

# Combinatorial Models of Quantum Algebras

Advisor: Karel Casteels

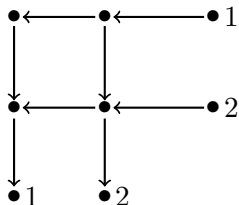
Very roughly speaking, a *quantization*  $A_q$  of a commutative  $\mathbb{K}$ -algebra  $A$  is a noncommutative algebra with relations depending on a parameter  $q \in \mathbb{K}^*$  such that when  $q = 1$ , we recover  $A$ .

One example of this is the quantization  $\mathcal{O}_q(M_{2,2}(\mathbb{K}))$  of the coordinate ring of  $2 \times 2$  matrices (AKA  $2 \times 2$  *quantum matrices*). This is the algebra generated by the entries of the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

subject to the relations:  $ab = qba$ ,  $cd = qdc$ ,  $ac = qca$ ,  $bd = qdb$ ,  $bc = cb$  and  $ad = da + (q - q^{-1})bc$ . We may similarly define the quantization of the coordinate ring of  $m \times n$  matrices. Using these quantizations we can form other “quantum algebras” such as the quantum general and special linear groups and the quantum Grassmannian. Moreover, these algebras have applications in a variety of areas, most recently in the fields of total non-negativity and KP theory.

Recently I have developed a combinatorial approach to quantum matrices by using paths in a certain directed graph. For example, in the  $2 \times 2$  quantum matrices case we consider the following directed graph:



We then look at directed paths from the vertices on the right (“row” vertices) to the vertices on the bottom (“column” vertices). To each such path we assign an element in a *quantum torus*, another quantum algebra

but one that is simpler than quantum matrices. For a fixed row and column vertex we sum up these elements over all paths from the row vertex to the column vertex. It turns out that these “sums over paths” generate a subalgebra of the quantum torus that is isomorphic to quantum matrices. Several aspects of quantum matrix theory turn out to have very nice combinatorial interpretations here. For example, there is a notion of quantum determinant/quantum minors that corresponds to sums over disjoint sets of paths in the obvious generalization of the above graph.

We will explore the above model’s applications, as well as try to find similar such models for other quantum algebras. Most essential for this project will be a basic knowledge of algebra/ring theory and a comfort with working with polynomials. Knowledge of Gröbner basis theory will be helpful as well.