# CONTINUOUS SYMMETRIC SOBOLEV INNER PRODUCTS

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January 23, 2003

Key words and phrases: Sobolev inner products, orthogonal polynomials, semiclassical linear functionals, recurrence relations.

2000 Mathematics Subject Classification. **42C05**

**Abstract**

In this paper we consider the sequence of monic polynomials $(Q_n)$ orthogonal with respect to a symmetric Sobolev inner product. If $Q_{2n}(x) = P_n(x^2)$ and $Q_{2n+1}(x) = xR_n(x^2)$, then we deduce the integral representation of the inner products such that $(P_n)$ and $(R_n)$ are, respectively, the corresponding sequences of monic orthogonal polynomials. In the semiclassical case, algebraic relations between such sequences are deduced. Finally, an application of the above results to Freud-Sobolev polynomials is given.

# 1 Introduction

Let U be a linear functional in the linear space $\mathbb{P}$ of polynomials with real coefficients. The sequence of real numbers $(\mu_n)_{n\in\mathbb{N}}$ where $\mu_n = U(x^n)$ is said to be the sequence of the moments associated with the linear functional.

Let consider the bilinear functional $\varphi_U : \mathbb{P} \times \mathbb{P} \to \mathbb{R}$ such that

$$\varphi(p, q) = U(pq), \quad p, q \in \mathbb{P}.$$

The Gram matrix of $\varphi_U$ with respect to the canonical basis $(x^n)_{n\in\mathbb{N}}$ is a Hankel matrix (see [3]). If the principal submatrices of the Hankel matrix are nonsingular, then the linear functional U is said to be quasi-definite.

For a quasi-definite linear functional U there exists a sequence of monic polynomials $\{T_n\}$ such that ( [3])

1. $deg(T_n) = n, \quad n \in \mathbb{N}.$
2. $\varphi_U(T_n, T_m) = k_n\delta_{nm}, \quad k_n \neq 0.$

This sequence of polynomials satisfies a three-term recurrence relation

$$xT_n(x) = T_{n+1}(x) + b_nT_n(x) + c_nT_{n-1}(x), \quad n \geq 0,$$

with initial conditions

$$T_{-1}(x) = 0, \quad T_0(x) = 1, \quad and \quad c_n \neq 0, \quad \forall n \in \mathbb{N}.$$

The linear functional is said to be positive definite if the principal submatrices of the associated Hankel matrix are positive definite. In such conditions, there exists a positive Borel measure $\mu$ supported in the real line such that the following integral representation for the linear functional U holds:

$$U(p) = \int_{\mathbb{R}} p(x)d\mu(x), \quad p \in \mathbb{P}. \tag{1.1}$$

A linear functional U is said to be symmetric if $U(x^{2n+1}) = 0, \quad n \in \mathbb{N}$. In particular, if U is positive definite and symmetric, then the support of the measure $\mu$ in (1.1) is a symmetric set with respect to the origin in the real line and the measure $\mu$ is associated with an even function in $\mathbb{R}$.

If U is a quasi-definite linear functional and $(T_n)$ denotes the corresponding sequence of monic orthogonal polynomials, then

$$T_{2n}(x) = S_n(x^2), \quad n \in \mathbb{N},$$

and

$$T_{2n+1}(x) = xS_n^*(x^2), \quad n \in \mathbb{N}.$$

Here $(S_n)$ and $(S_n^*)$ are, respectively, sequences of monic polynomials orthogonal with respect to two quasi-definite linear functionals $V$ and $V^*$ such that

$$V(x^n) = U(x^{2n}), \quad n \in \mathbb{N},$$
$$V^*(x^n) = V(x^{n+1}), \quad n \in \mathbb{N},$$

(see [3]).

Conversely, given a quasi-definite linear functional $V$ such that $S_n(0) \neq 0$ for the corresponding sequence of monic orthogonal polynomials, the linear functional U satisfying

$$U(x^{2n}) = V(x^n), \quad U(x^{2n+1}) = 0$$

is said to be the symmetrized linear functional associated with U. Notice that in this situation the sequence $(T_n)$ satisfies a three-term recurrence relation

$$xT_n(x) = T_{n+1}(x) + c_n T_{n-1}(x), \quad n \geq 0,$$

with initial conditions

$$T_0(x) = 1, \quad T_1(x) = x, \quad and \quad c_n \neq 0, \quad \forall n \in \mathbb{N}.$$

As a very well known example of symmetrization process, the Hermite polynomials are the symmetrized of Laguerre polynomials with parameter $\alpha = -1/2$, i.e.

$$H_{2n}(x) = L_n^{-\frac{1}{2}}(x^2)$$
$$H_{2n+1}(x) = xL_n^{\frac{1}{2}}(x^2)$$

In a recent work [1], the symmetrized linear functionals associated with semiclassical linear functionals are studied. A semiclassical linear functional U satisfies a distributional Pearson equation $D(\phi U) = \tau U$ where $\phi$ and $\tau$ are polynomials with $deg(\tau) \geq 1$. They constitute an extension of classical linear functionals (Hermite, Laguerre, Jacobi, and Bessel) and they have been extensively analyzed during the last two decades (see [4], [6]).

The aim of our contribution is to analyze the symmetrization process for a kind of inner products which have received some attention very recently, the so-called Sobolev inner products. Consider two positive definite linear functionals $U_0$ and $U_1$ in the linear space $\mathbb{P}$ of the polynomials with real coefficients. We introduce a bilinear functional $< \cdot, \cdot >$ in $\mathbb{P} \times \mathbb{P}$

$$< p, q >= U_0(pq) + U_1(p'q') \tag{1.2}$$

with $p, q \in \mathbb{P}$.

Using the Gram-Schmidt method for the canonical basis $(x^n)_{n\in\mathbb{N}}$ in $\mathbb{P}$, we obtain a sequence $(Q_n)$ of monic polynomials with $deg(Q_n) = n$ which are orthogonal with respect to the inner product (1.2).

Unfortunately, these polynomials do not satisfy recurrence relations as those associated with a linear functional. Nevertheless, under some assumptions for the linear functionals $U_0$ and $U_1$ it is possible to deduce some higher order recurrence relations ( see [5]) for the polynomials $Q_n$.

The starting point of our contribution is to assume that $U_0$ and $U_1$ are symmetric positive definite linear functionals. Then, $Q_{2n}(x) = P_n(x^2)$ as well as $Q_{2n+1}(x) = xR_n(x^2)$. In section 3 we deduce the integral representation for the inner products such that $(P_n)$ and $(R_n)$ are, respectively, the corresponding sequences of monic orthogonal polynomials. Thus, non-diagonal Sobolev inner products appear in a natural way.

In section 4 we assume that $U = U_0 = U_1$ and $U$ is a semiclassical linear functional. Then, algebraic relations between $(P_n)$ and $(R_n)$ are deduced as well as higher order recurrence relations for $(P_n)$ and $(R_n)$. Finally, as an example, we show the application of our results and techniques for the so-called Freud-Sobolev orthogonal polynomials [2].

# 2 Semiclassical Orthogonal Polynomials. Symmetrization and class.

Consider a quasi-definite linear functional U in the linear space $\mathbb{P}$ of polynomials with real coefficients and let $\{P_n\}$ be the sequence of monic polynomials orthogonal with respect to U.

U is said to be a *semiclassical linear functional* if

$$D(\phi U) = \tau U \tag{2.1}$$

where $\phi$ and $\tau$ are polynomials with $deg(\phi) = t \geq 0$ and $deg(\tau) = p \geq 1$.

**Theorem 2.1** *[1] The following statements are equivalent:*

1. *$U$ is a semiclassical linear functional.*
2. *The Stieltjes function $S_U(z) = -\sum_{n=0}^{\infty} \frac{\mu_n}{z^{n+1}}$ with $\mu_n = U(x^n)$ satisfies*

$$\phi(z)S_U'(z) = C(z)S_U(z) + D(z) \tag{2.2}$$

*where*

$$C(z) = -\phi'(z) + \tau(z) \tag{2.3}$$

$$D(z) = -(U\theta_0\phi)'(z) + (U\theta_0\tau)(z) \tag{2.4}$$

*and*

$$(U\theta_0 p)(c) =< U, \theta_c p >, \quad (U\theta_0 p)'(c) =< U, \theta_c^2 p >$$

$$\theta_c p = \frac{p(z) - p(c)}{z - c}.$$

The condition of being semiclassical can also be characterized in terms of a weight function.

**Proposition 2.2** *[4] Let $U$ be a semiclassical linear functional with integral representation*

$$U(p) = \int_{\mathbb{R}} p\omega(x)dx$$

*where $\omega$ is a continuosly differentiable function in an interval $[a, b]$ satisfying some extra boundary conditions and such that $D(\phi U) = \tau U$. Then*

$$(\phi\omega)' = \tau\omega \tag{2.5}$$

*and $\omega$ is said to be a semiclassical weight function.*

**Remark 2.3** *Observe that (2.1) holds for an infinite family of pairs of polynomials $(\phi, \tau)$. In particular, if $(\phi_1, \tau_1)$ satisfies (2.1), $(\pi\phi_1, \pi\tau_1 + \pi'\phi_1)$ with $\pi$ any polynomial, will also satisfy (2.1).*

**Definition 2.1** *[1] Let $(\phi, \tau)$ be the pair of polynomials with minimum degree that satisfy (2.1). Then, the class of $U$ is defined as*

$$s = max\{deg(\phi) - 2, deg(\tau) - 1\}. \tag{2.6}$$

It is possible to characterize those pairs of polynomials $(\phi, \tau)$ that define the class of a semiclassical functional.

**Proposition 2.4** *[1] Let $C$ and $D$ be the polynomials defined in (2.3) and (2.4). Then, $(\phi, \tau)$ is the pair of polynomials of minimum degree that satisfy (2.1) if and only if $(\phi, C, D)$ are coprime.*

**Theorem 2.5** *[1] Let $\Psi$ be a semiclassical linear functional of class $s$ such that $D(\phi\Psi) = \tau\Psi$ and let $U$ be its symmetrized. Then, $U$ is also semiclassical of class $\tilde{s}$ and*

1. *$\tilde{s} = 2s$ if $\phi(0) = 0, [\phi(z) = zE(z)]$ and $2C(0) + E(0) = 0$, $[2C(z) + E(z) = zG(z)]$.*
   *Furthermore, $D(\tilde{\phi}U) = \tilde{\tau}U$ and*
$$\tilde{\phi}(z) = E(z^2) \tag{2.7}$$
$$\tilde{\tau}(z) = z[G(z^2) + 2E'(z^2)]. \tag{2.8}$$

2. *$\tilde{s} = 2s + 1$ if $\phi(0) = 0, [\phi(z) = zE(z)]$ and $2C(0) + E(0) \neq 0$.*
   *Moreover*
$$\tilde{\phi}(z) = zE(z^2) \tag{2.9}$$
$$\tilde{\tau}(z) = 2[E(z^2) + z^2E'(z^2) + C(z^2)]. \tag{2.10}$$

3. *$\tilde{s} = 2s + 3$ if $\phi(0) \neq 0$ and*
$$\tilde{\phi}(z) = z\phi(z^2) \tag{2.11}$$
$$\tilde{\tau}(z) = 2[\phi(z^2) + z^2\phi'(z^2) + z^2C(z^2)]. \tag{2.12}$$

**Proposition 2.6** *Let $\boldsymbol{U}$ be a symmetric and semiclassical linear functional of class $\tilde{s}$ such that:*
$$D(\tilde{\phi}\mathbf{U}) = \tilde{\tau}\mathbf{U}.$$
*If $\tilde{s} = 2k$ for some $k \in \mathbb{N}$, then $\tilde{\phi}$ is an even polynomial. If $\tilde{s} = 2k + 1$, then $\tilde{\phi}$ is an odd polynomial.*

**Proof**

1. Suppose that $\mathbf{U}$ is the symmetrized of a linear functional $\mathbf{L}$ of class $s$. Moreover, assume that $\tilde{s}$ is even. Then, from Theorem 2.5 we get
$$\tilde{s} = 2s, \quad \tilde{s} = 2s + 1 \quad \text{or} \quad \tilde{s} = 2s + 3\,. \tag{2.13}$$
   It is easy to prove that, if $\tilde{s} = 2k$, then, necessarily $s = k$. Then, $\mathbf{L}$ is of class $k$ and $D(\phi\mathbf{L}) = \tau\mathbf{L}$ for certain polynomials $\phi, \tau$, and from (2.7)
$$\tilde{\phi}(x) = E(x^2)$$
   i.e., $\tilde{\phi}$ is an even polynomial.

2. Suppose now that $\tilde{s}$ is odd, namely, $\tilde{s} = 2k+1$ for some $k \in \mathbb{N}$. Then, because of (2.13) it may happen that $s = k$ or $s = k-1$ and $\mathbf{L}$ can be of class $k$ or $k-1$.

   - If $s = k$, from (2.9) it holds that
   $$\tilde{\phi}(x) = xE(x^2).$$
   Hence, $\tilde{\phi}$ is an odd polynomial.
   - If $s = k-1$, then from (2.11)
   $$\tilde{\phi}(x) = x\phi(x^2)$$
   and $\tilde{\phi}$ is an odd polynomial.

**Proposition 2.7** *Let* $\mathbf{U}$ *be a symmetric, semiclassical linear functional of class* $\tilde{s}$*. Assume* $\mathbf{U}$ *is the symmetrized of the semiclassical linear functional* $\mathbf{L}$ *of class* $s$*. If* $D(\tilde{\phi}U) = \tilde{\tau}U$*, where* $\tilde{\phi}$ *and* $\tilde{\tau}$ *are polynomials, then*

1. *For* $\tilde{s}$ *even,* $\tilde{\tau}$ *is an odd polynomial.*

2. *For* $\tilde{s}$ *odd,* $\tilde{\tau}$ *is an even polynomial.*

**Proof**

1. If $\tilde{s}$ is even, namely, $\tilde{s} = 2k$ for some $k \in \mathbb{N}$, then $s = k$ (see proposition 2.6). Moreover, for (2.8)
$$\tilde{\tau}(x) = x[G(x^2) + 2E'(x^2)]$$
for certain polynomials $G(x)$ and $E(x)$. Thus $\tilde{\tau}$ is an odd polynomial.

2. If $\tilde{s}$ is odd, namely, $\tilde{s} = 2k+1$ for some $k \in \mathbb{N}$, then one of the following statements holds

   - $s = k$ and $\tilde{\tau}(x) = 2[E(x^2) + x^2E'(x^2) + C(x^2)]$ for certain polynomials $E(x)$ and $C(x)$. As a consequence, $\tilde{\tau}$ is an even polynomial.

   - $s = k-1$ and $\tilde{\tau}(x) = 2[\phi(x^2) + x^2\phi'(x^2) + x^2C(x^2)]$ for certain polynomials $\phi(x), C(x)$. Thus $\tilde{\tau}$ is an even polynomial. ■.

# 3 Symmetric Sobolev Inner products.

Consider two positive Borel measures $\mu_0$,$\mu_1$ supported on the real line such that

$$\int_{\mathbb{R}} x^n d\mu_i < \infty \quad i = 0, 1, \quad n \in \mathbb{N}.$$

Consider an inner product in the linear space $\mathbb{P}$ of polynomials with real coefficients

$$< p, q >_s = \int_{\mathbb{R}} pq d\mu_0 + \int_{\mathbb{R}} p'q' d\mu_1. \tag{3.1}$$

This product is said to be a *Sobolev inner product.*

Furthermore, assume that $\mu_0$ and $\mu_1$ are supported on a subset of the real line which is symmetric with respect to the origin as well as the corresponding sequences of moments

$$c_n^{(i)} = \int_{\mathbb{R}} x^n d\mu_i, \quad i = 0, 1,$$

satisfy $c_{2n+1}^{(i)} = 0, \quad i = 0, 1, \quad n \in \mathbb{N}$.

Under these conditions, if we denote $\{Q_n\}$ the corresponding sequence of monic polynomials orthogonal with respect to (3.1), then

$$Q_{2n}(x) = P_n(x^2), \quad Q_{2n+1}(x) = xR_n(x^2)$$

for certain sequences of monic polynomials $\{P_n\}$ and $\{R_n\}$.

We are interested in the study of the orthogonality properties of the sequences $\{P_n\}$ and $\{R_n\}$, respectively.

First, observe that for $n \neq m$

$$0 = < Q_{2n}, Q_{2m} >_s = \int_{\mathbb{R}} P_n(x^2)P_m(x^2)d\mu_0 + \int_{\mathbb{R}} 4x^2 P_n'(x^2)P_m'(x^2)d\mu_1 =$$

$$= \int_0^\infty P_n(x)P_m(x)d\hat{\mu}_0 + \int_0^\infty P_n'(x)P_m'(x)d\hat{\mu}_1$$

where

$$d\hat{\mu}_0 = x^{-\frac{1}{2}} d\mu_0(x^{\frac{1}{2}}), \quad d\hat{\mu}_1 = 4x^{\frac{1}{2}} d\mu_1(x^{\frac{1}{2}}).$$

On the other hand,

$$0 \neq < Q_{2n}, Q_{2n} > = \int_{\mathbb{R}^+} P_n^2(x)d\hat{\mu}_0 + \int_{\mathbb{R}^+} [P_n'(x)]^2 d\hat{\mu}_1.$$

This means that $\{P_n\}$ is a sequence of monic polynomials orthogonal with respect to the Sobolev inner product

$$< p, q >_1 = \int_{\mathbb{R}^+} pq d\hat{\mu}_0 + \int_{\mathbb{R}^+} p'q' d\hat{\mu}_1. \tag{3.2}$$

Moreover, if $n \neq m$,

$$0 =< Q_{2n+1}, Q_{2m+1} >=$$

$$= \int_{\mathbb{R}} x^2 R_n(x^2) R_m(x^2) d\mu_0 + \int_{\mathbb{R}} [R_n(x^2) + 2x^2 R_n'(x^2)][R_m(x^2) + 2x^2 R_m'(x^2)] d\mu_1 =$$

$$= \int_{\mathbb{R}} \left[ \begin{array}{cc} R_n(x^2) & R_n'(x^2) \end{array} \right] \left[ \begin{array}{cc} x^2 d\mu_0 + d\mu_1 & 2x^2 d\mu_1 \\ 2x^2 d\mu_1 & 4x^4 d\mu_1 \end{array} \right] \left[ \begin{array}{c} R_m(x^2) \\ R_m'(x^2) \end{array} \right] =$$

$$= \int_{\mathbb{R}^+} \left[ \begin{array}{cc} R_n(x) & R_n'(x) \end{array} \right] \left[ \begin{array}{cc} x d\hat{\mu}_0 + \frac{d\hat{\mu}_1}{4x} & \frac{d\hat{\mu}_1}{2} \\ \frac{d\hat{\mu}_1}{2} & x d\hat{\mu}_1 \end{array} \right] \left[ \begin{array}{c} R_m(x) \\ R_m'(x) \end{array} \right]$$

and

$$0 \neq< Q_{2n+1}, Q_{2n+1} >= \int_{\mathbb{R}^+} \left[ \begin{array}{cc} R_n(x) & R_n'(x) \end{array} \right] \left[ \begin{array}{cc} x d\hat{\mu}_0 + \frac{d\hat{\mu}_1}{4x} & \frac{d\hat{\mu}_1}{2} \\ \frac{d\hat{\mu}_1}{2} & x d\hat{\mu}_1 \end{array} \right] \left[ \begin{array}{c} R_n(x) \\ R_n'(x) \end{array} \right].$$

This means that $\{R_n\}$ is a sequence of monic polynomials orthogonal with respect to the non-diagonal Sobolev inner product

$$< p, q >_2 = \int_{\mathbb{R}^+} \left[ \begin{array}{cc} p & p' \end{array} \right] d\Omega_2 \left[ \begin{array}{c} q \\ q' \end{array} \right] \tag{3.3}$$

where $d\Omega_2 = \left[ \begin{array}{cc} x d\hat{\mu}_0 + \frac{d\hat{\mu}_1}{4x} & \frac{d\hat{\mu}_1}{2} \\ \frac{d\hat{\mu}_1}{2} & x d\hat{\mu}_1 \end{array} \right]$.

Observe that $d\Omega_2$ is a matrix of measures related to the diagonal matrix of measures

$$d\Omega_1 = \left[ \begin{array}{cc} d\hat{\mu}_0 & 0 \\ 0 & d\hat{\mu}_1 \end{array} \right]$$

in the following way

$$d\Omega_2 = M d\Omega_1 M^t$$

with $M = \left[ \begin{array}{cc} x^{\frac{1}{2}} & \frac{1}{2x^{\frac{1}{2}}} \\ 0 & x^{\frac{1}{2}} \end{array} \right] = x^{\frac{1}{2}} \left[ \begin{array}{cc} 1 & \frac{1}{2x} \\ 0 & 1 \end{array} \right]$

namely,

$$d\Omega_2 = N \left[ \begin{array}{cc} x d\hat{\mu}_0 & 0 \\ 0 & x d\hat{\mu}_1 \end{array} \right] N^t$$

with $N = \begin{bmatrix} 1 & \frac{1}{2x} \\ 0 & 1 \end{bmatrix}$, or equivalently,

$$d\Omega_2 = \begin{bmatrix} x & \frac{1}{2} \\ 0 & x \end{bmatrix} \begin{bmatrix} \frac{1}{x}d\hat{\mu}_0 & 0 \\ 0 & \frac{1}{x}d\hat{\mu}_1 \end{bmatrix} \begin{bmatrix} x & 0 \\ \frac{1}{2} & x \end{bmatrix}.$$

In the sequel, we will analyze the particular case when $d\mu_0$ and $d\mu_1$ are equal and absolutely continuous measures. Moreover

- We will specify the orthogonality measures related to the sequences $\{P_n\}$ and $\{R_n\}$ .
- We will look for explicit algebraic relations between $\{P_n\}$ and $\{R_n\}$ .
- We will determine a recurrence relation that such sequences satisfy.

# 4 Symmetric Sobolev inner products with equal and absolutely continuous measures

The study of Sobolev inner products with respect to a measure was considered by F.Marcellán, T.E.Pérez, M.A.Piñar, and A.Ronveaux in [5]. Moreover, they took in consideration a semiclassical, positive definite linear functional U (2.1) to define the *Nth Sobolev inner product*

$$< p, q >_s^{(N)} = U(pq) + \sum_{m=1}^{N} \lambda_m U(p^{(m)} q^{(m)}), \quad \forall p, q \in \mathbb{P}. \tag{4.1}$$

Denote by

$$< p, q > = U(pq)$$

the standard inner product associated with U.

Considering $\{P_n\}$ the monic orthogonal polynomial sequence associated with the linear functional U and denoting $\{Q_n\}$ the MOPS with respect to the Sobolev inner product (4.1), they proved the following result:

**Proposition 4.1** *For every nonnegative integer number $n \geq Ns$, we get*

$$\phi(x)^N P_n(x) = \sum_{i=n-t}^{n+Ns} \alpha_{n,i} Q_i(x) \tag{4.2}$$

*where $s = deg(\phi)$, $\alpha_{n,n-t} \neq 0$ and $t = deg(F^{(N)}(x^n)) - n$.(Here $F^{(N)}$ denotes a differential operator introduced in [5]).*

We will consider the inner product

$$< p, q >_s = \int_{\mathbb{R}} pq\omega(x)dx + \int_{\mathbb{R}} p'q'\omega(x)dx \tag{4.3}$$

where $\omega(x)$ is an even weight function supported on an interval of the real line symmetric with respect to the origin. In this case, the corresponding odd moments satisfy

$$\mu_{2n+1} = 0, \quad \forall n \in \mathbb{N}\,.$$

Furthermore, suppose that $\omega(x)$ is a semiclassical weight, i.e,

$$(\phi\omega)' = \tau\omega \tag{4.4}$$

where $\phi, \tau$ are the polynomials of minimum degree that satisfy (4.4) with $deg(\phi) = s' \geq 0$ and $deg(\tau) = t > 0$.

Let $\{Q_n\}$ be the sequence of monic polynomials orthogonal with respect to the inner product (4.3). Then,

$$Q_{2n}(x) = P_n(x^2), \quad Q_{2n+1}(x) = xR_n(x^2) \tag{4.5}$$

for certain sequences of monic polynomials $\{P_n\}$ and $\{R_n\}$.

Consider the standard inner product

$$< p, q > = \int_{\mathbb{R}} pq\omega(x)dx \tag{4.6}$$

and let $\{T_n\}$ be the sequence of monic polynomials orthogonal with respect to (4.6). Then

**Proposition 4.2**

$$\phi(x)T_n(x) = \sum_{j=n-s}^{n+s'} \alpha_{nj}Q_j(x) \tag{4.7}$$

*with $\alpha_{n,n-s} \neq 0$, where $s = max\{\tilde{s}, s'\}$ and $\tilde{s}$ is the class of the semiclassical linear functional defined by $\omega$.*

**Proof** Let consider the Fourier expansion of $\phi T_n$ in terms of $\{Q_n\}$

$$\phi(x)T_n(x) = \sum_{j=0}^{n+s'} \alpha_{nj}Q_j(x)$$

Here $\alpha_{nj} = \frac{<\phi T_n, Q_j>_s}{\|Q_j\|_s^2}$. But

$$< \phi T_n, Q_j >_s = \int_{\mathbb{R}} \phi T_n Q_j \omega(x)dx + \int_{\mathbb{R}} \phi' T_n Q_j' \omega(x)dx + \int_{\mathbb{R}} \phi T_n' Q_j' \omega(x)dx.$$

Applying integration by parts to the third integral we get

$$= \int_{\mathbb{R}} \phi T_n Q_j \omega(x)dx + \int_{\mathbb{R}} \phi' T_n Q_j' \omega(x)dx - \int_{\mathbb{R}} T_n(\phi Q_j' \omega)' dx.$$

Since $\omega$ is a semiclassical weight, we obtain

$$= \int_{\mathbb{R}} \phi T_n (Q_j - Q_j'')\omega(x)dx - \int_{\mathbb{R}} T_n Q_j' (\tau - \phi')\omega(x)dx.$$

The first integral will vanish if $j < n - s'$, and the second one will vanish if $j < n - \tilde{s}$. Then $< \phi T_n, Q_j >_s = 0$ if $j < n - max\{s', \tilde{s}\}$. ■

Observe that $\phi(x)$ in (4.7) can be chosen in such a way that $\alpha_{n,n+s'} = 1$.

## 4.1 Orthogonality Measures for $\{P_n\}$ and $\{R_n\}$

Taking into account that $\{Q_n\}$ is orthogonal with respect to the Sobolev inner product (4.3), for $n \neq m$

$$0 =< Q_{2n}, Q_{2m} >_s = \int_{\mathbb{R}} P_n(x^2)P_m(x^2)\omega(x)dx + \int_{\mathbb{R}} 4x^2 P_n'(x^2)P_m'(x^2)\omega(x)dx =$$

$$= \int_0^{\infty} P_n(t)P_m(t)t^{-\frac{1}{2}}\omega(t^{\frac{1}{2}})dt + \int_0^{\infty} 4P_n'(t)P_m'(t)t^{\frac{1}{2}}\omega(t^{\frac{1}{2}})dt \,,$$

then $\{P_n\}$ is orthogonal with respect to the diagonal Sobolev inner product with matrix of measures

$$d\Omega_1 = \begin{bmatrix} 1 & 0 \\ 0 & 4t \end{bmatrix} t^{-\frac{1}{2}}\omega(t^{\frac{1}{2}})dt \,.$$

On the other hand, if $n \neq m$

$$0 =< Q_{2n+1}, Q_{2m+1} >_s = \int_{\mathbb{R}} x^2 R_n(x^2)R_m(x^2)\omega(x)dx +$$

$$+\int_{\mathbb{R}}[R_n(x^2)+2x^2R_n'(x^2)][R_m(x^2)+2x^2R_m'(x^2)]\omega(x)dx=$$

$$=\int_{\mathbb{R}}(x^2+1)R_n(x^2)R_m(x^2)\omega(x)dx+2\int_{\mathbb{R}}[R_n(x^2)R_m(x^2)]'x^2\omega(x)dx+$$

$$+4\int_{\mathbb{R}}R_n'(x^2)R_m'(x^2)x^4\omega(x)dx\,.$$

Changing the variable $t=x^2$

$$\int_0^\infty(t+1)R_n(t)R_m(t)t^{-\frac{1}{2}}\omega(t^{\frac{1}{2}})dt+2\int_0^\infty[R_n(t)R_m(t)]'t^{\frac{1}{2}}\omega(t^{\frac{1}{2}})dt+$$

$$+4\int_0^\infty R_n'(t)R_m'(t)t^{\frac{3}{2}}\omega(t^{\frac{1}{2}})dt\,. \quad (4.8)$$

Then $\{R_n\}$ is a sequence of monic polynomials orthogonal with respect to the Sobolev inner product with matrix of measures

$$d\Omega_2=\begin{bmatrix}1+t & 2t\\ 2t & 4t^2\end{bmatrix}t^{-\frac{1}{2}}\omega(t)dt\,.$$

The support of both measures, $d\Omega_1$, $d\Omega_2$ is contained in $\mathbb{R}^+$. Denote

$$\pi_1(t)=t\begin{bmatrix}0 & 0\\ 0 & 4\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$$

$$\pi_2(t)=t^2\begin{bmatrix}0 & 0\\ 0 & 4\end{bmatrix}+t\begin{bmatrix}1 & 2\\ 2 & 0\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}.$$

Thus,

$$d\Omega_1=\pi_1(t)t^{-\frac{1}{2}}\omega(t)dt$$

$$d\Omega_2=\pi_2(t)t^{-\frac{1}{2}}\omega(t)dt\,.$$

Taking into account the calculations done in (4.8) and applying integration by parts to the second integral

$$0=\int_0^\infty tR_nR_mt^{-\frac{1}{2}}\omega(t^{\frac{1}{2}})dt-\int_0^\infty R_nR_m\omega'(t^{\frac{1}{2}})dt+$$

$$+4\int_0^\infty R_n'(t)R_m'(t)t^{\frac{3}{2}}\omega(t^{\frac{1}{2}})dt\,.$$

If $\omega$ satisfy $\omega' = \tau\omega$, (Freud weights), then

$$0 = \int_0^\infty [t - \tau(t^{\frac{1}{2}})t^{\frac{1}{2}}] R_n R_m t^{-\frac{1}{2}} \omega(t^{\frac{1}{2}}) dt +$$
$$+4 \int_0^\infty R_n' R_m' t^{\frac{3}{2}} \omega(t^{\frac{1}{2}}) dt \,.$$

In such a case, $\{R_n\}$ is orthogonal with respect to a diagonal Sobolev inner product with matrix of measures

$$d\Omega_2 = \begin{bmatrix} t - \tau(t^{\frac{1}{2}})t^{\frac{1}{2}} & 0 \\ 0 & 4t^2 \end{bmatrix} t^{-\frac{1}{2}} \omega(t^{\frac{1}{2}}) dt \,.$$

If $\omega' = \tau\omega$, then the semiclassical functional defined by $\omega(t)$ is of even class. Thus, from Proposition 2.7, $\tau(x)$ is an odd polynomial and so $\tau(t^{\frac{1}{2}})t^{\frac{1}{2}}$ is a polynomial in $t$.

## 4.2 Explicit Algebraic Relations between $\{P_n\}$ and $\{R_n\}$

The sequence $\{T_n\}$, which is orthogonal with respect to the inner product(4.6), satisfies a three-term recurrence relation

$$xT_n(x) = T_{n+1}(x) + c_n T_{n-1}(x), \quad n \geq 1, \tag{4.9}$$
$$T_{-1}(x) \equiv 0, \quad T_0(x) \equiv 1, \quad c_n > 0 \,.$$

Furthermore,

$$T_{2n}(x) = S_n(x^2), \quad T_{2n+1}(x) = xS_n^*(x^2) \tag{4.10}$$

for a certain sequence of orthogonal polynomials $\{S_n\}$, where $\{S_n^*\}$ is the sequence of kernel polynomials $\{K_n(x;0)\}$ associated with the sequence $\{S_n\}$ [3].

Taking into account (4.7) for $n = 2m$ we get

$$\phi(x)T_{2m}(x) = Q_{2m+s'}(x) + \alpha_{2m,2m+s'-1}Q_{2m+s'-1}(x) + \ldots + \alpha_{2m,2m-s}Q_{2m-s}(x) \tag{4.11}$$

- Suppose that $\tilde{s}$ is even, then, $s'$ and $s$ are even, i.e., $s = 2k$ and $s' = 2k'$ with $k, k' \in \mathbb{N}$. In this case, because of Proposition 2.6, $\phi$ is an even polynomial. Taking into account this result,(4.11) may be simplified

$$\phi(x)T_{2m}(x) = Q_{2m+2k'}(x) + \alpha_{2m,2m+2k'-2}Q_{2m+2k'-2}(x) + \ldots$$

$$\cdots + + \alpha_{2m,2m-2k} Q_{2m-2k}(x) \, .$$

From (4.5) and (4.10)we get

$$\phi(x) S_m(x^2) = P_{m+k'}(x^2) + \sum_{j=m-k}^{m+k'-1} \alpha_{2m,2j} P_j(x^2) \, .$$

Since $\phi$ is an even polynomial, $\phi(x) = \tilde{\phi}(x^2)$ for a certain polynomial $\tilde{\phi}$ and then

$$\tilde{\phi}(x) S_m(x) = P_{m+k'}(x) + \sum_{j=m-k}^{m+k'-1} \alpha_{2m,2j} P_j(x) \, . \tag{4.12}$$

For $n = 2m + 1$ (4.7) becomes

$$\phi(x) T_{2m+1}(x) = Q_{2m+2k'+1}(x) + \alpha_{2m+1,2m+2k'-1} Q_{2m+2k'-1}(x) + \dots$$

$$\cdots + \alpha_{2m+1,2m+1-2k} Q_{2m-2k+1}(x) \, .$$

Because of (4.5) and (4.10)

$$\phi(x) x S^*_m(x^2) = x R_{m+k'}(x^2) + \sum_{j=m-k}^{m+k'-1} \alpha_{2m+1,2j+1} x R_j(x^2) \, .$$

Then,

$$\tilde{\phi}(x) S^*_m(x) = R_{m+k'}(x) + \sum_{j=m-k}^{m+k'-1} \alpha_{2m+1,2j+1} R_j(x) \, . \tag{4.13}$$

Taking into account the recurrence relation (4.9) for $n = 2m$, it holds

$$x T_{2m}(x) = T_{2m+1}(x) + c_{2m} T_{2m-1}(x), \quad m \geq 1 \, .$$

Because of (4.10)

$$S_m(x) = S^*_m(x) + c_{2m} S^*_{m-1}(x) \, . \tag{4.14}$$

Multiplying both hand sides of (4.14) by $\tilde{\phi}$ and applying (4.12) and (4.13),

$P_{m+k'}(x)+\sum_{j=m-k}^{m+k'-1}\alpha_{2m,2j}P_j(x) = R_{m+k'}(x)+[\alpha_{2m+1,2m+2k'-1}+c_{2m}]R_{m+k'-1}(x)+$
$+\sum_{j=m-k}^{m+k'-2}[\alpha_{2m+1,2j+1}+c_{2m}\alpha_{2m-1,2j+1}]R_j(x)+\dots$

$$\cdots + c_{2m}\alpha_{2m-1,2m-2k-1}R_{m-k-1}(x) \tag{4.15}$$

- Assume now that $\tilde{s}$ is odd. Let $k, k' \in \mathbb{N}$ be such that $s = 2k+1$ and $s' = 2k'+1$. Furthermore, $\phi$ is an odd polynomial from Proposition 2.7 and, as a consequence, (4.11) may be simplified to get
$\phi(x)T_{2m}(x) = Q_{2m+2k'+1}(x) + \alpha_{2m,2m+2k'-1}Q_{2m+2k'-1}(x) + \dots$

$$\cdots + \alpha_{2m,2m-2k-1}Q_{2m-2k-1}(x)\ .$$

Taking into account (4.5) and (4.10)

$$\phi(x)S_m(x^2) = xR_{m+k'}(x^2) + \sum_{j=m-k-1}^{m+k'-1}\alpha_{2m,2j+1}xR_j(x^2)\ .$$

Since $\phi$ is an odd polynomial, $\phi(x) = x\hat{\phi}(x^2)$, then

$$\hat{\phi}(x)S_m(x) = R_{m+k'}(x) + \sum_{j=m-k-1}^{m+k'-1}\alpha_{2m,2j+1}R_j(x)\ . \tag{4.16}$$

Writing (4.7) for $n = 2m+1$,

$\phi(x)T_{2m+1}(x) = Q_{2m+2k'+2}(x)+\alpha_{2m+1,2m+2k'}Q_{2m+2k'}(x)+\cdots+\alpha_{2m,2m-2k}Q_{2m-2k}(x).$

Because of (4.5) and (4.10)

$$\phi(x)xS_n^*(x^2) = P_{m+k'+1}(x^2) + \sum_{j=m-k}^{m+k'}\alpha_{2m+1,2j}P_j(x^2).$$

Then we get

$$x\hat{\phi}(x)S_n^*(x) = P_{m+k'+1}(x) + \sum_{j=m-k}^{m+k'}\alpha_{2m+1,2j}P_j(x)\ . \tag{4.17}$$

On the other hand, from

$$xT_{2n+1}(x) = T_{2n+2}(x) + c_{2n+1}T_{2n}(x), n \geq 0$$

we get

$$xS_n^*(x) = S_{n+1}(x) + c_{2n+1}S_n(x)\ . \tag{4.18}$$

Multiplying both sides by $\hat{\phi}$ and applying (4.16) and (4.17),

$P_{m+k'+1}(x) + \sum_{j=m-k}^{m+k'} \alpha_{2m+1,2j}P_j(x) = R_{m+k'+1}(x) + [\alpha_{2m+2,2m+2k'+1} + c_{2m+1}]R_{m+k'}(x) + + \sum_{j=m-k}^{m+k'-1}[\alpha_{2m+2,2j+1} + c_{2m+1}\alpha_{2m,2j+1}]R_j(x) + \dots$

$$\dots + \alpha_{2m,2m-2k-1}c_{2m+1}R_{m-k-1}(x)\ . \tag{4.19}$$

## 4.3 Recurrence Relations

If the linear functional associated with the weight function $\omega(x)$ is of class $\tilde{s}$ and $\tilde{s}$ is even, (4.15) holds as well as (4.18). Multiplying both hand sides of (4.18) by $\tilde{\phi}$ and considering (4.13) and (4.14) we get

$$x[R_{m+k'}(x)+\textstyle\sum_{i=m-k}^{m+k'-1} \alpha_{2m+1,2i+1}R_i(x)] = [P_{m+k'+1}(x)+\sum_{i=m-k+1}^{m+k'} \alpha_{2m+2,2i}P_i(x)]+$$

$$\dots + c_{2m+1}[P_{m+k'}(x) + \sum_{i=m-k}^{m+k'-1} \alpha_{2m,2i}P_i(x)]\ . \tag{4.20}$$

Substituting (4.15) in (4.20) in a convenient way

$$x[R_{m+k'}(x)+\sum_{i=m-k}^{m+k'-1} \alpha_{2m+1,2i+1}R_i(x)] = R_{m+k'+1}(x)+\sum_{i=m-k+1}^{m+k'} \alpha_{2m+3,2i+1}R_i(x)+$$

$$+c_{2m+2}[R_{m+k'}(x)+\sum_{i=m-k}^{m+k'-1} \alpha_{2m+1,2i+1}R_i(x)]+c_{2m+1}[R_{m+k'}(x)+\sum_{i=m-k}^{m+k'-1} \alpha_{2m+1,2i+1}R_i(x)+$$

$$+c_{2m}(R_{m+k'-1}(x) + \sum_{i=m-k-1}^{m+k'-2} \alpha_{2m-1,2i+1}R_i(x))]\ .$$

Then we get the following $(k+k'+2)$-term recurrence relation for $\{R_n\}$

$$\begin{aligned}
&R_{m+k'+1}(x) = (x - \alpha_{2m+3,2m+2k'+1} - c_{2m+2} - c_{2m+1})R_{m+k'}(x)+ \\
&+[x\alpha_{2m+1,2m+2k'-1} - \alpha_{2m+3,2m+2k'-1} - (c_{2m+2}+c_{2m+1})\alpha_{2m+1,2m+k'-1}- \\
&-c_{2m+1}c_{2m}]R_{m+k'-1}(x) + + \sum_{m-k+1}^{m+k'-2}[x\alpha_{2m+1,2i+1} - \alpha_{2m+3,2i+1}- \\
&-(c_{2m+2}+c_{2m+1})\alpha_{2m+1,2i+1} - c_{2m+1}c_{2m}\alpha_{2m-1,2i+1}]R_i(x) + [x\alpha_{2m+1,2m-2k+1}- \\
&\quad -(c_{2m+2}+c_{2m+1})\alpha_{2m+1,2m-2k+1} - c_{2m+1}c_{2m}\alpha_{2m-1,2m-2k-1}]R_{m-k}(x) + \\
&\qquad\qquad +c_{2m}c_{2m+1}\alpha_{2m-1,2m-2k-1}R_{m-k-1}(x)\ . \qquad (4.21)
\end{aligned}$$

Multiplying both sides of (4.15) by $x$ and replacing (4.20) in (4.15)

$$x[P_{m+k'}(x) + \sum_{i=m-k}^{m+k'-1}\alpha_{2m,2i}P_i(x)] = P_{m+k'+1}(x) + \sum_{i=m-k+1}^{m+k'}\alpha_{2m+2,2i}P_i(x)+$$

$$+c_{2m+1}[P_{m+k'}(x) + \sum_{i=m-k}^{m+k'-1}\alpha_{2m,2i}P_i(x)] + c_{2m}[P_{m+k'}(x) + \sum_{i=m-k}^{m+k'-1}\alpha_{2m,2i}P_i(x)+$$

$$+c_{2m-1}(P_{m+k'-1}(x) + \sum_{i=m-k-1}^{m+k'-2}\alpha_{2m-2,2i}P_i(x))]\ .$$

Thus a $(k+k'+2)$-term recurrence relation for $\{P_n\}$ follows.

$$\begin{aligned}
&P_{m+k'+1}(x) = [x - \alpha_{2m,2m+2k'} - c_{2m+1} - c_{2m}]P_{m+k'}(x)+ \\
&+[x\alpha_{2m,2m+2k'-2} - \alpha_{2m+2,2m+2k'-2} - (c_{2m-1}+c_{2m})\alpha_{2m,2m+2k'-2} - c_{2m}c_{2m-1}]P_{m+k'-1}(x)+ \\
&+\sum_{i=m-k+1}^{m-k'-2}[x\alpha_{2m,2i} - \alpha_{2m+2,2i} - (c_{2m+1}+c_{2m})\alpha_{2m,2i} - c_{2m}c_{2m-1}\alpha_{2m-2,2i}]P_i(x)+ \\
&+[x\alpha_{2m,2m-2k} - c_{2m+1}\alpha_{2m,2m-2k} - c_{2m}\alpha_{2m,2m-2k} - c_{2m}c_{2m-1}\alpha_{2m-2,2m-2k}]P_{m-k}(x) + \\
&c_{2m}c_{2m-1}\alpha_{2m-2,2m-2k-2}P_{m-k-1}(x)\ . \qquad (4.22)
\end{aligned}$$

On the other hand, it has also been proved that if $\tilde{s}$ is odd and $s = 2k+1$, $s' = 2k'+1$, then (4.19) holds. Let also remember (4.14). Multiplying both hand sides by $\hat{\phi}$ and substituting (4.16) and (4.17) there, we get

$x[R_{m+k'}(x)+\sum_{j=m-k-1}^{m+k'-1} \alpha_{2m,2j+1} R_j(x)] = P_{m+k'+1}(x)+\sum_{j=m-k}^{m+k'} \alpha_{2m+1,2j} P_j(x)+$

$$+c_{2m}[P_{m+k'}(x) + \sum_{j=m-k-1}^{m+k'-1} \alpha_{2m-1,2j} P_j(x)] \, . \tag{4.23}$$

Replacing (4.19) in (4.23)

$x[R_{m+k'}(x)+\sum_{j=m-k-1}^{m+k'-1} \alpha_{2m,2j+1} R_j(x)] = [R_{m+k'+1}(x)+\sum_{j=m-k}^{m+k'} \alpha_{2m+2,2j+1} R_j(x)]+$
$+c_{2m+1}[R_{m+k'}(x)+\sum_{j=m-k-1}^{m+k'-1} \alpha_{2m,2j+1} R_j(x)]+c_{2m}[R_{m+k'}(x)+\sum_{j=m-k-1}^{m+k'-1} \alpha_{2m,2j+1} R_j(x)$
$+c_{2m-1}(R_{m+k'-1}(x) + \sum_{j=m-k-2}^{m-k'-2} \alpha_{2m-2,2j+1} R_j(x))] \, .$

Finally

$R_{m+k'+1}(x) = [x-\alpha_{2m+2,2m+2k'+1}-c_{2m+1}-c_{2m}]R_{m+k'}(x)+[x\alpha_{2m,2m+2k'-1}-$
$-\alpha_{2m+2,2m+2k'-1} - (c_{2m+1} + c_{2m})\alpha_{2m,2m+2k'-1} - c_{2m}c_{2m-1}]R_{m+k'-1}(x)+$
$+\sum_{j=m-k}^{m+k'-2} [x\alpha_{2m,2j+1} - \alpha_{2m+2,2j+1} - (c_{2m+1} + c_{2m})\alpha_{2m,2j+1}-$
$-c_{2m}c_{2m-1}\alpha_{2m-2,2j+1}]R_j(x) + [x\alpha_{2m,2m-2k-1} - (c_{2m+1} + c_{2m})\alpha_{2m2m-2k-1}-$

$-c_{2m}c_{2m-1}\alpha_{2m-2,2m-2k-1}]R_{m-k-1}(x) - c_{2m}c_{2m-1}\alpha_{2m-2,2m-2k-3}R_{m-k-2}(x)$ (4.24)

Multiplying both hand sides of (4.19) by $x$ and substituting (4.23) in the resulting expression, a $(k + k' + 3)$-term recurrence relation for $\{P_n\}$ is obtained.

$P_{m+k'+2}(x) = [x - \alpha_{2m+1,2m+2k'} - c_{2m} - c_{2m-1}]P_{m+k'}(x) + [x\alpha_{2m-1,2m+2k'-2}$
$-\alpha_{2m+1,2m+2k'-2} - (c_{2m} + c_{2m-1})\alpha_{2m-1,2m+2k'-2} - c_{2m-1}c_{2m-2}]P_{m+k'-1}(x)+$
$+\sum_{j=m-k}^{m+k'-2} [x\alpha_{2m-1,2j} - \alpha_{2m+1,2j} - (c_{2m} + c_{2m+1})\alpha_{2m-1,2j}-$
$-c_{2m-1}c_{2m-2}\alpha_{2m-3,2j}]P_i(x)+[x\alpha_{2m-1,2m-2k-2}-(c_{2m}+c_{2m-1})\alpha_{2m-1,2m-2k-2}-$

$-c_{2m-1}c_{2m-2}\alpha_{2m-3,2m-2k-2}]P_{m-k-1}(x)--c_{2m-1}c_{2m-2}\alpha_{2m-3,2m-2k-4}P_{m-k-2}(x).$
(4.25)

### 4.3.1 Recurrence Relation for $\{Q_n\}$

Let start from (4.7). Multiplying both hand sides of (4.7) by $x$ and using the three-term recurrence relation for $\{T_n\}$,

$$\phi(x)(T_{n+1}(x) + c_n T_{n-1}(x)) = \sum_{j=n-s}^{n+s'} x\alpha_{nj} Q_j(x) \, , \tag{4.26}$$

thus, the substitution of (4.7) in (4.26) yields a recurrence relation for $\{Q_n\}$.

$$\begin{aligned} Q_{n+s'+1}(x) &= [x-\alpha_{n+1,n+s'}]Q_{n+s'}(x)+ \\ &+\sum_{j=n-s+1}^{n+s'-1}[x\alpha_{nj}-\alpha_{n+1,j}-c_n\alpha_{n-1,j}]Q_j(x)+ \\ &+[x\alpha_{n,n-s}-c_n\alpha_{n-1,n-s}]Q_{n-s}(x)-c_n\alpha_{n-1,n-s-1}Q_{n-s-1}(x)\,. \end{aligned} \tag{4.27}$$

## 4.4 Application: Freud-Sobolev Polynomials

A particular example of the Sobolev inner product given in (4.3) is

$$< p,q >_s = \int_{\mathbb{R}} pq e^{-x^4}dx + \int_{\mathbb{R}} p'q'e^{-x^4}dx \tag{4.28}$$

This kind of inner product has been introduced in [2]. Let $\{Q_n\}$ be the sequence of monic polynomials orthogonal with respect to (4.28). These polynomials are called *Freud-Sobolev Polynomials.*

Obviously (4.28) is a symmetric inner product, hence $\{Q_n\}$ satisfy (4.5).

### 4.4.1 Orthogonality Measures associated with $\{P_n\}$ and $\{R_n\}$ in the Freud-Sobolev Case.

Taking into account that $\{Q_n\}$ is orthogonal with respect to the inner product (4.28), for $n \neq m$,

$$\begin{gathered} 0 = < Q_{2n}, Q_{2m} >_s = \\ = \int_0^\infty P_n P_m t^{-\frac{1}{2}} e^{-t^2} dt + \int_0^\infty P'_n P'_m 4t^{\frac{1}{2}} e^{-t^2} dt\,. \end{gathered}$$

Thereupon, $\{P_n\}$ is orthogonal with respect to the diagonal Sobolev inner product given by the matrix of measures

$$d\Omega_1 = \begin{bmatrix} 1 & 0 \\ 0 & 4t \end{bmatrix} t^{-\frac{1}{2}} e^{-t^2} dt\,. \tag{4.29}$$

On the other hand, for $n \neq m$

$$0 = < Q_{2m+1}, Q_{2n+1} >_s =$$

$$= \int_0^\infty R_n R_m (t+4t^2) t^{-\frac{1}{2}} e^{-t^2} dt + 4 \int_0^\infty R_n' R_m' t^{\frac{3}{2}} e^{-t^2} dt \, . \tag{4.30}$$

Hence, $\{R_n\}$ is a sequence of monic polynomials orthogonal with respect to the diagonal Sobolev inner product given by the matrix of measures

$$d\Omega_2 = \begin{bmatrix} 1+4t & 0 \\ 0 & 4t \end{bmatrix} t^{\frac{1}{2}} e^{-t^2} dt \, . \tag{4.31}$$

Moreover, the support of the measures $d\Omega_1$ and $d\Omega_2$ is $\mathbb{R}^+$. Denote

$$\pi_1(t) = t \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\pi_2(t) = t^2 \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} + t \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} .$$

Then

$$d\Omega_1 = \pi_1(t) t^{-\frac{1}{2}} e^{-t^2} dt,$$
$$d\Omega_2 = \pi_2(t) t^{-\frac{1}{2}} e^{-t^2} dt \, .$$

Next we give an explicit relation between the sequences $\{P_n\}$ and $\{R_n\}$. Consider the standard inner product

$$< p, q >= \int_{\mathbb{R}} p q e^{-x^4} dx \tag{4.32}$$

and let $\{T_n\}$ be the sequence of monic polynomials orthogonal with respect to (4.32), namely, the sequence of the so-called *Freud Polynomials*. Then, it can be proved (see [2]) that

$$T_n(x) = Q_n(x) + a_n Q_{n-2}(x), \quad n \geq 3, \tag{4.33}$$
$$T_0(x) = Q_0(x), \quad T_1(x) = Q_1(x), \quad T_2(x) = Q_2(x),$$

where

$$a_n = 4n \frac{\| T_{n+2} \|^2}{\| Q_n \|_s^2}, \quad n \geq 1,$$
$$a_0 = a_{-1} = a_{-2} = 0 \, .$$

Furthermore,

$$T_{2n}(x) = S_n(x^2), \quad T_{2n+1}(x) = x S_n^*(x^2), \tag{4.34}$$

for a certain sequence of orthogonal polynomials $\{S_n\}$ where $S_n^*$ denotes the $nth$ kernel polynomial $K_n(x;0)$ normalized to be monic, associated with the sequence $\{S_n\}$.

Overwriting (4.33) for $n = 2m$ and for $n = 2m+1$, respectively,

$$S_m(x) = P_m(x) + a_{2m} P_{m-1}(x), \quad m \geq 2, \tag{4.35}$$

and $S_0(x) = P_0(x)$, $S_1(x) = P_1(x)$.

$$S_m^*(x) = R_m(x) + a_{2m+1} R_{m-1}(x), \quad m \geq 1, \tag{4.36}$$

and $S_0^*(x) = R_0(x)$.

On the other hand, the sequence $\{T_n\}$ satisfies the three-term recurrence relation

$$\begin{gathered} xT_n(x) = T_{n+1}(x) + c_n T_{n-1}(x), \quad n \geq 1, \\ T_0(x) = 1, \quad T_1(x) = x \end{gathered} \tag{4.37}$$

where

$$n = 4c_n(c_{n+1} + c_n + c_{n-1}), \quad n \geq 1,$$

with initial conditions $c_0 = 0, c_1 = \frac{\Gamma(3/4)}{\Gamma(1/4)}$ .

For $n = 2m+1$, the expression (4.37) yields

$$xS_m^*(x) = S_{m+1}(x) + c_{2m+1} S_m(x) \, .$$

Taking into account (4.35) and (4.36), for $m \geq 2$ we get

$$x[R_m(x) + a_{2m+1} R_{m-1}(x)] = P_{m+1}(x) + (a_{2m+2} + c_{2m+1}) P_m(x) + a_{2m} c_{2m+1} P_{m-1}(x), \tag{4.38}$$

Repeating the above procedure for $n = 2m$, $m \geq 2$

$$P_m(x) + a_{2m} P_{m-1}(x) = R_m(x) + (a_{2m+1} + c_{2m}) R_{m-1}(x) + c_{2m} a_{2m-1} R_{m-2}(x), \tag{4.39}$$

### 4.4.2 Recurrence Relations in Freud-Sobolev Case

Consider again (4.38) and (4.39). Substituting (4.39) in (4.38), we get

$$\begin{aligned} xR_m(x) &= R_{m+1}(x) + [a_{2m+3} + c_{2m+1} + c_{2m+2}] R_m(x) + [c_{2m} c_{2m+1} \\ &+ a_{2m+1}(c_{2m+1} + c_{2m+2} - x)] R_{m-1}(x) + a_{2m-1} c_{2m} c_{2m+1} R_{m-2}(x) \end{aligned} \tag{4.40}$$

for $m \geq 1$ with fixed initial conditions.

Multiplying (4.39) by $x$

$$x[P_m(x)+a_{2m}P_{m-1}(x)] = x[R_m(x)+a_{2m+1}R_{m-1}(x)]+xc_{2m}[R_{m-1}(x)+a_{2m-1}R_{m-2}(x)].$$

Because of (4.38),

$$xP_m(x) = P_{m+1}(x) + (a_{2m+2} + c_{2m} + c_{2m+1})P_m(x) + [c_{2m-1}c_{2m} + a_{2m}(c_{2m}+ \\ c_{2m+1} - x)]P_{m-1}(x) + c_{2m-1}c_{2m}a_{2m-2}P_{m-2}(x) \; .$$

As a conclusion, the sequences $\{P_n\}$ and $\{R_n\}$ satisfy the following recurrence relations

$$P_{m+1}(x) = [x - (a_{2m+2} + c_{2m} + c_{2m+1})]P_m(x) + [a_{2m}(x - c_{2m} - c_{2m+1}) - \\ -c_{2m-1}c_{2m}]P_{m-1}(x) - c_{2m-1}c_{2m}a_{2m-1}P_{m-2}(x), \quad (4.41)$$

$$R_{m+1}(x) = [x - (a_{2m+3} + c_{2m+1} + c_{2m+2})]R_m(x) + [a_{2m+1}(x - c_{2m+1} - \\ -c_{2m+2}) - c_{2m}c_{2m+1}]R_{m-1}(x) - c_{2m}c_{2m+1}a_{2m-2}R_{m-2}(x) \; . \quad (4.42)$$

We can summarize our main results

- We have identified $\{P_n\}$ and $\{R_n\}$ as sequences of polynomials orthogonal with respect to a diagonal Sobolev inner product.
- We have proved two algebraic relations between $\{P_n\}$ and $\{R_n\}$.
- We have proved that $\{P_n\}$ and $\{R_n\}$ satisfy four-term recurrence relations.

Finally , observe that multiplying (4.33) by $x$ and applying the recurrence relation (4.37)

$$T_{n+1}(x) + c_nT_{n-1}(x) = xQ_n(x) + a_nxQ_{n-2}(x) \; .$$

Replacing (4.33) in the previous equation

$$Q_{n+1}(x)+a_{n+1}Q_{n-1}(x)+c_n[Q_{n-1}(x)+a_{n-1}Q_{n-3}(x)] = xQ_n(x)+a_nxQ_{n-2}(x).$$

Then, $\{Q_n\}$ satisfy the following five-term recurrence relation

$$Q_{n+1}(x) = xQ_n(x) - (a_{n+1} + c_n)Q_{n-1}(x) + xa_nQ_{n-2}(x) - c_na_{n-1}Q_{n-3}(x)\,. \tag{4.43}$$

**Acknowledgements**
The work of the second author was partially supported by Dirección General de Investigación (Ministerio de Ciencia y Tecnología) of Spain under grant BFM 2000-0206-C04-01 and INTAS project INTAS 2000-272.

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