# ON THE EXPONENT OF $R$-REGULAR PRIMITIVE MATRICES. *

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**Abstract.** Let $P_{nr}$ be the set of $n$-by-$n$ $r$-regular primitive $(0,1)$-matrices. In this paper we find an explicit formula in terms of $n$ and $r$ for the minimum exponent achieved by matrices in $P_{nr}$. Moreover, we give matrices achieving that exponent. Gregory and Shen [6] conjectured that $b_{nr} = \lfloor \frac{n}{r} \rfloor^2 + 1$ is an upper bound for the exponent of matrices in $P_{nr}$. We present matrices achieving the exponent $b_{nr}$ when $n$ is not a multiple of $r$. In particular, we show that $b_{2r+1,r}$ is the maximum exponent attained by matrices in $P_{2r+1,r}$. When $n$ is a multiple of $r$ we conjecture that the maximum exponent achieved by matrices in $P_{nr}$ is strictly smaller than $b_{nr}$ and give matrices attaining the conjectured maximum exponent in that set. We also show that our conjecture is true when $n = 2r$.

**Key words.** $r$-regular matrices, primitive matrices, exponent of primitive matrices.

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**1. Introduction.** A nonnegative square matrix $A$ is called *primitive* if there exists a positive integer $k$ such that $A^k$ is positive. The smallest such $k$ is called the *exponent of* $A$. We denote the exponent of a primitive matrix $A$ by $exp(A)$.

A (0,1)-matrix $A$ is said to be $r$-regular if every column sum and every row sum is constantly $r$.

Consider the set $P_{nr}$ of all primitive r-regular $(0,1)$-matrices of order $n$, where $2 \leq r \leq n$. Notice that, for $n > 1$, $n$-by-$n$ 1-regular matrices are permutation matrices, which are not primitive. An interesting problem is to find the following two positive integers:

$$l_{nr} = \min\{exp(A) : A \in P_{nr}\}, \quad \text{and} \quad u_{nr} = \max\{exp(A) : A \in P_{nr}\},$$

as well as finding matrices attaining those exponents. In this paper, we call the integers $l_{nr}$ and $u_{nr}$ the optimal lower bound and the optimal upper bound for the exponent of matrices in $P_{nr}$, respectively.

The problem of finding an upper bound for the exponent of matrices in $P_{nr}$ has been considered by several authors in Discrete Mathematics, in particular, by some researchers in Graph Theory [2, 4, 5, 6]. In the literature, several such bounds can be found. In [4], it is shown that $exp(A) \leq \frac{2n(3n-2)}{(r+1)^2} - \frac{n+2}{r+1}$. In [6], it is shown that, if $A \in P_{nr}$, then $exp(A) \leq 3n^2/r^2$. Also, it is conjectured there that, if $A \in P_{nr}$, then $exp(A) \leq \lfloor \frac{n}{r} \rfloor^2 + 1$, where $\lfloor \ \rfloor$ denotes the *floor* function, that rounds a number to the next smaller integer. J. Shen proved that this conjecture is true when $r = 2$ [5], however it remains open for $r > 2$. Concerning a lower bound for the exponent of matrices in $P_{nr}$, S. G. Lee and collaborators [4] proved Lemma 3.1 in Section 3.

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In this paper, we give an explicit expression for $l_{nr}$ in terms of $n$ and $r$, and construct matrices attaining that exponent. We also construct matrices whose exponent is $\lfloor \frac{n}{r} \rfloor^2 + 1$ when $n = gr + c$, with $0 < c < r$, which proves that $u_{nr} \geq \lfloor \frac{n}{r} \rfloor^2 + 1$ in those cases. Moreover, we prove that $u_{nr} = \lfloor \frac{n}{r} \rfloor^2 + 1$ when $g = 2$ and $c = 1$. When $n = gr$, with $g = 2$, we determine $u_{nr}$; when $g \geq 3$, we give a conjecture for the value of $u_{nr}$ and give matrices achieving the conjectured optimal upper bound exponent. According to this conjecture, $u_{nr}$ would be smaller than $\lfloor \frac{n}{r} \rfloor^2 + 1$.

**2. Notation and Auxiliary Results.** In the sequel we will use the following notation: If $A$ is an $n$-by-$m$ matrix, we denote by $A(i,j)$ the entry of $A$ in the position $(i,j)$. By $A(i_1 : i_2, j_1 : j_2)$, with $i_2 \geq i_1$ and $j_2 \geq j_1$, we denote the submatrix of $A$ lying in rows $i_1, i_1+1, \ldots, i_2$ and columns $j_1, j_1+1 \ldots, j_2$. We abbreviate $A(i_1 : i_1, j_1 : j_2)$ to $A(i_1, j_1 : j_2)$ and $A(1 : n, j_1 : j_2)$ to $A(:, j_1 : j_2)$. Similar abbreviations are used for the columns of $A$. The $m$-by-$n$ matrix whose entries are all equal to one is denoted by $J_{mn}$. Unspecified entries in matrices are represented by a $*$.

Some of the proofs in this paper involve the concept of digraph associated with a $(0,1)$- matrix.

Definition 2.1. *Let $A$ be a $(0,1)$-matrix of size $n$-by-$n$. The digraph $G(A)$ associated with $A$ is the directed graph with vertex set $V = \{1, 2, ..., n\}$ and arc set $E$ where $(i,j) \in E$ if and only if $A(i,j) = 1$.*

Notice from the previous definition that $A$ is the adjacency matrix of $G(A)$.

A digraph $G$ is said to be $r$-regular if and only if its adjacency matrix is an $r$-regular matrix. Note that the outdegree and the indegree of each vertex of an $r$-regular digraph are exactly $r$. A digraph is said to be primitive if and only if its adjacency matrix is primitive. Clearly, for $A \in P_{nr}$, $exp(A) = k$ if and only if any two vertices in $G(A)$ are connected by a walk of length $k$ and, if $k > 1$, there are at least two vertices that are not connected by a walk of length $k-1$.

It is important to notice that if $A$ is an $r$-regular primitive matrix and $B = P^T A P$ for some permutation matrix $P$, then, for any positive integer $k$, $B^k = P^T A^k P$. Thus, $exp(A) = exp(B)$. Also $G(A)$ and $G(B)$ are isomorphic digraphs. Therefore, throughout the paper, we will work on the set of equivalence classes under permutation similarity. Notice also that $A \in P_{nr}$ if and only $A^t \in P_{nr}$.

Next we include some simple observations about $r$-regular primitive matrices that will be useful to prove some of the main results in the paper.

Lemma 2.2. *Let $A \in P_{nr}$ and let $k > 1$ be a positive integer. If $A^k(i,j) = 0$, then there are at least $r$ zero entries in the $i$-th row of $A^{k-1}$; also there are at least $r$ zero entries in the $j$-th column of $A^{k-1}$.*

*Proof.* Notice that $A^k(i,j) = A^{k-1}(i,:)A(:,j) = 0$. Since $A$ is $r$-regular, $r$ entries of $A(:,j)$ are ones. Taking into account that $A^{k-1}(i,:) \geq 0$, the first result follows. The second claim can be proven in a similar way taking into account that $A^k(i,j) = A(i,:)A^{k-1}(:,j) = 0$. □

Lemma 2.3. *Let $A \in P_{nr}$ and $i \in \{1, \ldots, n\}$. Then, the number of nonzero entries in the $i$-th row (column) of $A^k$, $k \geq 1$, is a nondecreasing sequence in $k$.*

*Proof.* Suppose that in the $i$-th row of $A^k$ there are exactly $s$ nonzero entries. We want to show that in the $i$-th row of $A^{k+1}$ there are at least $s$ nonzero entries. Denote by $S$ the set $\{j \in \{1,...,n\} : A^k(i,j) \neq 0\}$. Since the outdegree of each node of $G(A)$ is exactly $r$, there are $rs$ arcs with origin in the vertices in $S$. Since the indegree of each node of $G$ is exactly $r$, then the $rs$ arcs with origin in $S$ have their terminus in at least $rs/r = s$ vertices. Thus, with origin in the $i$-th node of $G(A)$, there are walks of length $k+1$ to at least $s$ distinct vertices. The result for columns follows taking into account that $A^t \in P_{nr}$. □

Note that the last lemma implies that each row (column) of $A^k$ has at least $r$ nonzero entries.

If $i \in \{1,...,n\}$ is such that $A(i,i) = 1$, then Lemma 2.3 may be refined. We consider this situation in the next lemma, as it will allow us to get an interesting corollary. We assume that $n \geq 2r$ since, by Lemma 2.2, if $n < 2r$, $A^2(i,:)$ is positive.

Lemma 2.4. *Let $A \in P_{nr}$, with $n \geq 2r$, and $i \in \{1,\ldots,n\}$. Suppose that $A(i,i) = 1$. Let $s_k$ be the number of nonzero entries in $A^k(i,:)$, $k \geq 1$. If $s_k < n$, then the number of nonzero entries in the $i$-th row of $A^{k+1}$ is at least $s_k + 1$. In particular, the $i$-th row of $A^{n-2r+3}$ is positive.*

*Proof.* By a possible permutation similarity of $A$, we assume that $i = 1$ and $A(1,:) = \begin{bmatrix} J_{1r} & 0 \end{bmatrix}$. Let $k \in \{2,\ldots,n\}$. Clearly, the first $r$ entries of $A^k(1,:)$ are nonzero. If $k = 2$, since $A$ is not reducible, $A^2(1,:)$ has more than $r$ nonzero entries. Now suppose that $k > 2$ and $s_k < n$. With a possible additional permutation similarity, we assume, without loss of generality, that $A^k(1,:) = \begin{bmatrix} a_1 & \cdots & a_{s_k} & 0 \end{bmatrix}$, where $a_i > 0$, $i = 1,\ldots,k$. We show that $s_{k+1} \geq s_k + 1$. Suppose that $A^{k-1}(1,:) = \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix}$, where $b_1, b_2, \ldots, b_r, b_{i_1}, \ldots, b_{i_{s_{k-1}-r}}$ are positive integers, with $r < i_1 < \cdots < i_{s_{k-1}-r} \leq n$. Because $A^k = AA^{k-1}$, then $i_{s_{k-1}-r} \leq s_k$; also, as $A^k = A^{k-1}A$ then

$$A = \begin{bmatrix} J_{1r} & 0 & 0 \\ * & R_{11} & 0 \\ * & R_{21} & R_{22} \\ * & R_{31} & R_{32} \end{bmatrix},$$

for some blocks $R_{ij}$, where $R_{11}$ and $R_{22}$ are $(r-1)$-by-$(s_k - r)$ and $(s_k - r)$-by-$(n - s_k)$ matrices, respectively. Since all the entries of

$$\begin{bmatrix} b_2 & \cdots & b_n \end{bmatrix} \begin{bmatrix} R_{11}^t & R_{21}^t & R_{31}^t \end{bmatrix}^t$$

are nonzero, then also all the entries of

$$\begin{bmatrix} a_2 & \cdots & a_{s_k} & 0 \end{bmatrix} \begin{bmatrix} R_{11}^t & R_{21}^t & R_{31}^t \end{bmatrix}^t$$

are nonzero, which implies that $A^{k+1}(1,i) \neq 0$ for $i = 1,...,s_k$. Since $A$ is not reducible, it also follows that $R_{22}$ is nonzero. Therefore, $A^{k+1}(1,:)$ has at least $s_k + 1$ nonzero entries. Clearly, $A^{n-2r+2}(1,:)$ has at most $r-1$ zero entries, which implies, by Lemma 2.2, that $A^{n-2r+3}(1,:)$ is positive. □

The next result is a simple consequence of Lemma 2.4. It gives an upper bound for the exponent of matrices in $P_{nr}$ with nonzero trace. Another such upper bound can be found in [4]: if $A \in P_{nr}$ has $p$ nonzero diagonal entries, then $exp(A) \leq \max\{2(n-r+1)-p, n-r+1\}$. It is easy to check that there are values of $n$ and $r$ for which the upper bound given in Corollary 2.5 for the exponent of matrices with nonzero trace is smaller than those in [4] and [6]. Check with n=30 and r=15, for instance.

COROLLARY 2.5. *Let $A \in P_{nr}$, with $n \geq 2r$, and suppose that $trace(A) \neq 0$. Then, $exp(A) \leq 2n - 4r + 6$.*

*Proof.* Let $i \in \{1, \dots, n\}$ be such that $A(i,i) \neq 0$. According to Lemma 2.4, the $i$-th row and the $i$-th column of $A^{n-2r+3}$ have no zero entries. Therefore, from any vertex in $G(A)$ there is a walk of length $n-2r+3$ to vertex $i$; also, there is a walk of length $n-2r+3$ from vertex $i$ to any vertex. Thus, any two vertices are connected by a walk of length $2n-4r+6$. □

Finally, we include the following technical lemma.

LEMMA 2.6. *Let $D_{rk}$, $k < r$, denote an $r$-by-$k$ matrix with exactly $r-1$ nonzero entries in each column. Then, at least one row of $D_{rk}$ has no zero entries. Moreover, if $k < r-1$, then at least two rows of $D_{rk}$ have no zero entries.*

*Proof.* Notice that the number $t$ of nonzero entries in $D_{rk}$ is $k(r-1)$ since every column contains $r-1$ nonzero entries. Assume that all rows of $D_{rk}$ have at least one zero entry. Then, the number $m$ of zero entries in $D_{rk}$ would be at least $r$. This implies that

$$t = rk - m \leq rk - r < k(r-1),$$

which is a contradiction. The second claim can be proven in a similar way. □

**3. Optimal lower bound.** In this section we determine the optimal lower bound $l_{nr}$ for the exponent of matrices in $P_{nr}$ in terms of $n$ and $r$. We also present matrices achieving this exponent.

First, we include a preliminary lemma that appears in [4]. We give its proof here because, to our knowledge, [4] has not been published yet. Moreover, our proof follows a matrix approach while theirs uses graph theory techniques.

LEMMA 3.1. *[4] Let $A \in P_{nr}$. Then, the exponent of $A$ satisfies:*

$$exp(A) \begin{cases} = 1, & \text{if } r = n, \\ = 2, & \text{if } (n+1)/2 \leq r \leq n-1, \\ \geq 2, & \text{if } \sqrt{n} \leq r \leq n/2, \\ \geq k+1, & \text{if } \sqrt[k+1]{n} \leq r \leq \sqrt[k]{n-1}, \quad k \geq 2. \end{cases}$$

*Proof.* If $r = n$, the result is straightforward since the only matrix in $P_{n,n}$ is $J_{nn}$, which is positive.

Suppose that $r \leq n-1$. Then $A$ is not positive which implies that $exp(A) \geq 2$.

Suppose that $r \geq (n+1)/2$. Then, for all $i \in 1, ..., n$, the $i$-th row of A has at most $(n-1)/2 < r$ zero entries and, therefore, by Lemma 2.2 , $A(i,:)A > 0$. Hence, $exp(A) \leq 2$.

Suppose that $r \leq \sqrt[k]{n-1}$. By a permutation similarity of $A$, we can assume, without loss of generality, that either the first $r$ entries or the last $r$ entries of $A(1,:)$

are ones. Suppose that the first case occurs (the proof is analogous in the second case). If $A^k(1,:) > 0$, then, in the first $r$ rows of $A^{k-1}$ there would exist at least $n$ nonzero entries. But, since each row of $A^{k-1}$ has at most $r^{k-1}$ nonzero entries (as the sum of the entries in each row is $r^{k-1}$), it follows that the first $r$ rows of $A^{k-1}$ have at most $r^k$ nonzero entries. Since, by hypothesis, $r^k < n$, then $A^k(1,:)$ is not positive, which implies that $exp(A) \geq k+1$, as desired.□

We now give a lower bound for the exponent of matrices in $P_{nr}$. By $\lceil . \rceil$ we denote the *ceil* function, that rounds a number to the next larger integer.

Lemma 3.2. *Let $A \in P_{nr}$. Then, $exp(A) \geq \lceil log_r(n) \rceil$.*

*Proof.* According to Lemma 3.1, if $n = r$, then $exp(A) = 1 = \lceil log_n(n) \rceil$.

If $(n+1)/2 \leq r \leq n-1$, then, by Lemma 3.1, $exp(A) = 2$. Since

$$\frac{log(n)}{log(r)} \leq \frac{log(2r-1)}{log(r)} \leq 2,$$

then $\lceil log_r(n) \rceil \leq 2$. Thus, $exp(A) = 2 \geq \lceil log_r(n) \rceil$.

If $\sqrt{n} \leq r \leq n/2$ then, by Lemma 3.1, $exp(A) \geq 2$. Since

$$\frac{log(n)}{log(r)} \leq \frac{log(r^2)}{log(r)} = 2,$$

$exp(A) \geq 2 \geq \lceil log_r(n) \rceil$.

Finally, if $n^{1/k+1} \leq r \leq (n-1)^{1/k}$, with $k \geq 2$, by Lemma 3.1, $exp(A) \geq k+1$. Since

$$\frac{log(n)}{log(r)} \leq \frac{log(r^{k+1})}{log(r)} = k+1,$$

$exp(A) \geq k+1 \geq \lceil log_r(n) \rceil$. □

Next we prove that there exist matrices in $P_{nr}$ whose exponent is $\lceil log_r(n) \rceil$.

Definition 3.3. *Let $B = [b_{ij}]$ be an $m$-by-$n$ real (complex) matrix. We call the indicator matrix of $B$, which we denote by $M(B)$, the $m$-by-$n$ $(0,1)$-matrix $[\mu_{ij}]$, with $\mu_{ij} = 1$ if $b_{ij} \neq 0$ and $\mu_{ij} = 0$ if $b_{ij} = 0$.*

Definition 3.4. *Let $v = (v_1, v_2, ...., v_n)$ be a row vector in $\mathbb{R}^n$. Let $s$ be an integer such that $0 < s \leq n$. Define the $s$-shift operator $f_s : \mathbb{R}^n \to \mathbb{R}^n$ by*

$$f_s(v_1, v_2, ..., v_n) = (v_{n-s+1}, v_{n-s+2}, ..., v_n, v_1, v_2, ...., v_{n-s}).$$

*The $s$-generalized circulant matrix associated with $v$ is the $n$-by-$n$ matrix whose $k$-th row is given by $f_s^{k-1}(v)$, for $k = 1, ..., n$, where $f_s^{k-1}$ denotes the composition of $f_s$ with itself $k-1$ times.*

Note that $f_s^n(v_1, ..., v_n) = (v_1, ..., v_n)$, as the position of $v_1$ after $n$ $s$-shifts is $ns+1$ modulo $n$, that is, 1.

Let $0 < s \leq r$ be an integer. We denote by $T_s^{nr}$ the $s$-generalized circulant matrix associated with $u_r = \sum_{i=1}^{r} e_i^t$, where $e_i$ denotes the $i$-th column of the $n$-by-$n$ identity matrix. For instance,

$$T_1^{52} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

LEMMA 3.5. *For $r \geq 2$, the matrix $T_r^{nr}$ is r-regular and primitive. Moreover, $exp(T_r^{nr}) = \lceil log_r(n) \rceil$.*

*Proof.* First we prove that $T_r^{nr}$ is an $r$-regular matrix. By construction, it is easy to see that the row sum is constantly $r$. In order to determine the column sum note that there are exactly $nr$ entries equal to one in $T_r^{nr}$. We denote by $s_i$, $i \geq 1$, the remainder of the division of $i$ by $n$, if $i$ is not a multiple of $n$, and $s_i = n$ otherwise. By construction again, the ones in the $i$-th row occur in positions $s_{(i-1)r+1}, ..., s_{ir}$. The sequence of columns in which the ones occur, starting in the first row, then the second row and so on, is just the sequence $s_1, s_2, s_3, ..., s_{nr}$, that is, $1, ..., n, 1, ..., n, ..., 1, ..., n$. Clearly, each $j \in \{1, 2, ..., n\}$ appears exactly $r$ times in that sequence.

Now we prove that $T_r^{nr}$ is primitive by computing its exponent. We first show, by induction on $k$, that the first $\min\{n, r^k\}$ entries of the first row of $(T_r^{nr})^k$ are nonzero and, if $r^k < n$, the last $n - r^k$ entries of the first row of $(T_r^{nr})^k$ are zero. If $k = 1$, this claim is trivially true. Now suppose that the claim is valid for $k = p$. Note that, for each integer $1 \leq k \leq n$, all the columns of the submatrix of $T_r^{nr}$ indexed by the first $r^k$ rows and the first $\min\{n, r^{k+1}\}$ columns are nonzero. Also, if $r^{k+1} < n$, the submatrix of $T_r^{nr}$ indexed by the first $r^k$ rows and the last $n - r^{k+1}$ columns is 0. Taking into account this observation, it follows that the first $\min\{n, r^{p+1}\}$ entries of $(T_r^{nr})^{p+1}(1:) = (T_n^{nr})^p(1,:)T_r^{nr}$ are nonzero while the last $n - \min\{n, r^{p+1}\}$ are zero.

Using similar arguments, we can show that, in general, the $i$-th row of $M((T_r^{nr})^k)$ is $f_r^{(i-1)r^{k-1}}(u_k)$, where $u_k = \sum_{j=1}^{\min\{r^k,n\}} e_j^t$.

Therefore, any row of $(T_r^{nr})^k$ has exactly $\min\{r^k, n\}$ nonzero entries. Thus, $(T_r^{nr})^k$ is positive if and only if $r^k \geq n$, which implies the result.
$\square$

THEOREM 3.6. *Suppose that $2 \leq r \leq n$. Then, $l_{rn} = \lceil log_r(n) \rceil$.*

*Proof.* Follows from Lemma 3.2 and Lemma 3.5. $\square$

**4. Optimal upper bound.** Although stated in terms of graphs, the following conjecture is given in [6]: If $A \in P_{nr}$, then $exp(A) \leq \left\lfloor \frac{n}{r} \right\rfloor^2 + 1$. In [5] this conjecture was proven for $r = 2$. Notice that this conjecture is trivially true for $r \geq \frac{n+1}{2}$. Just take into account Lemma 3.1 and note that in this case $\left\lfloor \frac{n}{r} \right\rfloor^2 + 1 = 2$. Hence, in the sequel we assume that $n \geq 2r$.

Given any $g \geq 2$, an r-regular primitive digraph with $n = gr + 1$ vertices whose exponent is $\lfloor \frac{n}{r} \rfloor^2 + 1$ can be found in [6]. A matrix with such a graph is the following:

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & J_{rr} \\ J_{rr} & 0 & \cdots & 0 & 0 & 0 \\ 0 & J_{rr} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & J_{1r} & 0 & 0 \\ 0 & 0 & \cdots & T_1^{r,r-1} & J_{r1} & 0 \end{bmatrix}. \tag{4.1}$$

In the next two subsections we generalize the structure of the matrix $A$ by defining the matrices $E_{nr}$ for all possible combinations of $n$ and $r$.

**4.1. The case in which $n$ is not a multiple of $r$.** Generalizing the structure of the matrix in (4.1), in this section we define the $n$-by-$n$ matrices $E_{nr}$, when $n = gr + c$ for some positive integers $g \geq 2$ and $0 < c < r$, as follows:

$$E_{nr} = \begin{bmatrix} 0 & 0 & J_{rr} \\ J_{cr} & 0 & 0 \\ T_1^{r,r-c} & J_{rc} & 0 \end{bmatrix}, \quad \text{if } n = 2r + c, \tag{4.2}$$

$$E_{nr} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & J_{rr} \\ J_{rr} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & J_{rr} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & J_{rr} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & J_{cr} & 0 & 0 \\ 0 & 0 & \cdots & 0 & T_1^{r,r-c} & J_{rc} & 0 \end{bmatrix}, \tag{4.3}$$

$$\text{if } n = gr + c, \text{ with } g \geq 3. \tag{4.4}$$

Note that we can replace $T_1^{r,r-c}$ by any matrix in $P_{r,r-c}$ without changing the exponent of $E_{nr}$.

Next we show that $\exp(E_{nr}) = \lfloor \frac{n}{r} \rfloor^2 + 1$, which implies that $u_{nr} \geq \lfloor \frac{n}{r} \rfloor^2 + 1$. We then prove the equality when $g = 2$ and $c = 1$.

LEMMA 4.1. *If $n = 2r + c$, where $0 < c < r$, then $\exp(E_{nr}) = \lfloor \frac{n}{r} \rfloor^2 + 1 = 5$.*

*Proof.* It is easy to check that

$$M(E_{nr}^2) = \begin{bmatrix} J_{rr} & J_{rc} & 0 \\ 0 & 0 & J_{cr} \\ J_{rr} & 0 & J_{rr} \end{bmatrix}, \quad M(E_{nr}^3) = \begin{bmatrix} J_{rr} & 0 & J_{rr} \\ J_{cr} & J_{cc} & 0 \\ J_{rr} & J_{rc} & J_{rr} \end{bmatrix},$$

$$M(E_{nr}^4) = \begin{bmatrix} J_{rr} & J_{rc} & J_{rr} \\ J_{cr} & 0 & J_{cr} \\ J_{rr} & J_{rc} & J_{rr} \end{bmatrix}.$$

Finally, we get that $M(E_{nr}^5) = J_{nn}$, which implies the result. □

Lemma 4.2. *If* $n = gr+c$, *with* $g \geq 3$ *and* $0 < c < r$, *then* $exp(E_{nr}) = \left\lfloor \frac{n}{r} \right\rfloor^2 + 1 = g^2 + 1$.

*Proof.* Consider the digraph $G$ associated with $E_{nr}$. Let us group the vertices of $G$ in the following way: We call $B_1$ the set of vertices from $(g-1)r + c + 1$ to $gr + c$; we call $B_2$ the set of vertices from $(g-1)r + 1$ to $(g-1)r + c$; we call $B_i$, $i = 3, \ldots, g+1$, the set of vertices from $(g-i+1)r + 1$ to $(g-i+2)r$.

Suppose that $u$ and $v$ are two vertices in the same block $B_i$. Then there is a path from $u$ to $v$ of length $g$ and another one of length $g+1$, except if $u, v \in B_2$, in which case there is just a path of length $g+1$. Therefore, a walk from $u$ to $v$ has length $t$ if and only $t = \alpha g + \beta(g+1)$, for some nonnegative integers $\alpha, \beta$, with $\beta > 0$ if $u, v \in B_2$. In particular, no vertex in $B_2$ lies on a closed walk of length $g^2$ since $\alpha g + \beta(g+1) = g^2$ implies $\beta = 0$. Thus, $exp(E_{nr}) > g^2$.

Because

$$g^2 + 1 = (g-1)g + (g+1),$$

it follows that there is a walk of length $g^2 + 1$ from any vertex to any other in the same block $B_i$, $i = 1, , \ldots, g+1$.

Now consider a vertex $u$ in $B_i$ and a vertex $v$ in $B_j$, where $i, j \in \{1, ..., g+1\}$ and $i \neq j$. Let $s$ be the distance from $u$ to $v$. Note that $s \leq g$. We will show that there is a walk of length $g^2 + 1$ from $u$ to $v$. Suppose that $s > 1$. In this case we have

$$g^2 - s + 1 = (s-2)g + (g-s+1)(g+1).$$

Thus, $u$ lies on a closed walk of length $g^2 - s + 1$, which implies that there is a walk of length $g^2 + 1$ from $u$ to $v$.

Now suppose that $s = 1$. If $u \notin B_2$, $u$ lies on a closed walk of length $g^2$, which implies that there is a walk of length $g^2 + 1$ from $u$ to $v$. If $u \in B_2$, then $v \in B_3$ and $v$ lies on a close walk of length $g^2$, which implies that there is a walk of length $g^2 + 1$ from $u$ to $v$.

We have shown that the vertices in $B_2$ do not lie on any closed walk of length $g^2$. On the other hand, between any two vertices there is a walk of length $g^2 + 1$. Thus $E_{nr}^{g^2}$ is not positive, while $E_{nr}^{g^2+1}$ is positive. Therefore, $\exp(E_{nr}^{g^2+1}) = g^2 + 1$. □

The following theorem follows in a straightforward way from Lemmas 4.1 and 4.2.

Theorem 4.3. *If* $n = gr + c$, *with* $0 < c < r$, *then* $u_{nr} \geq \lfloor \frac{n}{r} \rfloor^2 + 1$.

We now show that, when $n = 2r + 1$, $u_{nr} = \lfloor \frac{n}{r} \rfloor^2 + 1$.

Theorem 4.4. *Let* $n = 2r + 1$. *Then,* $u_{nr} = \lfloor \frac{n}{r} \rfloor^2 + 1 = 5$.

*Proof.* Clearly, by Theorem 4.3, $u_{nr} \geq 5$. We now show that if $A \in P_{nr}$ and $\exp(A) > 4$, then $\exp(A) = 5$, which means that there are no matrices in $P_{nr}$ with exponent greater than 5, and, therefore, $u_{nr} = 5$. The strategy we follow allows us to characterize, up to a permutation similarity, all the matrices in $P_{nr}$ that achieve exponent 5.

Suppose that $\exp(A) \geq 5$. Then, there is a zero entry in $A^4$. Without loss of generality, we can assume that $A^4(1, i) = 0$ for some $i \in \{1, ..., n\}$. Applying Lemma 2.2 repeatedly, we deduce that there are at least $r$ zero entries in the first row of $A^3$ and $A^2$.

By a convenient permutation similarity on $A$, we can reduce the proof to the next two cases (and subcases). Throughout the proof, we denote by $D_{rk}$ an $r$-by-$k$ matrix with exactly $r-1$ nonzero entries in each column and by $C_{rr}$ a matrix in $P_{r,r-1}$.

**Case 1.** Let us assume that $A(1,:) = [J_{1r} \quad 0]$. Then, $A^2(1,i) \neq 0$ for $i = 1, ..., r$ and we can assume that $A^2(1, r+2:n) = 0$. Therefore,

$$A = \begin{bmatrix} J_{1r} & 0 & 0_{1r} \\ * & R_1 & 0_{r-1,r} \\ * & * & D_{r+1,r} \end{bmatrix},$$

for some $(r-1)$-by-1 block $R_1$. If $R_1$ is zero, clearly $A$ is reducible, which is a contradiction. If $R_1$ is nonzero, then $M(A^2)(1,:) = [J_{1,r+1} \quad 0_{1,r}]$ and $A^3(1,i) = A^2(1,:)A(:,i) \neq 0$ for $i = 1, ..., r+1$. Since $A^3(1,:)$ contains at least $r$ zero entries then $M(A^3)(1,:) = [J_{1,r+1} \quad 0_{1,r}]$, which implies that $D_{r+1,r}(1,:) = 0$. Thus,

$$A = \begin{bmatrix} J_{1r} & 0 & 0_{1r} \\ C_{rr} & J_{r1} & 0_{rr} \\ 0_{rr} & 0_{r1} & J_{rr} \end{bmatrix}$$

is reducible, which is again a contradiction.

**Case 2.** Let us assume now that $A(1,:) = [0 \quad J_{1r} \quad 0_{1r}]$. Notice that there is $i \in \{r+2, ..., n\}$ such that $A^2(1,i) \neq 0$, otherwise $A(1:r+1, r+2:n) = 0$, and $A$ would be reducible. This observation leads to the following subcases:

Subcase 2.1. Assume that $A^2(1,i) = 0$ for $i = 1, r+2, ..., n-1$. Then

$$A = \begin{bmatrix} 0 & J_{1r} & 0 & 0 \\ 0 & C_{rr} & 0 & J_{r1} \\ J_{r1} & 0 & J_{r,r-1} & 0 \end{bmatrix}.$$

A calculation shows that $\exp(A) = 3$, which is a contradiction.

Subcase 2.2. Let us assume that $A^2(1,i) = 0$ for $i = 1, ..., k+1, r+2, ..., 2r-k$, with $0 < k < r-1$. Then,

$$A = \begin{bmatrix} 0 & J_{1k} & J_{1,r-k} & 0_{1,r-k-1} & 0_{1,k+1} \\ 0_{r1} & 0_{rk} & R_1 & 0_{r,r-k-1} & R_2 \\ J_{r1} & D_{rk} & * & J_{r,r-k-1} & R_3 \end{bmatrix},$$

for some blocks $R_i$, $i = 1, 2, 3$. Taking into account Lemma 2.6, each column of $R_1$ and $R_2$ is nonzero, which implies that $A^2(1,i) \neq 0$ for $i = k+2, ..., r+1, 2r-k+1, ..., n$. Since $A^2(1,:)$ has at least $r$ entries equal to zero, then $M(A^2)(1,:) = [0_{1,k+1} \quad J_{1,r-k} \quad 0_{r-k-1} \quad J_{1,k+1}]$. Note that the submatrix of $\begin{bmatrix} R_2^t & R_3^t \end{bmatrix}^t$ indexed by rows $k+1, \ldots, r, 2r-k, \ldots, 2r$ has all columns nonzero, otherwise $A$ would not be $r$-regular. Thus, $A^3(1,i) = A^2(1,:)A(:,i) \neq 0$ for $i = 1, .., k+1, r+2, ..., n$, and

$A^3(1,:)$ would not have $r$ zero entries, a contradiction.

Subcase 2.3. Let us assume that $A^2(1,i) = 0$ for $i = 1, ..., r$. Then,

$$A = \begin{bmatrix} 0 & J_{1,r-1} & 1 & 0_{1r} \\ 0_{r1} & 0_{r,r-1} & R_1 & R_2 \\ J_{r1} & D_{r,r-1} & * & * \end{bmatrix},$$

for some blocks $R_i$, $i = 1, 2$. Taking into account Lemma 2.6, all columns of $R_2$ are nonzero, which implies that $A^2(1,i) \neq 0$ for $i = r+2, ..., n$. If $R_1 = 0$, then

$$A = \begin{bmatrix} 0 & J_{1r} & 0 \\ 0 & 0 & J_{rr} \\ J_{r1} & C_{rr} & 0 \end{bmatrix},$$

and $exp(A) = 5$. If $R_1$ is nonzero, then, $M(A^2)(1,:) = [0_{1r} \quad J_{1,r+1}]$ and $A^3(1,:) = A^2(1,:)A$ has at most one nonzero entry, which is a contradiction. (Note that the last row of $[R_1 R_2]$ has exactly one zero entry.)

Subcase 2.4. Assume that $A^2(1,i) = 0$ for $i = 2, ..., r+1$. Then,

$$A = \begin{bmatrix} 0 & J_{1r} & 0 \\ * & 0 & * \\ * & D_{rr} & * \end{bmatrix}.$$

Note that, by Lemma 2.3, $A^2(1,:)$ has at least $r$ nonzero entries.

- Let us assume that $A^2(1,:)$ has exactly $r$ nonzero entries. If $M(A^2)(1,:) = [0_{1,r+1} \quad J_{1r}]$, then

$$A = \begin{bmatrix} 0 & J_{1r} & 0 \\ 0 & 0 & J_{rr} \\ J_{r1} & C_{rr} & 0 \end{bmatrix}; \tag{4.5}$$

  if $M(A^2)(1,:) = [1 \quad 0_{1,r+1} \quad J_{1,r-1}]$, then

$$A = \begin{bmatrix} 0 & J_{1r} & 0 & 0 \\ J_{r1} & 0 & 0 & J_{r,r-1} \\ 0 & C_{rr} & J_{r1} & 0 \end{bmatrix}. \tag{4.6}$$

  A straightforward computation shows that in both cases $\exp(A) = 5$.
- Let us assume that $A^2(1,:)$ has exactly $r+1$ nonzero entries. Then, $M(A^2)(1,:) = [1 \quad 0_{1r} \quad J_{1r}]$ and $A$ has the form

$$A = \begin{bmatrix} 0 & J_{1r} & 0 \\ R_1 & 0 & R_2 \\ R_3 & D_{rr} & R_4 \end{bmatrix}, \tag{4.7}$$

  where $R_1$ and $R_2$ are $r$-by-1 and $r$-by-$r$ matrices, respectively, with all columns nonzero. Notice also that, since not all rows of $D_{rr}$ sum $r$, either

$R_3$ or some column in $R_4$ is nonzero. A calculation shows that $A^3(1,i) \neq 0$ for $i = 2, \ldots, r+1$. Moreover, there is another nonzero entry in $A^3(1,:)$. If $A^3(1,:) = [J_{1,r+1} \quad 0_{1r}]$, then

$$A = \begin{bmatrix} 0 & J_{1r} & 0 \\ 0 & 0 & J_{rr} \\ J_{r1} & C_{rr} & 0 \end{bmatrix};$$

if $A^3(1,:) = [0 \quad J_{1,r+1} \quad 0_{1,r-1}]$, then

$$A = \begin{bmatrix} 0 & J_{1r} & 0 & 0 \\ J_{r1} & 0 & 0 & J_{r,r-1} \\ 0 & C_{rr} & J_{r1} & 0 \end{bmatrix}.$$

In both cases, $\exp(A) = 5$.

Subcase 2.5. Let us assume that $A^2(1,i) = 0$ for $i = 2, ..., k+1, r+2, ..., 2r-k+1$, with $0 < k < r$. Then,

$$A = \begin{bmatrix} 0 & J_{1k} & J_{1,r-k} & 0_{1,r-k} & 0_{1k} \\ R_1 & 0_{rk} & * & 0_{r,r-k} & R_2 \\ * & D_{rk} & * & J_{r,r-k} & * \end{bmatrix},$$

for some blocks $R_i$, $i = 1, 2$. Taking into account Lemma 2.6, each column of $R_1$ and $R_2$ is nonzero. Then, $A^2(1,i) \neq 0$ for $i = 1, 2r-k+2, ..., n$, which implies that $A^3(1,i) = A^2(1,:)A(:,i) \neq 0$, for $i = 2, ..., 2r-k+1$. Since $A^3(1,:)$ has at least $r$ zero entries, then $r-1 \leq k < r$, that is, $k = r-1$. Therefore,

$$M(A^2)(1,:) = [1 \quad 0_{1,r-1} \quad * \quad 0 \quad J_{1,r-1}].$$

- If $M(A^2)(1,:) = [1 \quad 0_{1,r-1} \quad 0 \quad 0 \quad J_{1,r-1}]$, then

$$A = \begin{bmatrix} 0 & J_{1r} & 0 & 0_{1,r-1} \\ J_{r1} & 0_{rr} & 0_{r1} & J_{r,r-1} \\ 0_{r1} & C_{rr} & J_{r1} & 0_{r,r-1} \end{bmatrix}.$$

  A calculation shows that $\exp(A) = 5$.
- If $M(A^2)(1,:) = [1 \quad 0_{1,r-1} \quad 1 \quad 0 \quad J_{1,r-1}]$, then

$$A = \begin{bmatrix} 0 & J_{1,r-1} & 1 & 0 & 0_{1,r-1} \\ * & 0_{r,r-1} & * & 0_{r1} & * \\ * & D_{r,r-1} & * & J_{r1} & * \end{bmatrix}$$

  and $M(A^3)(1, 2:r+2) = J_{1,r+1}$. Because $A^3(1,:)$ has at least $r$ zero entries, it follows that $M(A^3)(1,:) = [0 \quad J_{1,r+1} \quad 0_{1,r-1}]$. Since $A^2(1,r+1) \neq 0$, then $A^3(1,i) = A^2(1,:)A(:,i) = 0$ implies $A(r+1,i) = 0$. Thus, $A(r+1,i) = 0$, for $i = 1, .., r, r+2, ..., n$, and the $(r+1)$-th row of $A$ would have at least $2r$ entries equal to 0, which contradicts the fact that $A$ is $r$-regular.

□

Notice that, according to the proof of Theorem 4.4, the only "types" of matrices in $P_{2r+1,r}$ (up to a permutation similarity) that achieve maximum exponent are

$$A_1 := \begin{bmatrix} 0 & 0 & J_{rr} \\ J_{1r} & 0 & 0 \\ C_{rr} & J_{r1} & 0 \end{bmatrix} \quad \text{and} \quad A_2 := \begin{bmatrix} 0 & 0 & J_{rr} \\ C_{rr} & J_{r1} & 0 \\ J_{1r} & 0 & 0 \end{bmatrix}.$$

Clearly, if $C_{rr}$ is chosen equal to $T_1^{r,r-1}$, then $A_1 = E_{2r+1,r}$.

Note that the matrix $A_2$ has nonzero trace and has maximum exponent among the matrices in $P_{2r+1,r}$. However, Corollary 2.5 shows that, for most combinations of $n$ and $r$, $u_{nr}$ is not attained by matrices with nonzero trace. In particular, this is true if $n = gr + c$, with $0 < c < r$ and $g > r + \sqrt{r^2 - 4r + 5 + 2c}$, as $2n - 4r + 6 < g^2 + 1$ and, by Theorem 4.3, $u_{nr} \geq g^2 + 1$.

**4.2. The case in which $n$ is a multiple of $r$.** Suppose that $n = gr$, for some positive integer $g \geq 2$. Denote by $E_{nr}$ the $n \times n$ matrix given by

$$E_{nr} = H_{2r,r}, \quad \text{if } n = 2r, \tag{4.8}$$

$$E_{nr} = \begin{bmatrix} 0 & J_{rr} \\ H_{2r,r} & 0 \end{bmatrix}, \quad \text{if } n = 3r, \tag{4.9}$$

$$E_{nr} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & J_{rr} \\ J_{rr} & 0 & \cdots & 0 & 0 & 0 \\ 0 & J_{rr} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & J_{rr} & 0 & 0 \\ 0 & 0 & \cdots & 0 & H_{2r,r} & 0 \end{bmatrix}, \quad \text{if } n = gr, \text{ with } g \geq 4, \tag{4.10}$$

where

$$H_{2r,r} = \begin{bmatrix} J_{r-1,r-1} & J_{r-1,1} & 0_{r-1,1} & 0_{r-1,r-1} \\ J_{1,r-1} & 0 & 1 & 0_{1,r-1} \\ 0_{1,r-1} & 1 & 0 & J_{1,r-1} \\ 0_{r-1,r-1} & 0_{r-1,1} & J_{r-1,1} & J_{r-1,r-1} \end{bmatrix}.$$

We will show that $u_{2r,r} = exp(E_{2r,2})$. Taking into account the result of some numerical experiments, we also conjecture that, when $n = gr$ for some $g \geq 3$, the matrices $E_{nr}$ achieve the maximum exponent in the set $P_{nr}$. This conjecture is also reinforced by the following observation. Let us say that the exponent of an $n$-by-$n$ $r$-regular matrix $A$ is infinite if $A$ is not primitive. Given $n = gr$, with $g \geq 3$, consider the following cyclic matrix:

$$P_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & J_{rr} \\ J_{rr} & 0 & \cdots & 0 & 0 \\ 0 & J_{rr} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & J_{rr} & 0 \end{bmatrix}$$

which is irreducible but not primitive and, therefore, has infinite exponent. In [3] it was proven that given two $n$-by-$n$ $r$-regular matrices $A$ and $B$, then $B$ can be gotten from $A$ by a sequence of interchanges on 2-by-2 submatrices of $A$:

$$L_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \leftrightarrow I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrix $E_{nr}$ we have constructed has been obtained by applying just one of these interchanges to $P_1$. Notice, however, that not any arbitrary interchange in $P_1$ produces a matrix with maximum exponent.

In particular, our conjecture implies that $u_{nr} < \lfloor \frac{n}{r} \rfloor^2 + 1$. It is worth to point out that Shen [5] proved that $u_{n2} < \lfloor \frac{n}{2} \rfloor^2 + 1$.

Next we show that, if $n = 2r$, then $u_{nr} = \frac{n(n-r)}{2r^2} + 2 = 3$.

THEOREM 4.5. *Let $r \geq 2$. Then, $u_{2r,r} = 3$.*

*Proof.* Let $A \in P_{2r,r}$ and suppose that $exp(A) > 3$. Then, there must exist a zero entry in $A^3$. Without loss of generality, we can assume that $A^3(1,i) = 0$ for some $i \in \{1, ..., n\}$. Applying Lemma 2.2, we deduce that there must be at least $r$ zero entries in the first row of $A^2$. Without loss of generality, we can assume that one of the next cases holds.

**Case 1.** Suppose that $A(1,:) = [J_{1r} \quad 0_{1r}]$. Then, for $A$ to have exponent larger than 3, $M(A^2)(1,:) = [J_{1r} \quad 0_{1r}]$. Taking into account the position of the zeros in the first row of $A^2$, we deduce that

$$A = \begin{bmatrix} J_{rr} & 0_{rr} \\ 0_{rr} & J_{rr} \end{bmatrix},$$

which is a reducible matrix.

**Case 2.** Suppose that $A(1,:) = [0 \quad J_{1r} \quad 0_{1,r-1}]$. If $A^2(1,1) = 0$ or $A^2(1,i) = 0$ for some $i \geq r+1$, then $A$ would not be $r$-regular. Therefore, for $A$ to have exponent larger than 3, $M(A^2)(1,:) = [1 \quad 0_{1r} \quad J_{1,r-1}]$. Then,

$$A = \begin{bmatrix} 0 & J_{1r} & 0_{1,r-1} \\ J_{r1} & 0_{rr} & J_{r,r-1} \\ 0_{r-1,1} & J_{r-1,r} & 0_{r-1,r-1} \end{bmatrix},$$

which is reducible.

In both cases, we get a contradiction. Thus, for any $A \in P_{2r,r}$, $exp(A) \leq 3$. Since $E^2_{2r,r}$ is not positive, then $exp(E_{2r,r}) = 3 = u_{2r,r}$. □

Next we give the exponent of the matrices $E_{nr}$ when $n = gr$ for some positive integer $g \geq 3$. Before we prove the result, we include a preliminary result.

Let $a_1, a_2, ..., a_p$ be positive integers such that $gcd(a_1, .., a_p) = 1$. The Frobenius-Schur index, $\phi(a_1, ..., a_p)$, is the smallest integer such that the equation $x_1 a_1 + ... + x_p a_p = l$ has a solution in nonnegative integers $x_1, x_2, ..., x_p$ for all $l \geq \phi(a_1, ..., a_p)$. The following result is due to Brauer in 1942.

Proposition 4.6. *[1] Let $y$ be a positive integer. Then*

$$\phi(y, y+1, ..., y+j-1) = y \left\lfloor \frac{y+j-3}{j-1} \right\rfloor .$$

Lemma 4.7. *Let $y > 1$ be a positive integer. Then,*

$$\phi(y, y+1, y+2) = \begin{cases} \frac{1}{2}y^2, & \text{if } y \text{ is even} \\ \frac{1}{2}(y-1)y, & \text{if } y \text{ is odd.} \end{cases}$$

*Moreover, there are nonnegative integers $a, b, c$ satisfying $\phi(y, y+1, y+2) - 2 = ay + b(y+1) + c(y+2)$ if and only if $y$ is even. If $y$ is odd, there are nonnegative integers $a, b, c$ satisfying $\phi(y, y+1, y+2) - 3 = ay + b(y+1) + c(y+2)$.*

*Proof.* The first claim follows from Proposition 4.6. Now we show the second claim. Clearly, if $y$ is even, $\phi(y, y+1, y+2) - 2 = (\frac{y}{2} - 1)(y+2)$ can be written as $ay + b(y+1) + c(y+2)$ for some nonnegative numbers $a, b, c$. If $y$ is odd

$$\phi(y, y+1, y+2) - 3 = \frac{1}{2}(y-1)y - 3 = \left(\frac{y-1}{2} - 1\right)(y+2).$$

which implies that $\phi(y, y+1, y+2) - 3$ can be written as $ay + b(y+1) + c(y+2)$ for some nonnegative integers $a, b, c$. To see that there are no nonnegative integers $a, b, c$ such that

$$\phi(y, y+1, y+2) - 2 = ay + b(y+1) + c(y+2),$$

notice that the largest number of the form $ay+b(y+1)+c(y+2)$, for some nonnegative integers $a, b, c$, smaller than $\phi(y, y+1, y+2)$ is $\left(\frac{y-1}{2} - 1\right)(y+2)$ and

$$\left(\frac{y-1}{2} - 1\right)(y+2) < \left(\frac{y-1}{2} - 1\right)(y+2) + 3 - 2 = \phi(y, y+1, y+2) - 2.$$

□

Theorem 4.8. *Let $n = gr$, with $g \geq 3$ and $r \geq 2$. Then,*

$$\exp(E_{nr}) = \begin{cases} \frac{n(n-r)}{2r^2} + 2, & \text{if } \frac{n}{r} \text{ is even} \\ \frac{1}{2}\left(\left(\frac{n}{r}\right)^2 + 1\right), & \text{if } \frac{n}{r} \text{ is odd.} \end{cases}$$

*Proof.* Consider the digraph $G$ associated with $E_{nr}$. We group the vertices of $G$ in the following way: for $i = 1, \ldots, g$, we call block $B_i$ the set of vertices from $(g-i)r+1$ to $(g-i+1)r$. For convenience, we denote the vertices $n-3r+1, \ldots, n-2r$ in $B_3$ by $w_1, \ldots, w_r$, resp; the vertices $n-2r+1, \ldots, n-r$ in $B_2$ by $v_1, \ldots, v_r$, resp., and the vertices $n-r+1, \ldots, n$ in $B_1$ by $u_1, \ldots, u_r$, resp. Let $B_1' = \{u_2, \ldots, u_r\}$, $B_2' = \{v_2, \ldots, v_{r-1}\}$ and $B_3' = \{w_1, \ldots, w_{r-1}\}$. Note that $B_2'$ is empty if $r = 2$. The digraph $G$ is given in figure 4.1.

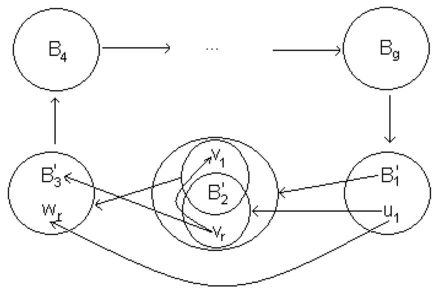


Fig. 4.1.

A directed edge in this graph from a set $S_1$ to a set $S_2$ means that there is an arc from each vertex in $S_1$ to each vertex in $S_2$.

Let $G'$ be the subgragh of $G$ induced by the vertices in $B_1 \cup B_2 \cup B_3$. The following table gives the possible lengths of a walk in $G'$ from a vertex in $B_1$ to a vertex in $B_3$.

| From | To | Possible lengths |
|---|---|---|
| $u_1$ | any vertex in $B_3'$ | $2, 3$ |
| $u_1$ | $w_r$ | $1, 2$ (if $r > 2$), $3$ |
| any vertex in $B_1'$ | any vertex in $B_3'$ | $2, 3$ |
| any vertex in $B_1'$ | $w_r$ | $2, 3$ |

Table 1.

Thus, for any $i \in \{1, \ldots, g\} \backslash \{2\}$, any walk in $G$ from a vertex $u \in B_i$ to a vertex $v \in B_i$ has length $t$ if and only if

$$t = a\,[(g-2)+1] + b\,[(g-2)+2] + c\,[(g-2)+3]\,, \tag{4.11}$$

for some nonnegative integers $a, b, c$, with $b + c > 0$ if either $u \in B_1'$ or $v \in B_3'$.

Taking into account Lemma 4.7, the smallest nonnegative integer $t_0$ such that, for any $t \geq t_0$, (4.11) holds for some nonnegative integers $a, b, c$ is

$$t_0 = \begin{cases} \frac{1}{2}(g-1)^2, & \text{if } g \text{ is odd} \\ \frac{1}{2}(g-2)(g-1), & \text{if } g \text{ is even.} \end{cases}$$

We will show that, if $g$ is odd, any two vertices $u, v$ in $G$ are connected by a walk of length $t_0 + g$ but not of length $t_0 + g - 1$; if $g$ is even, any two vertices $u, v$ in $G$ are connected by a walk of length $t_0 + g + 1$ but not of length $t_0 + g$. Denote by $d(u, v)$ the distance from the vertex $u$ to the vertex $v$. Clearly, $d(u, v) \leq g$.

If $u, v \in B_i$ for some $i \in \{1, \ldots, g\} \backslash \{2\}$, with $u = u_1$ if $i = 1$, and $v = w_r$ if $i = 3$, then, for any $t \geq t_0$, there is a walk of length $t$ from $u$ to $v$.

Suppose that $u, v \in B_2$. Clearly, there is a walk of length 1 from $u$ to some vertex in $B_3$. Also, there is a vertex $v'$ in $B_1$ such that there is a walk of length 1 from $v'$ to $v$. Taking into account these observations, and the fact that, for $t \geq t_0$, there is a walk of length $t$ from any vertex in $B_3$ to $w_r$, it follows that there is a walk of length $t + (g-2) + 2 = t + g$ from $u$ to $v$.

Suppose that $u \in B_1'$ and $v \in B_1$. Notice that there is a walk of length $g$ from $u$ to $u_1$. Since, for $t \geq t_0$, there is a walk of length $t$ from $u_1$ to $v$, it follows that there is a walk of length $t + g$ from $u$ to $v$.

Let $u \in B_3$ and $v \in B_3'$. Then, there is a walk of length $g$ from $w_r$ to $v$. Since, for $t \geq t_0$, there is a walk of length $t$ from $u$ to $w_r$, then there is a walk of length $t + g$ from $u$ to $v$.

Now suppose that $u \in B_i$ and $v \in B_j$, with $i \neq j$.

Suppose that $u \notin B_1' \cup B_2$. Let $w = u$ if $i \neq 3$, and $w = w_r$ otherwise. Then, for $t \geq t_0$, since $g - d(w, v) > 0$, $t + g - d(w, v) \geq t_0$ and there is a walk of length $t + g - d(w, v)$ from $u$ to $w$. This implies that there is a walk of length $t + g$ from $u$ to $v$.

Suppose that $u \in B_1'$ and $v \notin B_2 \cup B_3$. Note that $d(w_r, v) \leq g - 2$. Also, there is a walk of length 2 from $u$ to $w_r$. As, for $t \geq t_0$, $w_r$ lies on a closed walk of length $t+g-d(w_r, v)-2$, then there is a walk of length $2+(t+g-d(w_r, v)-2)+d(w_r, v) = t+g$ from $u$ to $v$.

Suppose that $u \in B_2$ and $v \notin B_3'$. Then $d(w_r, v) \leq g - 1$. As, for $t \geq t_0$, there is a walk of lenth $t + g - d(w_r, v) - 1$ from any vertex in $B_3$ to $w_r$, then there is a walk of length $1 + (t + g - d(w_r, v) - 1) + d(w_r, v) = t + g$ from $u$ to $v$.

We have shown that, for any $t \geq t_0$, there is a walk of length $t + g$ from $u$ to $v$, unless either $u \in B_2$ and $v \in B_3'$, or $u \in B_1'$ and $v \in B_2 \cup B_3$.

In order to determine the exponent of $E_{nr}$, we now consider two cases, depending on the parity of $g$.

**Case 1.** Suppose that $g$ is odd. Notice that every walk in $G$ from $v_1$ to $v_r$ of length $t > g$ contains a subgraph which is a walk of length $t - g$ from a vertex in $B_3$ to a vertex in $B_3$. Because there is no walk of length $t_0 - 1$ from a vertex in $B_3$ to a vertex in $B_3$, then there is no walk of length $t_0 + g - 1$ from $v_1$ to $v_r$.

We have already proven that there is a walk of length $t_0 + g$ from any vertex $u$ to any vertex $v$, unless either $u \in B_2$ and $v \in B_3'$, or $u \in B_1'$ and $v \in B_2 \cup B_3$, in which cases there is a walk of length $s_1$ from $u$ to some vertex in $B_3$ and there is a walk of length $s_2$ from some vertex in $B_1$ to $v$, with $s_1 + s_2 = 4$. By Lemma 4.7, there are nonnegative integers $a, b, c$ such that

$$t_0 - 2 = \frac{1}{2}(g-1)^2 - 2 = a(g-1) + bg + c(g+1).$$

Thus, from any vertex in $B_3$, there is a walk to $w_r$ of length $t_0 - 2$, which implies that there is a walk of length $(t_0 - 2) + (g - 2) + 4 = t_0 + g$ from $u$ to $v$. Therefore,

$$\exp(E_{n,r}) = t_0 + g = \frac{1}{2}(g^2 + 1) = \frac{1}{2}\left(\left(\frac{n}{r}\right)^2 + 1\right).$$

**Case 2.** Suppose that $g$ is even. First, consider the case $u \in B_1'$ and $v \in B_3$. Clearly, there is a walk of length 3 from $u$ to $w_r$; also, there is a walk of length 3 from some vertex in $B_1$ to $v$. Taking into account Lemma 4.7, $w_r$ lies on a closed walk of length $t_0-3$, which implies that there is a walk of length $(t_0-3)+(g-2)+6 = t_0+g+1$ from $u$ to $v$.

Now suppose that either $u \in B_2$ and $v \in B_3'$, or $u \in B_1'$ and $v \in B_2$. Then, there is a walk of length $s_1$ from $u$ to some vertex in $B_3$ and there is a walk of length $s_2$ from some vertex in $B_1$ to $v$, with $s_1 + s_2 = 3$. As, from any vertex in $B_3$, there is a walk of length $t_0$ to $w_r$, then there is a walk of length $t_0 + (g-2) + 3 = t_0 + g + 1$ from $u$ to $v$.

Now we show that there are two vertices not connected by a walk of length $t_0+g$. Note that $t_0 + g > g + 2$. Also, every walk of length $t > g + 2$ from $u \in B_1'$ to $v_r$ contains a subgraph which is a walk of length $t-g-1$ or $t-g-2$ from a vertex in $B_3$ to a vertex in $B_3$. By Lemma 4.7, for $k \in \{1, 2\}$, there are no nonnegative integers such that $t_0 - k = a(g-1) + bg + c(g+1)$. So, there is no walk of length $t_0 + g$ from $u \in B_1'$ to $v_r$.

Thus,

$$\exp(E_{n,r}) = t_0 + g + 1 = \frac{1}{2}(g^2 - g) + 2 = \frac{n(n-r)}{2r^2} + 2.$$

□

If $n = gr$, with $g \geq 3$ and $r \geq 2$, it follows from Theorem 4.8 that $u_{nr} \geq \exp(E_{nr})$. We conjecture that in this case the equality holds. Note that $\exp(E_{nr}) < \lfloor \frac{n}{r} \rfloor^2 + 1$.

Conjecture 1. *Let $n = gr$ with $g \geq 3$ and $r \geq 2$. Then,*

$$u_{nr} = \begin{cases} \frac{n(n-r)}{2r^2} + 2, & \text{if } \frac{n}{r} \text{ is even} \\ \frac{1}{2}\left(\left(\frac{n}{r}\right)^2 + 1\right), & \text{if } \frac{n}{r} \text{ is odd.} \end{cases}$$

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