CONTINUOUS SYMMETRIZED SOBOLEV INNER PRODUCTS OF ORDER N (II)

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Abstract. Given a symmetrized Sobolev inner product of order N, the corresponding sequence of monic orthogonal polynomials $\{Q_n\}$ satisfies $Q_{2n}(x) = P_n(x^2)$, $Q_{2n+1}(x) = xR_n(x^2)$ for certain sequences of monic polynomials $\{P_n\}$ and $\{R_n\}$. In this paper we consider the particular case when all the measures that define the symmetrized Sobolev inner product are equal, absolutely continuous and semiclassical. Under such restrictions, we give explicit algebraic relations between the sequences $\{P_n\}$ and $\{R_n\}$, as well as higher-order recurrence relations that they satisfy.

Key words. Sobolev inner product, orthogonal polynomials, semiclassical linear functionals, recurrence relation, symmetrization process

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1. Introduction. Let us consider the following inner product defined in the linear space $\mathbb{P}\times\mathbb{P}$, where $\mathbb{P}$ denotes the linear space of polynomials with real coefficients,

$$\langle p, q\rangle_s = \int_{\mathbb{R}} p\, q\, d\mu_0 + \sum_{i=1}^{N} \lambda_i \int_{\mathbb{R}} p^{(i)}\, q^{(i)}\, d\mu_i\ . \tag{1.1}$$

In the previous expression $\mu_0, \mu_1, ..., \mu_N$ denote positive and absolutely continuous Borel measures supported on a subset of the real line and such that the corresponding sequences of moments are finite, $p^{(i)}$ denotes the ith derivative of p, and λ_i are nonnegative real numbers ($\lambda_N \neq 0$). The inner product given in (1.1) is known as *Sobolev inner product of order* N [7]. Sobolev inner products and their corresponding sequences of orthogonal polynomials have been exhaustively studied during the last ten years, although most of the results have been obtained for $N = 1$.

The product $\langle\cdot,\cdot\rangle_s$ is said to be *symmetrized* if $\langle x^n, x^m\rangle_s = 0$ when $n + m$ is an odd nonnegative integer. In this case, μ_0, μ_1,..., μ_N are supported on a subset of the real line which is symmetric with respect to the origin and the measures themselves are also symmetric, so that

$$c^{(i)}_{2n+1} = \int_{\mathbb{R}} x^{2n+1} d\mu_i = 0\,, \quad i = 0, 1, ..., N\,, \quad n \in \mathbb{N}\,.$$

This concept extends the definition of *symmetric linear functional* [5] to the bilinear case.

Assume that $\langle\cdot,\cdot\rangle_s$ is quasi-definite, that is, there exists a sequence $\{Q_n\}$ of polynomials orthogonal with respect to $\langle\cdot,\cdot\rangle_s$. If $\langle\cdot,\cdot\rangle_s$ is symmetrized, then there exist another two sequences of polynomials $\{P_n\}$ and $\{R_n\}$ such that

$$Q_{2n}(x) = P_n(x^2), \quad Q_{2n+1}(x) = xR_n(x^2). \tag{1.2}$$

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In the sequel, we will refer to these two sequences as the *symmetric components of* $\{Q_n\}$.

In this paper we consider the bilinear symmetrization problem associated with (1.1), i.e. the analog in the bilinear case of Chihara's linear symmetrization problem [5], which consists in:

(1) finding the explicit expressions of the bilinear functionals such that $\{P_n\}$ and $\{R_n\}$ are the corresponding sequences of orthogonal polynomials,

(2) determining recurrence relations with a finite number of terms that $\{P_n\}$ and $\{R_n\}$ satisfy, and

(3) obtaining explicit algebraic relations between both sequences.

The problem (1) was solved for Sobolev inner products of order 1 in [3], and for general N in [4]. As regards problems (1) and (1), up to now they have only been solved in the particular case when $N = 1$ and the two measures that define the product $\langle \cdot, \cdot \rangle_s$ are equal, absolutely continuous and semiclassical [3]. In this paper we extend these results for an arbitrary $N \geq 1$, under the same restrictions on the measures involved.

The structure of the paper is the following: First, in Section 2, we present some auxiliary results related with semiclassical functionals and semiclassical measures. In Section 3, we find explicit algebraic relations between the sequences $\{P_n\}$ and $\{R_n\}$. In Section 4, we determine recurrence relations with a finite number of terms that the sequences $\{P_n\}$, $\{R_n\}$ and $\{Q_n\}$ satisfy. Finally, in Section 5, we apply our general results to a particular case of the so-called Freud-Sobolev polynomials [2].

2. Auxiliary results. Consider a quasi-definite linear functional $\mathbf{U}$ in $\mathbb{P}$, with integral representation

$$\mathbf{U}(p) = \int_{\mathbb{R}} p(x) d\mu = \int_{\mathbb{R}} p(x)\omega(x) dx \,, \tag{2.1}$$

where μ is an absolutely continuous positive Borel measure and ω is the corresponding weight function. The functional $\mathbf{U}$ is said to be a *semiclassical linear functional* if

$$D(\phi \mathbf{U}) = \psi \mathbf{U} \,, \tag{2.2}$$

where ϕ and ψ are polynomials with $\deg(\phi) = r \geq 0$ and $\deg(\psi) \geq 1$, and D denotes the derivative operator. The condition of being semiclassical can also be characterized in terms of the weight function ω:

PROPOSITION 2.1. *[6] Let* $\mathbf{U}$ *be a semiclassical linear functional with integral representation (2.1), where* ω *is a continuously differentiable function in an interval* $[a, b]$ *satisfying* $\lim_{x \to a,b} \phi(x)p(x)\omega(x) = 0$ *with* $p \in \mathbb{P}$. *Then,*

$$(\phi\omega)' = \psi\omega \,, \tag{2.3}$$

and ω *is said to be a* semiclassical weight function. *Equation (2.3) is the so-called Pearson equation.*

DEFINITION 2.2. *[8] Given a semiclassical linear functional* $\mathbf{U}$, *let* $\Pi_{\mathbf{U}}$ *be the set of all the pairs of polynomials satisfying (2.2). Then, the class* $\tilde{s}$ *of* $\mathbf{U}$ *is defined as*

$$\tilde{s} = \min_{(\phi,\psi) \in \Pi_{\mathbf{U}}} \left\{ \max\left\{ \deg(\phi) - 2 \,, \deg(\psi) - 1 \right\} \right\} . \tag{2.4}$$

LEMMA 2.3. *[7] If ω is a semiclassical weight function, then for every nonnegative integer N,*

$$\phi^N(x)D^N\big(\omega(x)\big) = \psi(x,N)\omega(x)\,, \tag{2.5}$$

where

$$\begin{aligned}&\psi(x,0) = 1\,,\\ &\psi(x,N) = \phi(x)\psi'(x,N-1) + \psi(x,N-1)[\psi(x) - N\phi'(x)]\,, \quad N \ge 1\,.\end{aligned} \tag{2.6}$$

LEMMA 2.4. *[7] Given a semiclassical weight function ω, the polynomials $\psi(x,N)$ defined in the previous lemma satisfy*

$$\deg\big(\psi(x,N)\big) \le N(\tilde{s}+1)\,, \quad N \ge 0\,, \tag{2.7}$$

where $\tilde{s}$ is the class of the semiclassical linear functional defined by ω.

Consider the following linear differential operator in the linear space $\mathbb{P}$:

$$F^{(N)} := \sum_{m=0}^{N} (-1)^m \lambda_m \sum_{i=0}^{m} \binom{m}{i} \phi^{N-m+i}(x)\psi(x,m-i)D^{m+i}\,, \tag{2.8}$$

where $\lambda_0 = 1$ and $D^0 = I$, the identity operator.

PROPOSITION 2.5. *Let $\mathbf{U}$ be a semiclassical linear functional with integral representation (2.1). Using the notations introduced above, for all nonnegative integers N and n,*

(1) If $\tilde{s} - r > 0$, then $\deg\big(F^{(N)}(x^n)\big) \le n + Nr + (\tilde{s}-r)\min\{N,n\}$.

(2) If $\tilde{s} - r \le 0$, then $\deg\big(F^{(N)}(x^n)\big) = n + Nr$.

Proof. From (2.8),

$$F^{(N)}(x^n) = \sum_{m=0}^{N} (-1)^m \lambda_m \sum_{i=0}^{m} \binom{m}{i} \phi^{N-m+i}(x)\psi(x,m-i)D^{m+i}(x^n)\,.$$

Notice that $D^{m+i}(x^n) = 0$ when $m+i > n$. It follows that the upper bounds on the summations over m and i can be replaced, respectively, by $\hat{N} = \min\{N,n\}$ and $\hat{m} = \min\{m, n-m\}$, with the condition that empty sum equals zero if $n < m$. Using Lemma 2.4,

$$\begin{aligned}\deg\big(F^{(N)}(x^n)\big) &\le \max_{\substack{i=0:\hat{m}\\ m=0:\hat{N}}} \{r(N-m+i) + (m-i)(\tilde{s}+1) + n - m - i\}\\ &= \max_{\substack{i=0:\hat{m}\\ m=0:\hat{N}}} \{n + Nr + m(\tilde{s}-r) - i(\tilde{s}+2-r)\}\,.\end{aligned}$$

From (2.4), we know that $\tilde{s}+2-r \ge 0$. Therefore, the maximum is attained for $i=0$,

$$\deg\big(F^{(N)}(x^n)\big) \le \max_{m=0:\hat{N}} \{n + Nr + m(\tilde{s}-r)\}\,.$$

If $\tilde{s} - r > 0$, then the maximum is attained for $m = \hat{N}$, and the result in (1) follows. On the other hand, if $\tilde{s} - r \le 0$, then the maximum is attained for $m=0$; since the term $m = i = 0$ in $F^{(N)}(x^n)$ has exact degree $n + Nr$, (2) holds. ∎

COROLLARY 2.6. *Let $s := \max\{r, \tilde{s}\}$. Then, for all nonnegative integers N, n,*

$$\deg\big(F^{(N)}(x^n)\big) \le n + Ns\,. \tag{2.9}$$

3. Explicit algebraic relations between $\{P_n\}$ and $\{R_n\}$. Let us consider a symmetrized quasi-definite Sobolev inner product given by (1.1) with $\mu_i = \mu$ for all i ($0 \le i \le N$), i.e.,

$$\langle p, q\rangle_s = \int_{\mathbb{R}} p\, q\, d\mu + \sum_{i=1}^{N} \lambda_i \int_{\mathbb{R}} p^{(i)} q^{(i)} d\mu \,. \tag{3.1}$$

We denote by $\{Q_n\}$ the sequence of monic polynomials orthogonal with respect to (3.1), and by $\{P_n\}$ and $\{R_n\}$ the corresponding symmetric components defined by (1.2). The weight function ω satisfying $d\mu = \omega(x)dx$ and the corresponding linear functional $\mathbf{U}$ given by (2.1) are both symmetric. Furthermore, the sequence $\{T_n\}$ of monic polynomials orthogonal with respect to $\mathbf{U}$ satisfies a three-term recurrence relation,

$$xT_n(x) = T_{n+1}(x) + c_n T_{n-1}(x)\,, \quad n \ge 1\,, \tag{3.2}$$

where $c_n \neq 0$, and there exists a sequence of monic polynomials $\{S_n\}$ such that

$$T_{2n}(x) = S_n(x^2)\,, \quad T_{2n+1}(x) = xS_n^*(x^2)\,, \tag{3.3}$$

where $\{S_n^*\}$ denotes the sequence of monic kernel polynomials associated with $\{S_n\}$ [5].

In the sequel, we assume that $\mathbf{U}$ is a semiclassical linear functional, and we use the notations introduced in the previous section. The following proposition states an algebraic relation between the sequences $\{Q_n\}$ and $\{T_n\}$ that will be useful to determine algebraic relations between $\{P_n\}$ and $\{R_n\}$.

LEMMA 3.1. *[7] Let p and q be arbitrary polynomials. Then,*

$$\langle \phi^N p, q\rangle_s = \mathbf{U}\big(pF^{(N)}(q)\big).$$

PROPOSITION 3.2. *For every nonnegative integer $n \ge Ns$, there exist real numbers $\alpha_{n,j}$ such that*

$$\phi^N(x)T_n(x) = \sum_{j=n-Ns}^{n+Nr} \alpha_{n,j}\, Q_j(x)\,. \tag{3.4}$$

Proof. Expanding the polynomial $\phi^N(x)T_n(x)$ in terms of the Sobolev polynomials we get

$$\phi^N(x)T_n(x) = \sum_{j=0}^{n+Nr} \alpha_{n,j}\, Q_j(x)\,.$$

Then, we use Lemma 3.1 to compute the coefficients $\alpha_{n,j}$,

$$\alpha_{n,j} = \frac{\langle \phi^N(x)T_n(x), Q_j(x)\rangle_s}{\langle Q_j(x), Q_j(x)\rangle_s} = \frac{\mathbf{U}\left(T_n(x)F^{(N)}\big(Q_j(x)\big)\right)}{\langle Q_j(x), Q_j(x)\rangle_s}.$$

Since $\{T_n\}$ is the sequence of polynomials orthogonal with respect to $\mathbf{U}$, Corollary 2.6 implies that $\alpha_{n,j} = 0$ if $0 \le j < n - Ns$. ∎

REMARK 3.3. *If in the above proof Proposition 2.5 is used instead of Corollary 2.6, a sharper lower bound can be obtained for the summation in (3.4). This means that the first terms of the sum in (3.4) may be zero. However, since the use of Proposition 2.5 would require two parallel developments in what follows, for the sake of brevity we use the general bound given by Corollary 2.6.*

3.1. Semiclassical functional U of even class.

Proposition 3.4. *If the class $\tilde{s}$ of the functional* $\mathbf{U}$ *is even, then the following explicit algebraic relation between the sequences $\{P_n\}$ and $\{R_n\}$ is obtained,*

$$\begin{aligned}\sum_{j=m-N\hat{s}}^{m+N\hat{r}} \alpha_{2m,2j} P_j(x) &= \alpha_{2m+1,2m+Nr+1} R_{m+N\hat{r}}(x) \\ &+ \sum_{j=m-N\hat{s}}^{m+N\hat{r}-1} (\alpha_{2m+1,2j+1} + c_{2m}\alpha_{2m-1,2j+1}) R_j(x) \\ &+ c_{2m}\alpha_{2m-1,2m-Ns-1} R_{m-N\hat{s}-1}(x),\end{aligned} \tag{3.5}$$

where $\hat{r} = r/2$ and $\hat{s} = s/2$.

Proof. Since $\tilde{s}$ is an even number, ϕ is an even polynomial [3, Prop. 2.6]. That is, there exists another polynomial $\tilde{\phi}$ such that

$$\phi(x) = \tilde{\phi}(x^2). \tag{3.6}$$

Therefore, r and s are also even numbers. We write $r = 2\hat{r}$ and $s = 2\hat{s}$.

For $n = 2m$, Proposition 3.2 reads

$$\phi^N(x) T_{2m}(x) = \sum_{j=2m-Ns}^{2m+Nr} \alpha_{2m,j} Q_j(x). \tag{3.7}$$

Since the term in the left-hand side of the previous identity is an even polynomial, we get

$$\phi^N(x) T_{2m}(x) = \sum_{j=m-N\hat{s}}^{m+N\hat{r}} \alpha_{2m,2j} Q_{2j}(x).$$

Taking into account (1.2), (3.3) and (3.6), the previous equation can be rewritten as

$$\tilde{\phi}^N(x) S_m(x) = \sum_{j=m-N\hat{s}}^{m+N\hat{r}} \alpha_{2m,2j} P_j(x). \tag{3.8}$$

In a similar way, for $n = 2m+1$ we obtain

$$\tilde{\phi}^N(x) S^*_m(x) = \sum_{j=m-N\hat{s}}^{m+N\hat{r}} \alpha_{2m+1,2j+1} R_j(x). \tag{3.9}$$

Taking into account (3.3), and (3.2) with n replaced by $2m$, we get

$$S_m(x) = S^*_m(x) + c_{2m} S^*_{m-1}(x). \tag{3.10}$$

Multiplying (3.10) by $\tilde{\phi}^N$ and using (3.8) and (3.9), we obtain

$$\begin{aligned}&\sum_{j=m-N\hat{s}}^{m+N\hat{r}} \alpha_{2m,2j} P_j(x) \\ &= \sum_{j=m-N\hat{s}}^{m+N\hat{r}} \alpha_{2m+1,2j+1} R_j(x) \ + \ c_{2m} \sum_{j=m-N\hat{s}-1}^{m+N\hat{r}-1} \alpha_{2m-1,2j+1} R_j(x).\end{aligned} \tag{3.11}$$

The result in (3.5) can be obtained in a straightforward way from the previous expression. ∎

3.2. Semiclassical functional U of odd class.

PROPOSITION 3.5. *Let the class $\tilde{s}$ of the functional* **U** *be an odd number. The following explicit algebraic relations between the sequences $\{P_n\}$ and $\{R_n\}$ are obtained:*

(1) If N is an even number, and $T = N/2$, then

$$\begin{aligned}\sum_{j=m-Ts}^{m+Tr} \alpha_{2m,2j} P_j(x) &= \alpha_{2m+1,2m+Nr+1} R_{m+Tr}(x) \\ &+ \sum_{j=m-Ts}^{m+Tr-1} (\alpha_{2m+1,2j+1} + c_{2m}\alpha_{2m-1,2j+1}) R_j(x) \\ &+ c_{2m}\alpha_{2m-1,2m-Ns-1} R_{m-Ts-1}(x).\end{aligned}$$

(2) If N is an odd number, and $T = (N-1)/2$, then

$$\begin{aligned}\sum_{j=m-Ts-\hat{s}}^{m+Tr+\hat{r}+1} \alpha_{2m+1,2j} P_j(x) &= \alpha_{2m+2,2m+Nr+2} R_{m+Tr+\hat{r}+1}(x) \\ &+ \sum_{j=m-Ts-\hat{s}}^{m+Tr+\hat{r}} (\alpha_{2m+2,2j+1} + c_{2m+1}\alpha_{2m,2j+1}) R_j(x) \\ &+ c_{2m+1}\alpha_{2m,2m-Ns} R_{m-Ts-\hat{s}-1}(x),\end{aligned}$$

where $\hat{r} = (r-1)/2$ and $\hat{s} = (s-1)/2$.

Proof. Since $\tilde{s}$ is an odd number, ϕ is an odd polynomial [3, Prop. 2.6]. Therefore, there exists another polynomial $\hat{\phi}$ such that

$$\phi(x) = x\hat{\phi}(x^2). \tag{3.12}$$

Since r and $\tilde{s}$ are odd numbers, so it is s. In the sequel, we write $r = 2\hat{r}+1$ and $s = 2\hat{s}+1$.

(1) Assume that N is even. Then, we write $N = 2T$, for some nonnegative integer T. From (3.4), we get

$$\phi^N(x) T_{2m}(x) = \sum_{j=m-Ts}^{m+Tr} \alpha_{2m,2j} Q_{2j}(x).$$

Taking into account (1.2), (3.12) and (3.3), we get

$$x^T \hat{\phi}^N(x) S_m(x) = \sum_{j=m-Ts}^{m+Tr} \alpha_{2m,2j} P_j(x). \tag{3.13}$$

Consider Proposition 3.2 with $n = 2m+1$ to obtain, in a similar way,

$$x^T \hat{\phi}^N(x) S_m^*(x) = \sum_{j=m-Ts}^{m+Tr} \alpha_{2m+1,2j+1} R_j(x). \tag{3.14}$$

By multiplying both sides of (3.10) by $x^T\hat{\phi}^N(x)$, (3.13) and (3.14) lead us to the following explicit algebraic relation between $\{P_n\}$ and $\{R_n\}$,

$$\begin{aligned}&\sum_{j=m-Ts}^{m+Tr} \alpha_{2m,2j} P_j(x) \\ &= \sum_{j=m-Ts}^{m+Tr} \alpha_{2m+1,2j+1} R_j(x) + c_{2m} \sum_{j=m-Ts-1}^{m+Tr-1} \alpha_{2m-1,2j+1} R_j(x),\end{aligned} \tag{3.15}$$

and the result follows in a straightforward way.

(2) Assume that N is odd. Then $N = 2T + 1$ for some nonnegative integer T. In this case, the term on the left-hand side of (3.7) is an odd polynomial, and taking into account (1.2), (3.12) and (3.3), we get

$$x^T \hat{\phi}^N(x) S_m(x) = \sum_{j=m-Ts-\hat{s}-1}^{m+Tr+\hat{r}} \alpha_{2m,2j+1} R_j(x). \tag{3.16}$$

A similar procedure gives, using Proposition 3.2 with $n = 2m+1$,

$$x^{T+1} \hat{\phi}^N(x) S_m^*(x) = \sum_{j=m-Ts-\hat{s}}^{m+Tr+\hat{r}+1} \alpha_{2m+1,2j} P_j(x). \tag{3.17}$$

From (3.2) with $n = 2m+1$, and (3.3),

$$x S_m^*(x) = S_{m+1}(x) + c_{2m+1} S_m(x). \tag{3.18}$$

Replacing (3.16) and (3.17) into (3.18), the following relation is obtained,

$$\begin{aligned}&\sum_{j=m-Ts-\hat{s}}^{m+Tr+\hat{r}+1} \alpha_{2m+1,2j} P_j(x) \\ &= \sum_{j=m-Ts-\hat{s}}^{m+Tr+\hat{r}+1} \alpha_{2m+2,2j+1} R_j(x) + c_{2m+1} \sum_{j=m-Ts-\hat{s}-1}^{m+Tr+\hat{r}} \alpha_{2m,2j+1} R_j(x),\end{aligned} \tag{3.19}$$

and the result follows straightforwardly. $\square$

4. Recurrence relations. In this section we deduce recurrence relations with a finite number of terms for the sequences $\{P_n\}$, $\{R_n\}$ and $\{Q_n\}$. The number of terms in these relations depends on the class of the functional $\mathbf{U}$ and the degree of the polynomial ϕ.

4.1. Recurrence relations for $\{P_n\}$.

PROPOSITION 4.1. *If the class $\tilde{s}$ of $\mathbf{U}$ is an even number, the sequence $\{P_n\}$ satisfies the following $[N(\hat{s}+\hat{r})+3]$-term recurrence relation,*

$$\begin{aligned}&\alpha_{2m+2,2m+Nr+2} P_{m+N\hat{r}+1}(x) \\ &= (x\alpha_{2m,2m+Nr} - \alpha_{2m+2,2m+Nr} - c_{2m+1}\alpha_{2m,2m+Nr} - c_{2m}\alpha_{2m,2m+Nr}) P_{m+N\hat{r}}(x) \\ &+ \sum_{j=m-N\hat{s}+1}^{m+N\hat{r}-1} [x\alpha_{2m,2j} - \alpha_{2m+2,2j} - c_{2m+1}\alpha_{2m,2j} - c_{2m}(\alpha_{2m,2j} + c_{2m-1}\alpha_{2m-2,2j})] P_j(x) \\ &+ [x\alpha_{2m,2m-Ns} - c_{2m+1}\alpha_{2m,2m-Ns} - c_{2m}(\alpha_{2m,2m-Ns} + c_{2m-1}\alpha_{2m-2,2m-Ns})] P_{m-N\hat{s}}(x) \\ &- c_{2m}c_{2m-1}\alpha_{2m-2,2m-Ns-2} P_{m-N\hat{s}-1}(x),\end{aligned}$$

where $\hat{r} = r/2$ *and* $\hat{s} = s/2$.

Proof. Assume that $\tilde{s}$ is an even number. Multiplying both sides of (3.18) by $\tilde{\phi}^N(x)$, and plugging (3.8) and (3.9) into it, we get

$$\begin{aligned} & x\left[\sum_{j=m-N\hat{s}}^{m+N\hat{r}} \alpha_{2m+1,2j+1} R_j(x)\right] \\ & = \sum_{j=m-N\hat{s}+1}^{m+N\hat{r}+1} \alpha_{2m+2,2j} P_j(x) \;+\; c_{2m+1} \sum_{j=m-N\hat{s}}^{m+N\hat{r}} \alpha_{2m,2j} P_j(x). \end{aligned} \tag{4.1}$$

Now, we multiply both sides of (3.11) by x and replace (4.1) in it to obtain

$$\begin{aligned} & x\left[\sum_{j=m-N\hat{s}}^{m+N\hat{r}} \alpha_{2m,2j} P_j(x)\right] \\ & = \sum_{j=m-N\hat{s}+1}^{m+N\hat{r}+1} \alpha_{2m+2,2j} P_j(x) \;+\; c_{2m+1} \sum_{j=m-N\hat{s}}^{m+N\hat{r}} \alpha_{2m,2j} P_j(x) \\ & + \; c_{2m}\left[\sum_{j=m-N\hat{s}}^{m+N\hat{r}} \alpha_{2m,2j} P_j(x) + c_{2m-1} \sum_{j=m-N\hat{s}-1}^{m+N\hat{r}-1} \alpha_{2m-2,2j} P_j(x)\right]. \end{aligned}$$

From the previous expression we get the $(N\hat{s} + N\hat{r} + 3)$-term recurrence relation for $\{P_n\}$ given in the statement of the proposition. ∎

Proposition 4.2. *Let the class* $\tilde{s}$ *of* $\mathbf{U}$ *be an odd number, and put* $\hat{r} = (r-1)/2$ *and* $\hat{s} = (s-1)/2$. *Then the sequence* $\{P_n\}$ *satisfies the following* $[N(\hat{s}+\hat{r}+1)+3]$*-term recurrence relations:*

(1) If N *is an even number, and* $T = N/2$, *then*

$$\begin{aligned} & \alpha_{2m+2,2m+Nr+2} P_{m+Tr+1}(x) \\ & = [\alpha_{2m,2m+Nr}(x - c_{2m+1} - c_{2m}) - \alpha_{2m+2,2m+Nr}] P_{m+Tr}(x) \\ & + \sum_{j=m-Ts+1}^{m+Tr-1} [(x - c_{2m+1} - c_{2m})\alpha_{2m,2j} - \alpha_{2m+2,2j} - c_{2m}c_{2m-1}\alpha_{2m-2,2j}] P_j(x) \\ & + [(x - c_{2m+1} - c_{2m})\alpha_{2m,2m-Ns} - c_{2m}c_{2m-1}\alpha_{2m-2,2m-Ns}] P_{m-Ts}(x) \\ & - c_{2m}c_{2m-1}\alpha_{2m-2,2m-Ns-2} P_{m-Ts-1}(x). \end{aligned}$$

(2) If N *is an odd number, and* $T = (N-1)/2$, *then*

$$\begin{aligned} & \alpha_{2m+3,2m+Nr+3} P_{m+Tr+\hat{r}+2} \\ & = (x\alpha_{2m+1,2m+Nr+1} - \alpha_{2m+3,2m+Nr+1} - c_{2m+2}\alpha_{2m+1,2m+Nr+1} - c_{2m+1}\alpha_{2m+1,2m+Nr+1}) P_{m+Tr+\hat{r}+1}(x) \\ & + \sum_{j=m-Ts-\hat{s}+1}^{m+Tr+\hat{r}} [x\alpha_{2m+1,2j} - \alpha_{2m+3,2j} - (c_{2m+2} + c_{2m+1})\alpha_{2m+1,2j} - c_{2m+1}c_{2m}\alpha_{2m-1,2j}] P_j(x) \\ & - (x\alpha_{2m+1,2m-Ns+1} - c_{2m+2}\alpha_{2m+1,2m-Ns+1} - c_{2m+1}\alpha_{2m+1,2m-Ns+1} - c_{2m+1}c_{2m}\alpha_{2m-1,2m-Ns+1}) P_{m-Ts-\hat{s}}(x) \\ & - c_{2m+1}c_{2m}\alpha_{2m-1,2m-Ns-1} P_{m-Ts-\hat{s}-1}(x). \end{aligned}$$

Proof. Again, we must distinguish between N being odd or even.

(1) Assume that N is even. Multiplying both sides of (3.18) by $x^T \hat{\phi}^N(x)$ and

applying (3.13) and (3.14) into it, we get

$$\begin{aligned} & x\left[\sum_{j=m-Ts}^{m+Tr} \alpha_{2m+1,2j+1} R_j(x)\right] \\ & = \sum_{j=m-Ts+1}^{m+Tr+1} \alpha_{2m+2,2j} P_j(x) + c_{2m+1} \sum_{j=m-Ts}^{m+Tr} \alpha_{2m,2j} P_j(x). \end{aligned} \tag{4.2}$$

Now, multiplying both sides of (3.15) by x and using (4.2) in the resulting equation, we obtain the following $T(r+s)+3 = (N\hat{r}+N\hat{s}+3+N)$-term recurrence relation for $\{P_n\}$,

$$\begin{aligned} & x\left[\sum_{j=m-Ts}^{m+Tr} \alpha_{2m,2j} P_j(x)\right] \\ & = \sum_{j=m-Ts+1}^{m+Tr+1} \alpha_{2m+2,2j} P_j(x) \ + \ c_{2m+1} \sum_{j=m-Ts}^{m+Tr} \alpha_{2m,2j} P_j(x) \\ & + \ c_{2m} \left[\sum_{j=m-Ts}^{m+Tr} \alpha_{2m,2j} P_j(x) + c_{2m-1} \sum_{j=m-Ts-1}^{m+Tr-1} \alpha_{2m-2,2j} P_j(x)\right], \end{aligned}$$

from which the statement (1) of the proposition follows straightforwardly.

(2) Assume now that N is odd. Considering (3.10), (3.16) and (3.17), we get

$$\begin{aligned} & x\left[\sum_{j=m-Ts-\hat{s}-1}^{m+Tr+\hat{r}} \alpha_{2m,2j+1} R_j(x)\right] \\ & = \sum_{j=m-Ts-\hat{s}}^{m+Tr+\hat{r}+1} \alpha_{2m+1,2j} P_j(x) + c_{2m} \sum_{j=m-Ts-\hat{s}-1}^{m+Tr+\hat{r}} \alpha_{2m-1,2j} P_j(x). \end{aligned} \tag{4.3}$$

Now plug the previous result in (3.19). We get the following $(N\hat{r}+N\hat{s}+3+N)$-term recurrence relation for $\{P_n\}$,

$$\begin{aligned} & x\left[\sum_{j=m-Ts-\hat{s}}^{m+Tr+\hat{r}+1} \alpha_{2m+1,2j} P_j(x)\right] \\ & = \sum_{j=m-Ts-\hat{s}+1}^{m+Tr+\hat{r}+2} \alpha_{2m+3,2j} P_j(x) + \ c_{2m+2} \sum_{j=m-Ts-\hat{s}}^{m+Tr+\hat{r}+1} \alpha_{2m+1,2j} P_j(x) \\ & + \ c_{2m+1} \left[\sum_{j=m-Ts-\hat{s}}^{m+Tr+\hat{r}+1} \alpha_{2m+1,2j} P_j(x) + \ c_{2m} \sum_{j=m-Ts-\hat{s}-1}^{m+Tr+\hat{r}} \alpha_{2m-1,2j} P_j(x)\right], \end{aligned}$$

which implies the result (2) in the statement of the proposition. □

4.2. Recurrence relations for $\{R_n\}$. In the sequel, and for the sake of brevity, we will omit the extended expression of the recurrence relations.

Proposition 4.3. *If the class $\tilde{s}$ of* $\mathbf{U}$ *is an even number, the sequence* $\{R_n\}$ *satisfies the following* $[N(\hat{s}+\hat{r})+3]$*-term recurrence relation,*

$$\begin{aligned} & x\left[\sum_{j=m-N\hat{s}}^{m+N\hat{r}} \alpha_{2m+1,2j+1} R_j(x)\right] \\ & = \sum_{j=m-N\hat{s}+1}^{m+N\hat{r}+1} \alpha_{2m+3,2j+1} R_j(x) + \ c_{2m+2} \sum_{j=m-N\hat{s}}^{m+N\hat{r}} \alpha_{2m+1,2j+1} R_j(x) \\ & + c_{2m+1}\left[\sum_{j=m-N\hat{s}}^{m+N\hat{r}} \alpha_{2m+1,2j+1} R_j(x) + \ c_{2m} \sum_{j=m-N\hat{s}-1}^{m+N\hat{r}-1} \alpha_{2m-1,2j+1} R_j(x)\right], \end{aligned}$$

where $\hat{r}=r/2$ *and* $\hat{s}=s/2$.

Proof. Replacing (3.11) into (4.1), we get the result. □

Proposition 4.4. *Let the class $\tilde{s}$ of* $\mathbf{U}$ *be an odd number, and put* $\hat{r}=(r-1)/2$ *and* $\hat{s}=(s-1)/2$*. Then the sequence* $\{R_n\}$ *satisfies the following* $[N(\hat{s}+\hat{r}+1)+3]$*-term recurrence relations:*

(1) If N *is even, and* $T=N/2$*, then*

$$\begin{aligned} & x\left[\sum_{j=m-Ts}^{m+Tr} \alpha_{2m+1,2j+1} R_j(x)\right] \\ & = \sum_{j=m-Ts+1}^{m+Tr+1} \alpha_{2m+3,2j+1} R_j(x) + \ c_{2m+2} \sum_{j=m-Ts}^{m+Tr} \alpha_{2m+1,2j+1} R_j(x) \\ & + c_{2m+1}\left[\sum_{j=m-Ts}^{m+Tr} \alpha_{2m+1,2j+1} R_j(x) + \ c_{2m} \sum_{j=m-Ts-1}^{m+Tr-1} \alpha_{2m-1,2j+1} R_j(x)\right]. \end{aligned}$$

(2) If N *is odd, and* $T=(N-1)/2$*, then*

$$\begin{aligned} & x\left[\sum_{j=m-Ts-\hat{s}-1}^{m+Tr+\hat{r}} \alpha_{2m,2j+1} R_j(x)\right] \\ & = \sum_{j=m-Ts-\hat{s}}^{m+Tr+\hat{r}+1} \alpha_{2m+2,2j+1} R_j(x) + \ c_{2m+1} \sum_{j=m-Ts-\hat{s}-1}^{m+Tr+\hat{r}} \alpha_{2m,2j+1} R_j(x) \\ & + c_{2m}\left[\sum_{j=m-Ts-\hat{s}-1}^{m+Tr+\hat{r}} \alpha_{2m,2j+1} R_j(x) + \ c_{2m-1} \sum_{j=m-Ts-\hat{s}-2}^{m+Tr+\hat{r}-1} \alpha_{2m-2,2j+1} R_j(x)\right]. \end{aligned}$$

Proof. We must distinguish again between N even or odd. In the case N even, replace (3.15) into (4.2), and in the case N odd, replace (3.19) into (4.3), and so the results are obtained. □

4.3. Recurrence relation for $\{Q_n\}$.

PROPOSITION 4.5. *The sequence $\{Q_n\}$ satisfies the following $(Ns+Nr+3)$-term recurrence relation,*

$$\begin{aligned}
&\alpha_{n+1,n+Nr+1}Q_{n+Nr+1}(x) = (x\alpha_{n,n+Nr} - \alpha_{n+1,n+Nr})Q_{n+Nr}(x)\\
&+ \sum_{j=n-Ns+1}^{n+Nr-1} (x\alpha_{n,j} - \alpha_{n+1,j} - c_n\alpha_{n-1,j})Q_j(x)\\
&+ (x\alpha_{n,n-Ns} - c_n\alpha_{n-1,n-Ns})Q_{n-Ns}(x) - c_n\alpha_{n-1,n-Ns-1}Q_{n-Ns-1}(x).
\end{aligned}$$

Proof. Multiplying both sides of (3.4) by x, from (3.2) we get

$$\phi^N(x)[T_{n+1}(x) + c_nT_{n-1}(x)] = \sum_{j=n-Ns}^{n+Nr} \alpha_{n,j}xQ_j(x).$$

Applying twice Proposition 3.2 to the previous expression, we obtain

$$\sum_{j=n-Ns+1}^{n+Nr+1} \alpha_{n+1,j}Q_j(x) + c_n \sum_{j=n-Ns-1}^{n+Nr-1} \alpha_{n-1,j}Q_j(x) = \sum_{j=n-Ns}^{n+Nr} \alpha_{n,j}xQ_j(x),$$

from which the result follows straightforwardly. ☐

5. Example: Freud-Sobolev orthogonal polynomials. Consider the following inner product of type (1.1),

$$\langle p,q\rangle_s = \int_{\mathbb{R}} p\,q\,e^{-x^4}dx \;+\; \sum_{i=1}^{N}\lambda_i \int_{\mathbb{R}} p^{(i)}\,q^{(i)}\,e^{-x^4}dx\;, \tag{5.1}$$

and let us apply the results obtained in the previous sections to this particular example. The polynomials $\{Q_n\}$ orthogonal with respect to (5.1) are a particular case of the so-called Freud-Sobolev polynomials [2].

It is well known that $\omega(x) = e^{-x^4}$ is a semiclassical weight function [2]. In fact, $\omega(x)$ satisfies the Pearson equation (2.3) with $\phi(x) = 1$ and $\psi(x) = -4x^3$. Hence, using the notations in Section 2, $r = 0$, $\tilde{s} = 2$ and $s = 2$. Since $\tilde{s}$ is an even number, we can state that

(1) The sequences $\{P_n\}$ and $\{R_n\}$ satisfy $(N+3)$-term recurrence relations (see Propositions 4.1 and 4.3).

(2) The sequence $\{Q_n\}$ satisfies a $(2N+3)$-term recurrence relation (see Proposition 4.5).

(3) An explicit algebraic relation between $\{P_n\}$ and $\{R_n\}$ can be established. This relation involves $N+1$ terms of the sequence $\{P_n\}$ and $N+2$ terms of the sequence $\{R_n\}$ (see Proposition 3.4).

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