Sec 3.2 Tangents, Velocity, and Acceleration

- Def (Differentiation and integration)
  For a vector-valued function \( \mathbf{z}(t) = (x(t), y(t), z(t)) \) in \( \mathbb{R}^3 \),
  \( \mathbf{z}'(t) = (x'(t), y'(t), z'(t)) \) and
  \[ \int \mathbf{z}'(t) \, dt = (\int x'(t) \, dt, \int y'(t) \, dt, \int z'(t) \, dt). \]

Example: Find a parametrization of the tangent line to
  \( \mathbf{z}(t) = (2t, t^3, 0) \) at the point \((4, 8, 0)\).

Solution: Notice that \( \mathbf{z}'(2) = (2, 8, 0) \) so the tangent line has
  slope \( \mathbf{z}'(2) = (2, 3t^2, 0) \bigg|_{t=2} = (2, 12, 0) \).

  \[ \mathbf{r}(t) = (4, 8, 0) + t (2, 12, 0). \]

Example: If
  \( \mathbf{z}(t) = (t^2, \ t) \)
  \( \mathbf{z}(0) = (0, 2, -3) \)
  \( \mathbf{z}'(0) = (4, 2, -6) \)
then
  \[ \mathbf{v}(t) = \int \mathbf{z}'(t) \, dt = \left( \frac{t^3}{3}, \frac{t^3}{2} + B, \frac{t^5}{5} + C \right) \]
  \[ = \left( \frac{t^3}{3}, \frac{t^3}{2} + 2, \frac{t^5}{2} - 3 \right), \text{ using } \mathbf{z}'(0), \]
  so
  \[ \mathbf{z}(t) = \int \mathbf{v}(t) \, dt = \left( \frac{t^3}{6} + D, \frac{t^4}{12} + 2t + E, \frac{t^5}{6} + 3t + F \right) \]
  \[ = \left( \frac{t^3}{6} + 4, \frac{t^4}{12} + 2t + 2, \frac{t^5}{6} - 3t - 6 \right), \text{ using } \mathbf{z}(0). \]

Sec 3.3 Length of a curve

- Def: A function is \( C^1 \) or continuously differentiable if
  all of its partial derivatives exist and are continuous.
Def Let \( \vec{z} : [a, b] \to \mathbb{R}^3 \) be \( C^1 \). Then
\[
\text{length}(\vec{z}) = L(\vec{z}) = \int_a^b \| \vec{c}'(t) \| \, dt
\]

Example \( \vec{z}(t) = (r \cos t, r \sin t), \ t \in [0, 2\pi] \)
\[
L(\vec{z}) = \int_0^{2\pi} \sqrt{(r \cos t)^2 + (r \sin t)^2} \, dt = \int_0^{2\pi} r \, dt = 2\pi r.
\]
For \( \vec{z}(t) = (r \cos t, r \sin t), \ t \in [0, 4\pi], \) we have \( L(\vec{z}) = 4\pi r \).

Both paths parametrize the same curve, but only \( L(\vec{z}) \) gives the circumference.

So for \( L(\vec{z}) \), think distance traveled (in general).

Def If \( \vec{z} \) is \( C^1 \), \( \vec{c}'(t) \neq \vec{0} \), and \( \vec{z}(t) \) maps distinct points in \((a, b)\) to distinct points on the curve, then \( \vec{z} \) is called smooth.

Def A path is \text{closed} if \( \vec{z}(a) = \vec{z}(b) \).

A smooth path can be closed, but can't intersect, retrace itself, or have corners.
Example: Parametrize the graph of a $C^1$ function $f:[a,b] \to \mathbb{R}$ by $\gamma(t) = (t, f(t)), t \in [a,b]$. Then $\gamma$ is smooth and
\[
\ell(\gamma) = \int_a^b ||\gamma'(t)|| \, dt = \int_a^b \sqrt{1 + f'(t)^2} \, dt = \text{arc length formula}.
\]
Remark: If $\gamma$ is smooth, then $\ell(\gamma)$ is what one might expect, and not just distance traveled.

* Definition: Let $\gamma: [a,b] \to \mathbb{R}^2$ or $\mathbb{R}^3$ be $C^1$. The arc length function $s(t)$ for $\gamma(t)$ is:
\[
s(t) = \int_a^t ||\gamma'(u)|| \, du \quad \text{distance traveled in time to } t.
\]

* Note: By the FTC,
\[
\frac{ds}{dt} = \frac{d}{dt} \left( \int_a^t ||\gamma'(u)|| \, du \right) = ||\gamma'(t)||
\]

i.e.,
\[
\text{change in distance} \quad \text{over change in time} = \text{speed}.
\]

* If we solve for $t$ in terms of $s$, then $\gamma = \gamma(t(s))$ is said to be parametrized by arc length.

Example: $\gamma(t) = (\cos t, \sin t, t), t \in [0, 2\pi]$, has arc length function
\[
s(t) = \int_0^t \sqrt{(-\sin u)^2 + (\cos u)^2 + 1^2} \, du = \int_0^t \sqrt{2} \, du = \sqrt{2} t
\]
so $t(s) = s/\sqrt{2}$; and
\[
\gamma(t) = \gamma(t(s)) = \gamma\left(\frac{s}{\sqrt{2}}\right) = (\cos \frac{s}{\sqrt{2}}, \sin \frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}), \quad s \in [0, 2\sqrt{2}\pi]
\]

Use: After traveling along the curve a distance of $s = \sqrt{2}\pi$, we're at the point $\left(\cos \frac{\sqrt{2}\pi}{\sqrt{2}}, \sin \frac{\sqrt{2}\pi}{\sqrt{2}}, \frac{\sqrt{2}\pi}{\sqrt{2}}\right) \approx \left( -1, 0, \pi \right)$. 
Sec 3.4 Acceleration and curvature

Example For the line $\mathbf{z}(t) = (1, -3) + t^2 (1, 2) = (1 + 2t^2, -3 + 2t^2)$,
$\mathbf{z}'(t) = (3t^2, 6t)$ is not constant (but its direction is)
$\mathbf{z}''(t) = \mathbf{a}(t) = (6t, 12t) = 6t (1, 2)$ is a nonzero vector parallel to the motion given by $\mathbf{z}'(t)$.

Example For the circle $\mathbf{z}(t) = (r \cos t, r \sin t)$
$\mathbf{z}'(t) = v(t) = (-r \sin t, r \cos t)$, so
the speed $||v(t)|| = r$ is constant
$\mathbf{z}''(t) = \mathbf{a}(t) = (-rcos t, -r \sin t)$ is a nonzero vector perpendicular to the motion.

These are the extreme cases. In general, $\mathbf{a}(t)$ has a component parallel to the motion, $\mathbf{a}_T$, and a component perpendicular to the motion, $\mathbf{a}_N$, so that $\mathbf{a} = \mathbf{a}_T + \mathbf{a}_N$

The tangential component $\mathbf{a}_T$ corresponds to changes in speed.
The normal component $\mathbf{a}_N$ corresponds to changes in direction (as in the examples).

Curvature: $\kappa_N$ is related to motion around a circle.

Goal: Find a circle that best approximates the motion near some point.
The unit tangent vector is \( \mathbf{T}(t) = \frac{\mathbf{c}'(t)}{||\mathbf{c}'(t)||} \).

The curvature of a smooth curve \( \mathbf{c} \) is

\[
K = \| \frac{d\mathbf{T}}{ds} \| , \text{ where } \mathbf{T} \text{ is the unit tangent vector, and } s \text{ is the arc length.}
\]

In words, the magnitude of the rate of change of the unit tangent vector with respect to arc length.

A more practical formula is

\[
K(t) = \frac{\| \mathbf{c}'(t) \times \mathbf{c}''(t) \|}{\| \mathbf{c}'(t) \|^3} = \frac{\| \mathbf{T}'(t) \|}{\| \mathbf{c}'(t) \|^3} .
\]

Example \( \mathbf{c}(t) = (r \cos t, r \sin t, 0) \),

\[
\mathbf{c}'(t) = (-r \sin t, r \cos t, 0),
\]

\[
\mathbf{c}''(t) = (-r \cos t, -r \sin t, 0).
\]

\[
\mathbf{c}' \times \mathbf{c}'' = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-r \sin t & r \cos t & 0 \\
-r \cos t & -r \sin t & 0
\end{vmatrix} = (0, 0, r^2),
\]

so \( \| \mathbf{c}' \times \mathbf{c}'' \| = r^2 \).

\[
\| \mathbf{c}'(t) \| = r \Rightarrow K(t) = \frac{\| \mathbf{c}' \times \mathbf{c}'' \|}{\| \mathbf{c}'(t) \|^3} = \frac{r^2}{r^3} = \frac{1}{r}
\]

\[ K = \frac{1}{r} \text{ big } \quad K = \frac{1}{R} \text{ small } \]

See 3.5 Intro to Differential Geometry.

The unit normal vector is \( \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\| \mathbf{T}'(t) \|} \).

Claim: \( \mathbf{N}(t) \) is orthogonal to \( \mathbf{T}(t) \), and hence the curve.

Proof: \( \mathbf{T}'(t) = \| \mathbf{T}(t) \|^2 = 1 \rightarrow 2 \mathbf{T}(t) \cdot \mathbf{T}'(t) = 0 \) by the product rule.

So \( \mathbf{T}'(t) \) is orthogonal to \( \mathbf{T}(t) \), and so \( \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\| \mathbf{T}'(t) \|} \) is as well.
Definition: The binormal vector \( \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) \) is orthogonal to both \( \mathbf{T} \) and \( \mathbf{N} \) and has unit length.

- \( \mathbf{T}, \mathbf{N}, \) and \( \mathbf{B} \) form a coordinate system called the Frenet frame at \( \mathbf{c}(t) \).

Differential geometry application: Analyzing derivatives of \( \mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t) \), which depend on curvature. Motion along a curve studied in terms of how \( \mathbf{T}, \mathbf{N}, \mathbf{B} \) change. Application: roller coasters.