Problem

Consider the equation $Ax^3 + (2 - A)x^2 - x - 1 = 0$, where $A$ is a real number for which the equation has three real roots, not necessarily distinct. For certain values of $A$, there is a repeated root $r$ and a distinct root $s$. List all such values of the triple $(A, r, s)$.

Solution Notice that $x = 1$ is a solution for every possible $A$. Factoring yields $(x - 1)(Ax^2 + 2x + 1) = 0$. Using the quadratic formula, we find that the other two solutions are

$$x = \frac{-2 \pm \sqrt{4 - 4A}}{2A} = \frac{-1 \pm \sqrt{1 - A}}{A}.$$

Note that for $A > 1$, the discriminant is negative, and the roots are not real. If $A = 1$, this formula gives $-1$ as a double root.

We can also have $1$ as a double root if

$$\frac{-1 \pm \sqrt{1 - A}}{A} = 1$$

$$\pm \sqrt{1 - A} = A + 1,$$

$$1 - A = A^2 + 2A + 1,$$

$$0 = A^2 + 3A = A(A + 3).$$

So either $A = 0$ (impossible) or $A = -3$, in which case the other root is $-1/3$. So the possible values of the triple are $(1, -1, 1)$ and $(-3, 1, -1/3)$. 