Numerical properties of linearizations in the family $\mathbb{SLD}(P)$.
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Before I explain the background of the problems we will work on, let me mention some important information:

- The only math background that you need to work on any of the two projects that I am proposing is upper-division Linear Algebra.
- Being capable of writing good proofs is a must.
- Having programming skills in Matlab is desirable. If you don't know how to program in Matlab, no worries. We will teach you here the basics.

When you read the description of the project below, you may feel overwhelmed by lots of technical words that you have not heard before. Do not worry. We will spend some time learning the background in depth before we start working on the projects. So, if you like Linear Algebra, this is your project!

Also note that my goal is to publish papers with you, in refereed journals if possible. You can check my webpage

`math.ucsb.edu/~mbueno/publications.html`

to see the papers that I have published or submitted with undergraduate students.

Description of the project.

Let

$$P(\lambda) = A_k\lambda^k + A_{k-1}\lambda^{k-1} + \cdots + A_0 \qquad (1)$$

be a matrix polynomial of degree $k \geq 2$, where the coefficients A_i are $n \times n$ matrices with entries in a field $\mathbb{F}$. In particular, if $k = 1$, $P(\lambda)$ is called a pencil.

A matrix pencil $L_P(\lambda) = \lambda L_1 - L_0$, with $L_1, L_0 \in M_{kn}(\mathbb{F})$, is a *linearization* of $P(\lambda)$ if there exist two unimodular matrix polynomials (i.e. matrix polynomials with constant nonzero determinant), $U(\lambda)$ and $V(\lambda)$, such that

$$U(\lambda)L(\lambda)V(\lambda) = \begin{bmatrix} I_{(k-1)n} & 0 \\ 0 & P(\lambda) \end{bmatrix}.$$

Here $I_{(k-1)n}$ denotes the $(k-1)n \times (k-1)n$ identity matrix.

Linearizations of a matrix polynomial $P(\lambda)$ share the finite eigenvalues of $P(\lambda)$, among other important properties. Beside other applications, linearizations of matrix polynomials are used in the study of the polynomial eigenvalue problem, that is, in the search of scalars λ and nonzero vectors x such that $P(\lambda)x = 0$. In the classical approach to this problem the original matrix polynomial $P(\lambda)$ is replaced by a linearization of $P(\lambda)$. Then, standard methods for linear eigenvalue problems are applied.

The *reversal* of the matrix polynomial $P(\lambda)$ in (1) is the matrix polynomial obtained by reversing the order of the matrix coefficients of $P(\lambda)$, that is,

$$rev(P(\lambda)) := \sum_{i=0}^{k} \lambda^i A_{k-i}.$$

A linearization $L_P(\lambda)$ of a matrix polynomial $P(\lambda)$ is called a *strong linearization* of $P(\lambda)$ if $rev(L_P(\lambda))$ is a linearization of $rev(P(\lambda))$. Strong linearizations of $P(\lambda)$ have the same finite and infinite elementary divisors as $P(\lambda)$, therefore they preserve not only the finite but the infinite eigenvalues. Moreover, if $P(\lambda)$ is regular (i.e. $det(P(\lambda)) \not\equiv 0$), any linearization with the same infinite elementary divisors as $P(\lambda)$ is a strong linearization.

In applications, many matrix polynomials have some structure: either all the matrix coefficients are symmetric, or Hermitian, or skew-symmetric, or the coefficients alternate between symmetric and skew-symmetric, etc. For each structured matrix polynomial $P(\lambda)$, many different linearizations can be constructed but, in practice, those sharing the structure of $P(\lambda)$ are the most convenient from the theoretical and computational point of view, since the structure of $P(\lambda)$ often implies some symmetries in its spectrum, which are meaningful in physical applications and that can be destroyed by numerical methods when the structure is ignored. For example, if a matrix polynomial is real symmetric or Hermitian, its spectrum is symmetric with respect to the real axis, and the sets of left and right eigenvectors coincide. Thus, it is important to construct linearizations that reflect the structure of the original problem. In the literature different kinds of structured linearizations have been considered: palindromic, symmetric, skew-symmetric, alternating, etc. Among the structured linearizations, those that are strong and can be easily constructed from the coefficients of the matrix polynomial $P(\lambda)$ are of particular interest.

In summer 2017 we will work on two projects. In the first project, we will focus on a family $\mathbb{SLD}(P)$ of block-symmetric pencils most of which are strong linearizations of a matrix polynomial $P(\lambda)$. We will study numerical properties of these pencils such as the conditioning of their eigenvalues and the backward error of approximate eigenpairs. In particular we will be interested in those pencils that are linearizations of matrix polynomials of even degree. Additionally, if $P(\lambda)$ is Hermitian, we will study the sign characteristic of the linearizations of $P(\lambda)$ in $\mathbb{SLD}(P)$.

In the second project, we will try to generalize some recent results about companion matrices by constructing linearizations of scalar polynomials with smallest bandwith.