_t}$ as the stability condition with central charge

$$Z_{\tilde{\alpha_t}}(\mathrm{ch}(E)) = -\int_X \mathrm{ch}(E)\tilde{\alpha_t} + \sqrt{-1}\int_X \mathrm{ch}(E)\tilde{\alpha_t}H,$$

where $\tilde{\alpha_t} = e^{-tH}\mathrm{td}(X) - (\Gamma_t - K^2)$. An important intermediate step that leads to the results in Theorem 2 was to establish the following duality theorem:

Theorem 3 ([Mar17]). *E is σ_{α_t}-semistable if and only if $E^D = R\mathcal{H}om(E, \omega_X)[1]$ is $\sigma^D_{\alpha_t}$-semistable. Moreover, the functor $E \mapsto E^D$ induces an isomorphism between the moduli spaces $M_{\sigma_{\alpha_t}}(v)$ and $M_{\sigma^D_{\alpha_t}}(v^D)$ provided these moduli spaces exist. In particular, $\mathcal{E} \mapsto \mathscr{E}xt^1(\mathcal{E}, \omega_X)$ induces an isomorphism between the Gieseker moduli spaces $M_H(0, c_1, \chi)$ and $M_H(0, c_1, -\chi)$.*

The first version of the duality theorem is due to Maican in [Mai10] who proved it for moduli spaces of Gieseker semistable 1-dimensional sheaves on the projective space. It was later proven by Saccà, in her thesis [Sac13], that Maican's result holds for moduli spaces of 1-dimensional Gieseker semistable sheaves on arbitrary smooth projective surfaces.

Our duality theorem was used by Arcara and Miles [AM16] to obtain the destabilizing walls for $\mathcal{O}[1]$ from the destabilizing walls for $\mathcal{O}$, and by Nuer [Nue16] to describe a bouncing wall when studying the geometry of the Hilbert scheme of points on an Enriques surface.

Also, our way of computing flipping loci based on the Maican stratifications of moduli spaces of 1-dimensional sheaves, introduced in a paper with Bertram and Wang [BMW14] and generalized in [Mar17], was used by Choi and Chung [CC15] to compute the Poincare polynomial for M_6, the moduli space of 1-dimensional semistable sheaves on $\mathbb{P}^2$ supported on a sextic and with Euler characteristic 1.

The definition of the stability conditions σ_{α_t} can be extended to include the notion of twisted Gieseker stability. More precisely, for every fixed Chern character v and every two positive line bundles $H, H' \in \mathrm{Amp}(X)$ one can consider stability conditions $\sigma_{\alpha_{s,t}}$ with central charges

$$Z_{\alpha_{s,t}}(\mathrm{ch}(\,\cdot\,)) = -\int_X \mathrm{ch}(\,\cdot\,)\alpha_{s,t} + \sqrt{-1}\int_X \mathrm{ch}(\,\cdot\,)\alpha_{s,t}H,$$

where $\alpha_{s,t} = e^{sH'+tH}\mathrm{td}(X) - \Gamma_{s,t}$, and

$$\Gamma_{s,t} = \frac{p_H(E\otimes\mathcal{O}(sH'))(t)}{r(E)} = \frac{p_{H'}(E\otimes\mathcal{O}(tH))(s)}{r(E)} \quad \text{with} \quad \mathrm{ch}(E) = v.$$

After a detailed study of the wall and chamber decomposition for the family $\{\sigma_{\alpha_{s,t}}\}_{s,t\geq 0}$, in my joint work with Bertram, we obtained a new proof of a result Matsuki and Wentworth in [MW97] explaining the relation between moduli spaces of Gieseker semistable sheaves for different embeddings of the surface. The precise statement is:

Theorem 4 ([BM15]). *The moduli spaces $M_H(v)$ and $M_{H'}(v)$ on a smooth surface X can be related by a sequence of "Thaddeus flips" involving only moduli spaces of twisted Gieseker semistable sheaves and obtained by varying the family $\{\sigma_{\alpha_{s,t}}\}_{s,t\geq 0}$ on the segment from $\sigma_{\alpha_{0,t}}$ with $t \gg 0$ to $\sigma_{\alpha_{s,0}}$ with $s \gg 0$.*

Similar techniques can be used to give an alternative proof of a result of Toda [Tod14] realizing every smooth MMP on a surface by varying a family of stability conditions.

Theorem 5 ([BM15]). *Every smooth MMP for a surface X can be obtained by varying a family of stability conditions on X, after regarding X as the parameter space for ideal sheaves of a point.*

2. Current research and future plans

When X is a threefold, even the question of whether the space of stability conditions is nonempty is still widely open. Besides the case when X has a full strong exceptional collection [Mac07], stability conditions have been constructed only in a handful of cases: X is a Fano threefold [BMT14], [Mac14], [Sch14], [Li15], [BMSZ17]; an abelian threefold [BMS16], [MP15, MP16]; a crepant resolution of an abelian threefold [BMS16]; or a product of projective spaces and abelian varieties [Kos17].

The conjectural construction of stability conditions on threefolds is due to Bayer, Macrì, and Toda [BMT14]. For every pair of $\mathbb{R}$-line bundles (B, ω) with ω positive, they construct the heart of a t-structure $\mathcal{A}_{B,\omega}$ as a double tilt of $\mathrm{Coh}(X)$ and assuming the positivity property of their central charge

$$Z_{B,\omega}(\mathrm{ch}(E)) = -\int_X e^{-B-i\omega}\,\mathrm{ch}(E),$$

they prove that $\sigma_{B,\omega} = (Z_{B,\omega}, \mathcal{A}_{B,\omega})$ is a stability condition.

In each of the cases mentioned above, the positivity property of $Z_{B,\omega}$ has been proven. However, it was first noticed by Schmidt [Sch17] that if E denotes the exceptional divisor of the blow up $X = \mathbf{bl}_p\mathbb{P}^3$, then one can find a pair of $\mathbb{R}$-line bundles (B, ω) on X such that $\mathcal{O}(E) \in \mathcal{A}_{B,\omega}$ violates the positivity property. More generally, in a recent paper with Schmidt [MSD17], we found a complete new class of counterexamples:

Theorem 6 ([MSD17])**.** *Let X be a smooth projective threefold. Suppose that there is an effective divisor D and an ample divisor H such that*

$$D^3 > \frac{(D\cdot H^2)^3}{4(H^3)^2} + \frac{3}{4}\frac{(D^2\cdot H)^2}{D\cdot H^2}.$$

Then there exists a pair $(\beta H, \alpha H)$ such that $\mathcal{O}_D$ violates the positivity property of $Z_{\beta H,\alpha H}$.

In particular, we obtain

Theorem 7 ([MSD17])**.** *Let $f : X \to Y$ be a birational morphism between projective threefolds, where X is smooth. Let $D \subset X$ be an effective divisor such that $-D$ is f-ample and $f(D)$ is a point. Then we can choose a polarization H on X such that $Z_{\beta H,\alpha H}$ does not satisfy the positivity property for some $\beta, \alpha \in \mathbb{R}$. In particular, $Z_{\beta H,\alpha H}$ is not the central charge of a stability condition on $\mathcal{A}_{\beta H,\alpha H}$ on any Weierstraß elliptic Calabi-Yau threefold over a del Pezzo surface.*

Even though the conjectural construction of stability conditions in [BMT14] fails in general, one can still ask if it is possible to modify the central charges $Z_{B,\omega}$ a little so that they satisfy the positivity property. Conjectural approaches to this problem appear in [BMSZ17] and [Piy17], and consist of a modification in codimension 2. More precisely, we have

Problem 1. Can we find a class $\Gamma \in A_1(X) \otimes \mathbb{R}$ such that

$$Z_{B,\omega}(\mathrm{ch}(E)) = -\int_X (e^{-i\omega} - \Gamma)e^{-B}\,\mathrm{ch}(E)$$

satisfies the positivity property on $\mathcal{A}_{B,\omega}$?

In [MSD17] we also give a partial answer to Problem 1:

Theorem 8 ([MSD17])**.** *Assume that X is a smooth projective threefold where Problem 1 has an affirmative answer for some positive line bundle ω and $\mathbb{R}$-divisor B. Then the induced upper half-plane of stability conditions on X embeds into the space of stability conditions on the blow up of X at an arbitrary point.*

For these stability conditions skyscraper sheaves $\mathbb{C}(x)$ on the blow up are all semistable. A skyscraper sheaf $\mathbb{C}(x)$ is stable if and only if x does not lie on the exceptional divisor.

In the proof of Theorem 8, the class Γ that is used to modify the construction of the central charges on the blow up is the pullback of the modifying class on the base plus a multiple of the square of the exceptional divisor. Our argument uses a noncommutative description of the derived category of the blow up due to Toda [Tod13], which does not apply to case of the Weierstraß elliptic Calabi-Yau threefold. This leads to the following

Problem 2. Can we find noncommutative descriptions of the derived category of an elliptic Calabi-Yau threefold? If so, can we use such descriptions to prove inequalities on the third Chern class of a tilt-semistable object?

On the other hand, in a series of papers by Lo and his collaborators [Lo16, CL16, LZ15, Lo17b, Lo17a, Lo17c] the authors study the behavior of stability under natural Fourier-Mukai transforms.

Problem 3. Study the behavior of tilt-semistable objects under Fourier-Mukai transforms on families of abelian surfaces. Can we use translate the inequality on the third Chern character of a tilt semistable object to an inequality on the second Chern character of its Fourier-Mukai image?

Now, even if we know the existence of stability conditions on a threefold X, it is not clear that we can construct a stability condition that recovers Gieseker stability.

Problem 4. Let H be a positive line bundle on a smooth projective threefold X. For which classes $\Gamma_t \in \mathrm{Num}^{\geq 2}(X) \otimes \mathbb{R}$ is there a value $t_0 > 0$ for which

$$Z_{\alpha_t}(\mathrm{ch}(E)) = -\int_X \mathrm{ch}(E)\alpha_t + \sqrt{-1}\int_X \mathrm{ch}(E)\alpha_t H,$$

where $\alpha_t = e^{tH}\mathrm{td}(X) - \Gamma_t$, satisfies the positivity property for all $t > t_0$?

There is an instance when we can identify Bridgeland stability with Gieseker stability. In an ongoing project with Schmidt, we have stablished the following:

Theorem 9. *Given $p(m) = dm + \chi \in \mathbb{Z}[m]$, there exists a two dimensional family of stability conditions on $\mathbb{P}^3$ with a finite wall and chamber structure, and a distinguished chamber such that the only σ-semistable objects on this chamber are sheaves supported on a curve of degree d with Euler characteristic χ and that are semistable in the Gieseker sense.*

However, the walls are not easy to compute and the challenge is that there are more numerical walls than actual walls.

Problem 5. Develop an optimal algorithm to compute the walls where sheaves get destabilized. What types of wall-crossing affect the component of the Hilbert scheme parametrizing structure sheaves of curves $\mathcal{O}_C$ of Hilbert polynomial $p(m)$ in $M_H(v)$?

Now, in the same spirit of the construction of flips for secant varieties of Veronese surfaces, one can attempt to construct flips for k-uple embeddings of $\mathbb{P}^3$ by varying stability conditions. For simplicity, suppose that $k \geq 4$ is even. Write $k = d-1$, the k-uple embedding of $\mathbb{P}^3$ naturally lands on $\mathbb{P}(\mathrm{Ext}^1(\mathcal{O}((d+3)/2), \mathcal{O}(-(d+3)/2)[2]))$. This projective space parametrizes extensions

$$0 \to \mathcal{O}(-(d+3)/2)[2] \to E \to \mathcal{O}((d+3)/2) \to 0. \tag{2}$$

We have $v = \mathrm{ch}(E) = (2, 0, ((d+3)/2)^2 H^2, 0)$, which is not the Chern character of any sheaf. However, just as in Theorem 9, there is a two-parameter family of stability conditions and a chamber for the class v where the complexes E are all stable. Some of the walls are produced by ideal sheaves $\mathcal{I}_p$, $\mathcal{I}_{p+q}, \ldots$, corresponding to the image of $\mathbb{P}^3$ by the k-uple embedding, and its secant varieties.

Problem 6. Describe the stable objects for stability conditions in the last chamber ($\omega = \alpha H$ with $0 < \alpha \ll 1$). What type of objects (besides ideal sheaves of points) can destabilize extensions of the form (2)? Is the wall-crossing in this case produced by variation of a GIT parameter?

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