STRUCTURED STRONG LINEARIZATIONS FROM FIEDLER PENCILS WITH REPETITION II. *

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Abstract. In this paper we give strong linearizations of a matrix polynomial $P(\lambda)$ preserving the skew-symmetry or T-alternating structure of $P(\lambda)$. The linearizations obtained are of the form $SL(\lambda)$, where $L(\lambda)$ is a block-symmetric Fiedler pencil with repetition and S is a direct sum of blocks of the form I or $-I$, with I the identity matrix. This paper is a continuation of [4], where the corresponding problem for $P(\lambda)$ with a symmetric structure was studied and, as a consequence, the block-symmetric Fiedler pencils with repetition were characterized.

Key words. Skew-symmetric linearization, alternating linearization, Fiedler pencils with repetition, matrix polynomial, companion form, polynomial eigenvalue problem.

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1. Introduction. In this paper we consider $n \times n$ matrix polynomials of degree $k \geq 2$ of the form

$$P(\lambda) = A_k\lambda^k + A_{k-1}\lambda^{k-1} + \cdots + A_0, \tag{1.1}$$

where the coefficients A_i are $n \times n$ matrices with entries in a field $\mathbb{F}$.

If $P(\lambda)$ has some structure (palindromic, symmetric, skew-symmetric...), it is important to find strong linearizations that preserve the structure of $P(\lambda)$. In the literature, some structured linearizations of structured matrix polynomials can be found [2, 5, 8, 12, 13, 17, 18, 19].

Due to their simplicity, structured linearizations whose matrix coefficients can be seen as block matrices whose blocks are of the form 0, $\pm I_n$, or $\pm A_i$ are of particular interest. In previous work [4, 5], we constructed such type of linearizations from the family of Fiedler pencils with repetition (FPR) preserving the palindromic (in case the matrix polynomial has odd degree) and the symmetric structure of the matrix polynomial $P(\lambda)$.

In this paper we assume that $P(\lambda)$ is a skew-symmetric (resp. T-even, T-odd) matrix polynomial and construct a family of skew-symmetric (resp. T-even, T-odd) pencils from the FPR. These pencils are of the form $SL(\lambda)$, where $L(\lambda)$ is a block-symmetric Fiedler pencil with repetition and S is a direct sum of blocks of the form I_n or $-I_n$, where I_n denotes the $n \times n$ identity matrix. Although not every pencil of this form is necessarily a strong linearization of $P(\lambda)$, we give conditions on $L(\lambda)$ and $P(\lambda)$ that ensure that $SL(\lambda)$ is a strong linearization of $P(\lambda)$.

This paper is a continuation of [4], where the corresponding problem when $P(\lambda)$ has a symmetric structure was studied. The results obtained in that paper are crucial in the present work. In particular, we will use the theory of tuples that we developed there as well as the characterization of block-symmetric FPR [4, Corollary 5.6]. To keep this paper as concise as possible, we will use here the definitions and results

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in [4] without reproducing them. However, we caution the reader that, in order to understand this second part, it is necessary to be familiar with the content of [4].

From now on we assume that the field $\mathbb{F}$ has characteristic different from 2. We observe that, when $\mathbb{F}$ has characteristic 2, skew-symmetric and T-alternating matrix polynomials are symmetric. However, in this case, the usual definition of a skew-symmetric matrix polynomial $P(\lambda)$ requires that all the diagonal entries of $P(\lambda)$ are zero. Though symmetric linearizations of symmetric matrix polynomials were considered in [4], this additional condition requires a special approach.

Under the assumption that the characteristic of $\mathbb{F}$ is 2, we now define skew-symmetric and T-alternating matrix polynomials.

Definition 1.1. *A matrix polynomial $P(\lambda)$ as in (1.1) is said to be skew-symmetric if $P^T(\lambda) = -P(\lambda)$ or, alternatively, $A_i^T = -A_i$, $i = 0, ..., k$.*

Definition 1.2. *Let $P(\lambda)$ be a matrix polynomial.*

1. *$P(\lambda)$ is said to be* T-even *if $P(-\lambda) = P(\lambda)^T$ or, alternatively, $A_i^T = (-1)^i A_i$, $i = 0, \ldots, k$.*
2. *$P(\lambda)$ is said to be* T-odd *if $P(-\lambda) = -P(\lambda)^T$ or, alternatively, $A_i^T = (-1)^{i+1} A_i$, $i = 0, \ldots, k$.*
3. *$P(\lambda)$ is said to be* T-alternating *if it is either T-even or T-odd.*

We say that two T-alternating matrix polynomials have the same parity *if they are both T-even or both T-odd.*

The existence and construction of skew-symmetric linearizations for skew-symmetric matrix polynomials have been considered in [2, 7, 19]. There exist skew-symmetric strong linearizations for all skew-symmetric matrix polynomials of odd degree [19, Lemma 6.11] as well as for skew-symmetric matrix polynomials of even degree when the size n is even [7, Theorem 7.22]. In particular, there are skew-symmetric strong linearizations for all regular skew-symmetric matrix polynomials of even degree, as they necessarily have even size. However, when $P(\lambda)$ is skew-symmetric of even degree and odd size, there is no skew-symmetric strong linearization of $P(\lambda)$ [7, Theorem 7.22]. Thus, it follows that not all singular skew-symmetric matrix polynomials of even degree have a skew-symmetric linearization. Lemma 6.11 in [19] gives one skew-symmetric strong linearization for each skew-symmetric matrix polynomial of odd degree over an arbitrary field. Also, skew-symmetric strong linearizations for regular skew-symmetric matrix polynomials of even (or odd) degree are constructed from the pencils in the vector space $\mathbb{DL}(P)$ introduced in [16]. Note that all the pencils in $\mathbb{DL}(P)$ are block-symmetric. Moreover, they are skew-symmetric if $P(\lambda)$ is. In [2], the authors also construct an example of a skew-symmetric linearization for regular skew-symmetric matrix polynomials.

In this paper, we construct a family of pencils strictly equivalent to block-symmetric FPR that are skew-symmetric when the matrix polynomial $P(\lambda)$ is. We show that under some conditions on the leading and constant matrix coefficients of $P(\lambda)$, the pencil is a strong linearization of $P(\lambda)$. We also prove that the only FPR that are skew-symmetric when $P(\lambda)$ is are those forming the standard basis for $\mathbb{DL}(P)$ introduced in [12]. We extend this family of skew-symmetric FPR by multiplying some block-symmetric FPR by a block-diagonal matrix whose main diagonal blocks are $\pm I_n$. Example 3.16 here gives all the 22 distinct skew-symmetric pencils from FPR that we obtain when $k = 5$. This family contains the 5 pencils in the standard basis for $\mathbb{DL}(P)$, and its matrix coefficients are block-matrices whose blocks are 0, $\pm I_n$, or $\pm A_i$.

The problem of giving necessary and sufficient conditions for a T-alternating matrix polynomial to have a T-alternating strong linearization was considered in [20]. There, the authors give one strong linearization for each matrix polynomial $P(\lambda)$ of odd degree. They also study the even degree case but no examples of linearizations are provided. Also, in [2], an example of a T-alternating linearization analogous to the one in [17] is presented. Here, we construct a family of pencils strictly equivalent to block-symmetric FPR that are T-odd (resp. T-even) when the matrix polynomial is. Again, under some conditions, these pencils are strong linearizations of $P(\lambda)$. Note that the linearizations that we construct do not appear in [2] and [20]. In fact, the linearizations in those papers do not allow repetitions of the coefficients A_i in the matrix coefficients of the T-alternating pencils while our pencils present such repetitions. (See Example 4.15). Note also that, for each T-alternating matrix polynomial of degree k, at least when A_0 and A_k are invertible, our family provides more than one single linearization with the same parity as the matrix polynomial. For example, for $k = 5$, we obtain 10 distinct T-even strong linearizations for a T-even matrix polynomial, assuming that A_0 and A_5 are invertible.

In summary, the structured linearizations from FPR that we give in this paper are easily constructed from the coefficients of the matrix polynomial and significantly enlarge the known families of linearizations preserving the skew-symmetric (resp. T-alternating) structure of a matrix polynomial $P(\lambda)$. Also, they are obtained over any field with characteristic different from 2. A natural question to be studied in future works is if our family contains linearizations with any relevant advantages over the previously known structured linearizations.

The paper is organized as follows. In Section 2 we introduce some notation and give results that will be used in the next two sections. None of the results there depend on the assumption that the matrix polynomial has some structure (skew-symmetric or T-alternating). In Section 3 we construct a family of strong linearizations preserving the skew-symmetric structure of $P(\lambda)$. The main result in that section is Theorem 3.15. In Section 4 we give a family of strong linearizations preserving the T-alternating (and parity) structure of $P(\lambda)$. The main result there is Theorem 4.14. We close the paper with Section 5, where a summary of the main results obtained in the paper is given.

2. General notation and definitions. The following notation, introduced in [4], will be used throughout the paper.

Given the tuple $\mathbf{t} = (a : b)$, we denote by $\mathbf{t}_{rev_c}$ the reverse-complement of $\mathbf{t}$, which was introduced in [4, Definition 2.4].

If $\mathbf{t}_1$ and $\mathbf{t}_2$ are two index tuples, we use the notation $\mathbf{t}_1 \sim \mathbf{t}_2$ to denote that the two tuples are equivalent [4, Definition 2.9].

A key concept in the development of skew-symmetric and T-alternating strong linearizations is the Successor Infix Property (SIP) presented in [4, Definition 2.16]. Given a tuple $\mathbf{t}$ with indices from $\{0 : h\}$ satisfying the SIP, we denote by $csf(\mathbf{t})$ the column standard form of $\mathbf{t}$, given in [4, Definition 2.18], which has the form

$$(a_s : t_s, a_{s-1} : t_{s-1}, \ldots, a_2 : t_2, a_1 : t_1, a_0 : t_0)\,,$$

with $h \geq t_s > t_{s-1} > \cdots > t_2 > t_1 > t_0 \geq 0$ and $0 \leq a_j \leq t_j$, for all $j = 0, 1, \ldots, s$. We call the indices $t_0, t_1, \ldots, t_s$ the *end points of* $\mathbf{t}$. Each subtuple of consecutive integers $(a_i : t_i)$ is called a string of $csf(\mathbf{t})$.

In addition to the notation and definitions introduced in [4], we will need the following.

If i, j, p are integers, with $p, j \geq 0$ and $j \equiv 0 \bmod p$, we denote by $i :_p i + j$ the sequence $i, i + p, i + 2p, \ldots, i + j$. If $p = 1$, we write $:$ for $:_1$. If $j < 0$, $i :_p i + j$ is empty.

Given a set T of integers and an integer a, we denote by $a \pm T$ the set obtained from T by adding to (subtracting from) a each element of T.

If J and H are nonempty sets of integers, we denote by $H \ominus J$ the symmetric difference of J and H, that is, $H \ominus J = (H - J) \cup (J - H)$.

Given an $nk \times nk$ matrix A, viewed as a $k \times k$ partitioned matrix with blocks of size $n \times n$, and a set $J \subseteq \{1 : k\}$, we denote by $A[J]$ the principal submatrix of A lying in the block-rows indexed by J. By $A(i, j)$ we denote the block of A in position (i, j). If $i = j$, we also write $A[i]$ for $A(i, i)$.

We will use $\star$ to denote an unspecified block of appropriate size in a block-matrix.

We denote by R the block-involutory matrix

$$R := \begin{bmatrix} 0 & \ldots & I_n \\ \vdots & \iddots & \vdots \\ I_n & \ldots & 0 \end{bmatrix}. \tag{2.1}$$

2.1. Block-signature matrices. The block-signature matrices, which we now define, will play a crucial role in our work.

Definition 2.1. *We say that $S \in M_{nk}$ is a block-signature matrix if $S = \epsilon_1 I_n \oplus \cdots \oplus \epsilon_k I_n$ for some $\epsilon_i \in \{1, -1\}$, $i = 1 : k$. We call $\epsilon_1, \ldots, \epsilon_k$* the parameters *of S. We denote by S_i the block-signature matrix such that $\epsilon_i = -1$ and $\epsilon_j = 1$ for $j \neq i$. By S_0, S_{k+1} and $S_\emptyset$ we denote the identity matrix I_{nk}. If $Z = \{i_1, ..., i_r\}$ is a subset of $\{1 : k\}$, then $S_Z := S_{i_1} \cdots S_{i_r}$.*

Note that a block-signature matrix S is exactly the product of the matrices S_i for which ϵ_i is a negative parameter of S.

The next result is immediate and will be used without comment. The matrices M_j in the statement are those defined in [4, Section 4.1].

Proposition 2.2. *The following holds:*

1. $S_{k-i} M_j = M_j S_{k-i}$, *for any* $i, j \in \{1 : k - 1\}$ *with* $i \neq j, j - 1$;
2. $S_{k-i} M_0 = M_0 S_{k-i}$ *and* $S_{k-i} M_{-k} = M_{-k} S_{k-i}$ *for any* $i \in \{1 : k\}$;
3. $S_{k-i} S_{k-i+1} M_i = M_i S_{k-i} S_{k-i+1}$ *for any* $i \in \{1 : k - 1\}$.

Remark 2.3. *If H and J are subsets of $\{1 : k\}$, the $nk \times nk$ matrix $S_{H \ominus J}$ is given by $S_H S_J$.*

2.2. Products of the matrices M_i for admissible tuples. The matrices M_i, defined in [4, Section 4.1], are the elementary matrices used in the construction of a FPR associated with a matrix polynomial $P(\lambda)$ as in (1.1), which is a pencil of the form

$$L(\lambda) = \lambda M_{\mathbf{l}_q, \mathbf{l}_z, \mathbf{z}, \mathbf{r}_z, \mathbf{r}_q} - M_{\mathbf{l}_q, \mathbf{l}_z, \mathbf{q}, \mathbf{r}_z, \mathbf{r}_q}, \tag{2.2}$$

where $h \in \{0 : k - 1\}$, $\mathbf{q}$ and $\mathbf{z}$ are permutations of $\{0 : h\}$ and $\{-k : -h - 1\}$, respectively, $\mathbf{l}_q$ and $\mathbf{r}_q$ are tuples with indices from $\{0 : h - 1\}$ such that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ satisfies the SIP, and $\mathbf{l}_z$ and $\mathbf{r}_z$ are tuples with indices from $\{-k : -h - 2\}$ such that $(\mathbf{l}_z, \mathbf{z}, \mathbf{r}_z)$ satisfies the SIP.

In what follows we will be working with products of the matrices M_i. The next observation will be useful in some of our proofs.

REMARK 2.4. *We observe that, by Lemmas 5.1 and 5.5 in [4], if* $\mathbf{s} = (s_1, \ldots, s_k)$ *is a symmetric tuple satisfying the SIP (see Definitions 2.16 and 3.1 in [4]) and* $P(\lambda)$ *is a matrix polynomial of the form (1.1), then* $M_{\mathbf{s}}(P)$ *is block-symmetric. In particular, if* $\mathbf{w}$ *is an admissible tuple and* $\mathbf{r}_w$ *is the symmetric complement of* $\mathbf{w}$ *(see Definitions 3.6 and 3.8 in [4]) then, because of Lemma 3.11 in [4], both* $M_{\mathbf{w},\mathbf{r}_w}(P)$ *and* $M_{\mathbf{r}_w}(P)$ *are block-symmetric.*

We now make some observations that will be useful in the proofs of Lemma 2.10 and Theorem 4.1.

Let $\mathbf{s}$ be a tuple with indices from $\{0 : k-1\}$ satisfying the SIP. Because of the SIP, all blocks in $M_{\mathbf{s}}$ are 0, I_n or $-A_i$, where i is an index in $\mathbf{s}$ [21]. Notice that the matrix $M_{\mathbf{s}}$ does not have a block-row or a block-column which is zero for every matrix polynomial, as otherwise it would be singular for every matrix polynomial $P(\lambda)$, which cannot occur because the matrices M_i, $i \neq 0$, are nonsingular independently of A_i, and M_0 is nonsingular if A_0 is. From this observation, taking into account the way the product of matrices is performed and noting that no cancellation can occur, it follows that $M_{\mathbf{s}}$ contains at least one block $-A_i$, for each index i in $\mathbf{s}$.

We now give some results regarding the products $M_{\mathbf{w},\mathbf{r}_w}$ and $M_{\mathbf{r}_w}$, where $\mathbf{w}$ is an admissible tuple and $\mathbf{r}_w$ is the symmetric complement of $\mathbf{w}$. These results will be useful when getting our structured linearizations. In particular, they will allow us to show the uniqueness (up to multiplication by -1) of the block-signature matrix S such that $SL(\lambda)$ has the same structure as the matrix polynomial $P(\lambda)$, where $L(\lambda)$ is a given block-symmetric FPR. In order to state these results, we need to introduce the following notation.

Let $\mathbf{s}$ be a symmetric tuple with indices from $\{0 : k-1\}$ satisfying the SIP. Since $M_{\mathbf{s}}$ is block-symmetric for any matrix polynomial $P(\lambda)$ of degree k, we can construct the undirected graph with vertex set $\{1 : k\}$ and an edge between i and j if and only if the block in position (i, j) in $M_{\mathbf{s}}(P)$ is nonzero for some matrix polynomial $P(\lambda)$ of degree k. We denote this graph by $G(\mathbf{s})$.

LEMMA 2.5. *Let* $\mathbf{w}$ *be an admissible tuple with indices from* $\{0 : h\}$, $0 \leq h < k$, *and* $\mathbf{r}_w$ *be the symmetric complement of* $\mathbf{w}$. *Then each connected component of* $G(\mathbf{r}_w)$ *and* $G(\mathbf{w}, \mathbf{r}_w)$ *contains a loop.*

Proof. We start by proving the result for $G(\mathbf{r}_w)$. If l is the index of $\mathbf{w}$, then $\mathbf{r}_w = (s_{(h-l)/2}, \ldots, s_1, \mathbf{s}_0)$, where $\mathbf{s}_0 = (0 : l)_{rev_c}$ and $s_i = l + 2i - 1$, for $i > 0$. Let G_i be the subgraph of $G(\mathbf{r}_w)$ with vertex set $\{k - s_i, k - s_i + 1\}$, for $i = 1 : (h-l)/2$, and G_0 be the subgraph of $G(\mathbf{r}_w)$ with vertex set $\{k - l + 1 : k\}$ if $l > 0$. The matrix $M_{\mathbf{r}_w}$ is the direct sum

$$I_{n(k-h)} \oplus \begin{bmatrix} -A_{s_{(h-l)/2}} & I_n \\ I_n & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} -A_{s_1} & I_n \\ I_n & 0 \end{bmatrix} \oplus T,$$

where the block T has the form

$$\begin{bmatrix} -A_{l-1} & -A_{l-2} & \cdots & -A_1 & -A_0 \\ -A_{l-2} & -A_{l-3} & \cdots & -A_0 & 0 \\ -A_{l-3} & -A_{l-4} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -A_0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{2.3}$$

(T is empty if $l = 0$.) Thus, each subgraph G_i is a connected component of $G(\mathbf{r}_w)$ and contains a loop. Moreover, for any $j \in \{1 : k-h\}$, the subgraph of $G(\mathbf{r}_w)$ with vertex set $\{j\}$ is a connected component of $G(\mathbf{r}_w)$ and contains a loop.

Next we prove the result for $G(\mathbf{w},\mathbf{r}_w)$. Clearly, for any $j\in\{1:k-h-1\}$, the subgraph of $G(\mathbf{w},\mathbf{r}_w)$ with vertex set $\{j\}$ is a connected component of $G(\mathbf{w},\mathbf{r}_w)$ and contains a loop. Thus, it is enough to show that the subgraph of $G(\mathbf{w},\mathbf{r}_w)$ with vertex set $\{k-h:k\}$ is connected and contains a loop. We prove the first claim by induction on $h-l$. If $h-l=0$, that is, $\mathbf{w}=(0:h)$, then $M_{(\mathbf{w},\mathbf{r}_w)}$ has the form $I_{n(k-h-1)}\oplus T$, where T has the form (2.3), with l replaced by $l+1$, and the result follows. Now suppose that $h>l$ and $csf(\mathbf{w})=(\mathbf{s}_{(h-l)/2},\mathbf{w}')$, where $\mathbf{w}'=(\mathbf{s}_{(h-l)/2-1},\ldots,\mathbf{s}_0)$, $\mathbf{s}_0=(0:l)$, and $\mathbf{s}_i=(l+2i-1,l+2i)$ for $i>0$. Let $\mathbf{r}_{w'}$ be the symmetric complement of $\mathbf{w}'$. Note that $(\mathbf{w},\mathbf{r}_w)\sim(h-1:h,\mathbf{w}',\mathbf{r}_{w'},h-1)$. The matrix $M_{\mathbf{w}',\mathbf{r}_{w'}}$ has the form

$$I_{n(k-h-1)}\oplus\begin{bmatrix} I_n & 0 & 0 & 0\\ 0 & I_n & 0 & 0\\ 0 & 0 & -A_{h-2} & R_1\\ 0 & 0 & R_1' & R_2\end{bmatrix},$$

for some matrices R_1,R_1',R_2 of appropriate sizes. By the induction hypothesis, the subgraph of $G(\mathbf{w}',\mathbf{r}_{w'})$ with vertex set $\{k-h+2:k\}$ is connected and contains a loop. In particular, because, by Remark 2.6, $M_{\mathbf{w}',\mathbf{r}_{w'}}$ is block-symmetric, this implies that R_1 and R_1' are nonzero. Note that the principal submatrix M' of $M_{\mathbf{w},\mathbf{r}_w}$ lying on the block-rows $k-h:k$ is given by

$$\begin{bmatrix} -A_h & I_n & 0 & 0\\ -A_{h-1} & 0 & I_n & 0\\ I_n & 0 & 0 & 0\\ 0 & 0 & 0 & I_{n(h-2)}\end{bmatrix}\begin{bmatrix} I_n & 0 & 0 & 0\\ 0 & I_n & 0 & 0\\ 0 & 0 & -A_{h-2} & R_1\\ 0 & 0 & R_1' & R_2\end{bmatrix}\begin{bmatrix} I_n & 0 & 0 & 0\\ 0 & -A_{h-1} & I_n & 0\\ 0 & I_n & 0 & 0\\ 0 & 0 & 0 & I_{n(h-2)}\end{bmatrix}$$
$$=\begin{bmatrix} -A_h & -A_{h-1} & I_n & 0\\ -A_{h-1} & -A_{h-2} & 0 & R_1\\ I_n & 0 & 0 & 0\\ 0 & R_1' & 0 & R_2\end{bmatrix}.$$

Clearly, the subgraph of $G(\mathbf{w},\mathbf{r}_w)$ with vertex set $\{k-h:k-h+2\}$ is connected. Thus, taking into account that M' is block-symmetric, it is enough to show that for any i in $\{k-h+3:k\}$ and any j in $\{k-h:k\}$, there is a path between i and j. Let $i\in\{k-h+3:k\}$.

If $j\in\{k-h+3:k\}$, by the induction hypothesis there is a path p in $G(\mathbf{w}',\mathbf{r}_{w'})$ between i and j. Clearly, this path just involves vertices in $\{k-h+2:k\}$. Since the submatrix of $M_{\mathbf{w}',\mathbf{r}_{w'}}$ in block-rows $k-h+2:k$ coincides with the submatrix of $M_{\mathbf{w},\mathbf{r}_w}$ in block-rows $k-h+1,k-h+3:k$, the path obtained from p by replacing the vertex $k-h+2$ by $k-h+1$, in case $k-h+2$ is a vertex in p, is a path between i and j in $G(\mathbf{w},\mathbf{r}_w)$.

Suppose that $j\in\{k-h:k-h+2\}$. By the induction hypothesis there is a path q in $G(\mathbf{w}',\mathbf{r}_{w'})$ between i and $k-h+2$. Replacing the vertex $k-h+2$ in q by $k-h+1$, we obtain a path in $G(\mathbf{w},\mathbf{r}_w)$ between i and $k-h+1$. Since in $G(\mathbf{w},\mathbf{r}_w)$ there is a path between $k-h+1$ and j, it follows that there is a path between i and j in $G(\mathbf{w},\mathbf{r}_w)$.

Clearly, the vertex $k-h$ of $G(\mathbf{w},\mathbf{r}_w)$ has a loop and the result follows. $\square$

The following observation follows from the proof of Lemma 2.7.

Remark 2.6. *Let* $\mathbf{w}$ *be an admissible tuple relative to* $\{0:h\}, 0\le h<k$, *with index* l. *The connected components of* $G(\mathbf{w},\mathbf{r}_w)$ *are the subgraphs of* $G(\mathbf{w},\mathbf{r}_w)$ *with vertex set* $\{k-h:k\}$ *and* $\{i\}$, $i=1:k-h-1$. *The connected components of* $G(\mathbf{r}_w)$ *are the subgraphs of* $G(\mathbf{r}_w)$ *with vertex set* $\{i\}$, $i=1:k-h$, $\{k-h+2i-1,k-h+2i\}$, $i=1:(h-l)/2$, *and* $\{k-l+1:k\}$ *if* $l>0$. *In particular, the set of vertices of any* *connected component of* $G(\mathbf{r}_w)$ *is contained in the set of vertices of some connected component of* $G(\mathbf{w}, \mathbf{r}_w)$.

COROLLARY 2.7. *Let* $\mathbf{w}$ *be an admissible tuple with indices from* $\{0 : h\}, 0 \leq h < k$, $\mathbf{r}_w$ *be the symmetric complement of* $\mathbf{w}$, *and* $\mathbf{t}$ *be a tuple (possibly empty) with indices from* $\{0 : h-1\}$ *such that* $(rev(\mathbf{t}), \mathbf{w}, \mathbf{r}_w, \mathbf{t})$ *satisfies the SIP. Then the connected components of* $G(rev(\mathbf{t}), \mathbf{w}, \mathbf{r}_w, \mathbf{t})$ *correspond to the same sets of vertices as the connected components of* $G(\mathbf{w}, \mathbf{r}_w)$ *and each of them contains a loop.*

Proof. We prove the result by induction on the length of $\mathbf{t}$. If $\mathbf{t}$ is the empty tuple then the result follows from Lemma 2.7. Now suppose that $\mathbf{t} = (\mathbf{t}', t)$, where $\mathbf{t}'$ is a tuple with indices from $\{0 : h-1\}$, and $(rev(\mathbf{t}), \mathbf{w}, \mathbf{r}_w, \mathbf{t})$ satisfies the SIP. Let $\mathbf{s} = (rev(\mathbf{t}'), \mathbf{w}, \mathbf{r}_w, \mathbf{t}')$. By the induction hypothesis and Remark 2.8, the connected components of $G(\mathbf{s})$ are the subgraphs of $G(\mathbf{s})$ with vertex set $\{k-h : k\}$ and $\{i\}$, $i = 1 : k-h-1$, and each contains a loop. By computing the product $M_{t,\mathbf{s},t}$ and taking into account that no cancellations occur, it follows that the graphs $G(t, \mathbf{s}, t)$ and $G(\mathbf{s})$ coincide if $t = 0$ and, otherwise, $G(t, \mathbf{s}, t)$ is obtained from the graph of $G(\mathbf{s})$ by interchanging the roles of the vertices $k-t$ and $k-t+1$ and, possibly, by adding some more edges connecting the vertices $k-t$, $k-t+1$ to other vertices in $\{k-h : k\}$. Thus the claim follows. □

In order to state the next lemma in a general way, we denote by $\mathcal{C}$ an arbitrary subclass of matrix polynomials of the form (1.1) with degree k such that, for any $l \in \{0 : k\}$, there is $P(\lambda) \in \mathcal{C}$ with the coefficient A_l nonzero. We note that an example of such a class $\mathcal{C}$ is the class of symmetric (resp. skew-symmetric, T-odd, T-even, or T-alternating) matrix polynomials of degree k.

LEMMA 2.8. *Let* $\mathbf{w}$ *be an admissible tuple with indices from* $\{0 : h\}, 0 \leq h < k$, $\mathbf{r}_w$ *be the symmetric complement of* $\mathbf{w}$, *and* $\mathbf{t}$ *be a tuple (possibly empty) with indices from* $\{0 : h-1\}$ *so that* $(rev(\mathbf{t}), \mathbf{w}, \mathbf{r}_w, \mathbf{t})$ *satisfies the SIP. Let either* $\mathbf{s} = (rev(\mathbf{t}), \mathbf{w}, \mathbf{r}_w, \mathbf{t})$ *or* $\mathbf{s} = \mathbf{r}_w$. *Let* S_1, S_1', S_2, S_2' *be* $nk \times nk$ *block-signature matrices. Suppose that* $S_1 M_{\mathbf{s}}^T S_1' = M_{\mathbf{s}}$ *for any matrix polynomial* $P(\lambda) \in \mathcal{C}$. *Then* $S_2 M_{\mathbf{s}}^T S_2' = M_{\mathbf{s}}$ *for any matrix polynomial* $P(\lambda) \in \mathcal{C}$ *if and only if* $S_1 S_1' = S_2 S_2'$ *and* $(S_1 S_2)[J] = \pm I$, *for any set* J *of vertices of a connected component of* $G(\mathbf{s})$.

Proof. Suppose that $M_{\mathbf{s}} = S_1 M_{\mathbf{s}}^T S_1' = S_2 M_{\mathbf{s}}^T S_2'$ for any matrix polynomial $P(\lambda) \in \mathcal{C}$. Then $S_1 S_2 M_{\mathbf{s}}^T S_1' S_2' = M_{\mathbf{s}}^T$. Let $S = S_1 S_2$ and $S' = S_1' S_2'$. To prove that $S_1 S_1' = S_2 S_2'$, and because $\mathbf{s}$ is symmetric, it is enough to show that

$$S M_{\mathbf{s}} S' = M_{\mathbf{s}} \tag{2.4}$$

implies $SS' = I_{nk}$. As already observed, the matrix $M_{\mathbf{s}}$ does not have a block-row which is zero for every matrix polynomial $P(\lambda)$. Thus, for any $i = 1 : k$, there exists j_i such that $M_{\mathbf{s}}(i, j_i)$ is nonzero for some $P(\lambda) \in \mathcal{C}$. Note that we can have $j_i = i$. Since, by Remark 2.6, $M_{\mathbf{s}}$ is block-symmetric, the blocks in positions (i, j_i) and (j_i, i) are equal. So, from (2.4) we have

$$S(i,i) M_{\mathbf{s}}(i, j_i) S'(j_i, j_i) = M_{\mathbf{s}}(i, j_i) = M_{\mathbf{s}}(j_i, i) = S(j_i, j_i) M_{\mathbf{s}}(i, j_i) S'(i,i)$$

for all $P(\lambda) \in \mathcal{C}$, which implies that

$$S(i,i) S'(j_i, j_i) = S(j_i, j_i) S'(i,i) = I_n. \tag{2.5}$$

In particular, (2.5) implies

$$S(i,i) S'(i,i) = S(j_i, j_i) S'(j_i, j_i) = \pm I_n. \tag{2.6}$$

Note that, from (2.5), if $j_i = i$, we have $S(i,i)S'(i,i) = I_n$. From (2.6), $S(i,i)S'(i,i) = S(j,j)S'(j,j)$ for any vertices i,j in a connected component of $G(s)$. Thus, to prove that each of these products is I_n, it is enough to note that each connected component of G has a loop, which follows from Corollary 2.9. Then, we have $SS' = I_{nk}$, proving the first claim. Because of the previous equality, (2.5) implies

$$S(i,i)S(j_i,j_i) = I_n.$$

Thus, if $i_1 \leftrightarrow \cdots \leftrightarrow i_k$ is a walk containing all vertices in a connected component of $G(s)$, we have

$$S(i_1,i_1)S(i_2,i_2) = \cdots = S(i_{k-1},i_{k-1})S(i_k,i_k) = I_n,$$

which implies that

$$S(i_1,i_1) = S(i_2,i_2) = \cdots = S(i_k,i_k),$$

proving the last claim.

The converse follows because $M_{\mathbf{s}}$ is a direct sum of matrices lying in the rows corresponding to the connected components of $G(\mathbf{s})$. Note that, from the hypothesis, if J is the set of vertices of a connected component of $G(\mathbf{s})$, then either $S_1[J] = S_2[J]$ and $S_1'[J] = S_2'[J]$ or $S_1[J] = -S_2[J]$ and $S_1'[J] = -S_2'[J]$. □

Observe that, if a solution S_1 and S_1' of the equation $S_1 M_{\mathbf{s}}^T S_1' = M_{\mathbf{s}}$ is known, Lemma 2.10 characterizes all other solutions.

We finish this section with a definition that will be useful in presenting some results in this paper.

Definition 2.9. *Let* $\mathbf{s}$ *be a tuple with indices from* $\{-k : k-1\}$. *We say that* $\mathbf{s}$ *is skew-symmetric (resp.* T*-even,* T*-odd) direct-transpose related if there exists a block-signature matrix* S *such that* $M_{\mathbf{s}}^T = S M_{\mathbf{s}} S$, *for any skew-symmetric (resp.* T*-even,* T*-odd) matrix polynomial* $P(\lambda)$ *of degree* k.

We say that $\mathbf{s}$ *is skew-symmetric (resp.* T*-even,* T*-odd) complement-transpose related if there exist block-signature matrices* S *and* S' *such that* $M_{\mathbf{s}}^T = S' M_{\mathbf{s}} S$, *for any skew-symmetric (resp.* T*-even,* T*-odd) matrix polynomial* $P(\lambda)$ *of degree* k, *and* $S'(i,i) = -S(i,i)$ *for any* $i \in \{1 : k\}$, *unless the block in position* (i,i) *is a direct summand of* $M_{\mathbf{s}}$ *equal to* I_n *for any* $P(\lambda)$ *(where* n *is the size of the matrix polynomial* $P(\lambda)$*).*

If $\mathbf{s}$ is as in Lemma 2.10, we can conclude that $\mathbf{s}$ cannot be simultaneously skew-symmetric (resp. T-even, T-odd) complement-transpose related and direct-transpose related.

3. Skew-symmetric strong linearizations.

In this section we construct strong linearizations of an $n \times n$ matrix polynomial $P(\lambda)$ of degree k of the form (1.1) that are skew-symmetric when $P(\lambda)$ is. These linearizations are obtained from Fiedler pencils with repetition.

3.1. Type 1 tuples.

Here we introduce some definitions that allow us to associate a simple tuple (that is, an index tuple with no repeated indices) to some index tuples with repetitions. This simple tuple has an important role in the construction of the skew-symmetric linearizations.

We recall that, if $\mathbf{q}$ is a permutation of $\{0 : h\}$, then $\mathbf{q}$ can be expressed in column standard form; more precisely,

$$csf(\mathbf{q}) = (t_{s+1}+1 : t_s, t_{s-2}+1 : t_s - 1, \ldots, t_0+1 : t_1, 0 : t_0),$$

with $h = t_s > t_{s-1} > \cdots > t_2 > t_1 > t_0 \geq 0$.

The next three definitions were presented in [5], where examples were also given.

Definition 3.1. *Let* $\mathbf{q}$ *be a permutation of* $\{0 : h\}$, $h \geq 0$. *Let* s *be an index from* $\{0 : h-1\}$. *The index* s *is said to be a* right index of type 1 relative to $\mathbf{q}$ *if there is a string* $(t_{d-1}+1{:}t_d)$ *in* $csf(\mathbf{q})$ *such that* $s = t_{d-1}+1 < t_d$.

Note that if s is a right index of type 1 relative to $\mathbf{q}$, then $(\mathbf{q}, s) \sim (s, \mathbf{q}')$ where $\mathbf{q}'$ is also a simple tuple. This observation justifies the following definition.

Definition 3.2. *Let* $\mathbf{q}$ *be a permutation of* $\{0 : h\}$, $h \geq 0$. *Let* $csf(\mathbf{q}) = (\mathbf{b}_w, \mathbf{b}_{w-1}, \ldots, \mathbf{b}_1)$, *where* $\mathbf{b}_i = (t_{i-1}+1 : t_i)$, $i = 1, \ldots, w$, *are the strings of* $csf(\mathbf{q})$. *We say that the* simple tuple associated with $\mathbf{q}$ *is* $csf(\mathbf{q})$ *and denote it by* $z(\mathbf{q})$. *If* s *is a right index of type 1 relative to* $\mathbf{q}$, *say* $s = t_{d-1}+1 < t_d$, *then we define the* simple tuple associated with $(\mathbf{q}, s)$ *as the simple tuple:*

- $z(\mathbf{q}, s) := \left(\mathbf{b}_w, \mathbf{b}_{w-1}, \ldots, \mathbf{b}_{d+1}, \widetilde{\mathbf{b}}_d, \widetilde{\mathbf{b}}_{d-1}, \mathbf{b}_{d-2}, \ldots, \mathbf{b}_1\right)$, *where*
$$\widetilde{\mathbf{b}}_d = (t_{d-1}+2 : t_d) \qquad \text{and} \qquad \widetilde{\mathbf{b}}_{d-1} = (t_{d-2}+1 : t_{d-1}+1),$$
if $s \neq 0$;
- $z(\mathbf{q}, s) := \left(\mathbf{b}_w, \mathbf{b}_{w-1}, \ldots, \mathbf{b}_d, \ldots, \widetilde{\mathbf{b}}_1, \widetilde{\mathbf{b}}_0\right)$, *where*
$$\widetilde{\mathbf{b}}_1 = (1 : t_1) \qquad \text{and} \qquad \widetilde{\mathbf{b}}_0 = (0),$$
if $s = 0$.

Note that $z(\mathbf{q}, s)$ is in column standard form by construction and is still a permutation of $\{0 : h\}$.

Definition 3.3. *Let* $\mathbf{q}$ *be a permutation of* $\{0 : h\}$, $h \geq 0$, *and* $\mathbf{t} = (s_1, ..., s_r)$ *be a tuple with indices* s_i *from* $\{0 : h-1\}$, *possibly with repetitions. We say that* $\mathbf{t}$ *is a* right index tuple of type 1 *relative to* $\mathbf{q}$ *if, for* $i = 1 : r$, s_i *is a right index of type 1 with respect to* $z(\mathbf{q}, (s_1, ..., s_{i-1}))$, *where* $z(\mathbf{q}, (s_1, ..., s_{i-1})) := z(z(\mathbf{q}, (s_1, ..., s_{i-2})), s_{i-1})$ *for* $i > 2$.

Remark 3.4. *If* $\mathbf{t}$ *is a right index tuple of type 1 relative to* $\mathbf{q}$, *then* $(\mathbf{q}, \mathbf{t}) \sim (\mathbf{t}, z(\mathbf{q}, \mathbf{t}))$. *In particular, this implies that the end points of* $(\mathbf{q}, \mathbf{t})$ *and* $z(\mathbf{q}, \mathbf{t})$ *coincide.*

Next we describe the column standard form of $(\mathbf{w}, \mathbf{r}_w)$, when $\mathbf{w}$ is an admissible tuple and $\mathbf{r}_w$ is the symmetric complement of $\mathbf{w}$. Note that $(\mathbf{w}, \mathbf{r}_w)$ can be expressed in column standard form because it satisfies the SIP, as stated in Lemma 3.11 in [4].

Lemma 3.5. *Suppose that* $\mathbf{w}$ *is an admissible tuple with index* l *relative to* $\{0 : h\}$, $h \geq 0$. *Let* $\mathbf{r}_w$ *be the symmetric complement of* $\mathbf{w}$ *and* $m = (h-l)/2$. *Then,*
$$csf(\mathbf{w}, \mathbf{r}_w) = (\mathbf{b}_m, \mathbf{b}_{m-1}, ..., \mathbf{b}_0),$$
where $\mathbf{b}_{m-i} = (h-2i-1 : h-2i+1)$, *for* $i = 1, ..., m-1$, $\mathbf{b}_m = (h-1 : h)$ *if* $m > 0$, *and*

- $\mathbf{b}_0 = (0)$, *if* $h = l = 0$;
- $\mathbf{b}_0 = (0 : 1)$, *if* $0 = l < h$;
- $\mathbf{b}_0 = (0 : l+1)_{rev_c}$, *if* $0 < l = h$;
- $\mathbf{b}_0 = (0 : l+1, (0 : l)_{rev_c})$, *if* $0 < l < h$.

Proof. We have $csf(\mathbf{w}) = (\tilde{\mathbf{b}}_m, ..., \tilde{\mathbf{b}}_0)$, where $\tilde{\mathbf{b}}_{m-i} = (h-2i-1 : h-2i)$ for $i = 0, ..., m-1$, and $\tilde{\mathbf{b}}_0 = (0 : l)$.

If $h = l = 0$, then $\mathbf{r}_w$ is empty and the result is trivially true. If $0 = l < h$, then
$$(\mathbf{w}, \mathbf{r}_w) \sim \left(\tilde{\mathbf{b}}_m, ..., \tilde{\mathbf{b}}_1, \tilde{\mathbf{b}}_0, h-1, h-3, \ldots, 1\right).$$

Note that in this case $m \geq 1$. If $l > 0$, then

$$(\mathbf{w}, \mathbf{r}_w) \sim \left(\tilde{\mathbf{b}}_m, ..., \tilde{\mathbf{b}}_0, h-1, h-3, ..., l+3, l+1, (0:l)_{rev_c}\right).$$

In both cases, using the commutativity relations for indices, it follows that $(\mathbf{w}, \mathbf{r}_w)$ has the claimed column standard form. □

In the next lemma we consider the tuples that will be key in the construction of the skew-symmetric linearizations and show that they satisfy the SIP, a condition that will be necessary for our purposes.

LEMMA 3.6. *Let* $\mathbf{w}$ *be an admissible tuple with indices from* $\{0:h\}$, $h \geq 0$, *and* $\mathbf{r}_w$ *be the symmetric complement of* $\mathbf{w}$. *If* $\mathbf{t}$ *is a right index tuple of type 1 relative to* $rev(\mathbf{w})$, *then* $(rev(\mathbf{t}), \mathbf{w}, \mathbf{r}_w, \mathbf{t})$ *satisfies the SIP.*

Proof. By Lemma 3.11 in [4], $(\mathbf{w}, \mathbf{r}_w)$ and $\mathbf{r}_w$ are symmetric. Thus,

$$(\mathbf{w}, \mathbf{r}_w) \sim rev(\mathbf{w}, \mathbf{r}_w) \sim (rev(\mathbf{r}_w), rev(\mathbf{w})) \sim (\mathbf{r}_w, rev(\mathbf{w})). \tag{3.1}$$

By Lemma 2.16 in [5], $(rev(\mathbf{w}), \mathbf{t})$ satisfies the SIP. By Lemma 3.11 in [4], $(\mathbf{w}, \mathbf{r}_w)$, and therefore $(\mathbf{r}_w, rev(\mathbf{w}))$, satisfies the SIP. Thus, $(\mathbf{w}, \mathbf{r}_w, \mathbf{t}) \sim (\mathbf{r}_w, rev(\mathbf{w}), \mathbf{t})$ satisfies the SIP, as $rev(\mathbf{w})$ is a permutation of $\{0:h\}$. Since $(rev(\mathbf{t}), \mathbf{w})$ also satisfies the SIP, and again because $\mathbf{w}$ is a permutation of $\{0:h\}$, the result follows. □

REMARK 3.7. *It can easily be seen that, if* $\mathbf{w}$ *is an admissible tuple relative to* $\{0:h\}$ *and* $\mathbf{r}_w$ *is the symmetric complement of* $\mathbf{w}$, *then* $\mathbf{r}_w$ *is a right index tuple of type* 1 *relative to* $\mathbf{w}$. *Moreover, because of Remark 3.4 and (3.1),* $z(\mathbf{w}, \mathbf{r}_w) \sim rev(\mathbf{w})$. *Thus, taking into account Remark 3.4, the end points of* $(\mathbf{w}, \mathbf{r}_w)$ *and* $rev(\mathbf{w})$ *coincide.*

3.2. Construction of skew-symmetric linearizations from FPR. As shown in [7, Theorem 7.22] and [19, Lemma 6.11], any skew-symmetric matrix polynomial of odd degree or of even degree with even size has a skew-symmetric strong linearization. However, no skew-symmetric matrix polynomial of even degree and odd size has a strong linearization with the same structure.

Here we construct a family of FPR with the following property: for each pencil $L_P(\lambda)$ in this family, there exists some block-signature matrix S (independent of $P(\lambda)$) such that $SL_P(\lambda)$ is a skew-symmetric strong linearization of the matrix polynomial $P(\lambda)$ whenever $P(\lambda)$ is skew-symmetric, as long as $L_P(\lambda)$ satisfies the nonsingularity conditions (see Definition 4.5 in [4]). This family includes all the FPR that are skew-symmetric as well. We observe that, for the singular skew-symmetric matrix polynomials of even degree for which no linearizations with the same structure exist, no pencil $L_P(\lambda)$ in our family satisfies the nonsingularity conditions.

From now on we assume that the matrix polynomial $P(\lambda)$ of degree $k \geq 2$ given in (1.1) is skew-symmetric, which implies that

$$M_i^T = S_{k-i} M_i S_{k-i+1} = S_{k-i+1} M_i S_{k-i}, \tag{3.2}$$

for $i = 0:k$, where the matrices S_j are the block-signature matrices given in Definition 2.1.

LEMMA 3.8. *Let* $L_P(\lambda)$ *be a FPR of the form (2.2) with degree* $k \geq 2$, *depending on the coefficients of* $P(\lambda)$. *Let* S *be a fixed* $nk \times nk$ *block-signature matrix. If* $SL_P(\lambda)$ *is skew-symmetric for any skew-symmetric matrix polynomial* $P(\lambda)$, *then* $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$, $(\mathbf{l}_q, \mathbf{r}_q)$, $(\mathbf{l}_z, \mathbf{z}, \mathbf{r}_z)$, *and* $(\mathbf{l}_z, \mathbf{r}_z)$ *are symmetric tuples.*

Proof. Assume that $SL_P(\lambda)$ is skew-symmetric for some block-signature matrix S. Here we focus on the tuples $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ and $(\mathbf{l}_z, \mathbf{r}_z)$. The proof that $(\mathbf{l}_z, \mathbf{z}, \mathbf{r}_z)$ and

$(\mathbf{l}_q, \mathbf{r}_q)$ are symmetric can be done similarly. Since $SL_P(\lambda)$ is skew-symmetric, we have

$$(SM_{\mathbf{l}_z,\mathbf{l}_q,\mathbf{q},\mathbf{r}_q,\mathbf{r}_z})^T = -SM_{\mathbf{l}_z,\mathbf{l}_q,\mathbf{q},\mathbf{r}_q,\mathbf{r}_z}$$

or equivalently,

$$M^T_{\mathbf{l}_z,\mathbf{r}_z} M^T_{\mathbf{l}_q,\mathbf{q},\mathbf{r}_q} = -SM_{\mathbf{l}_z,\mathbf{r}_z} M_{\mathbf{l}_q,\mathbf{q},\mathbf{r}_q} S. \tag{3.3}$$

Since $(\mathbf{l}_z, \mathbf{r}_z)$ is a tuple with indices from $\{-k : -h-2\}$ and $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ has indices from $\{0 : h\}$, we have that

$$M_{\mathbf{l}_z,\mathbf{r}_z} = H_1 \oplus I_{n(h+1)} \text{ and } M_{\mathbf{l}_q,\mathbf{q},\mathbf{r}_q} = I_{n(k-h-1)} \oplus H_2,$$

for some $H_1 \in M_{n(k-h-1)}$ and $H_2 \in M_{n(h+1)}$. Let $S = S' \oplus S''$, with $S' \in M_{n(k-h-1)}$. Then, (3.3) is equivalent to

$$M^T_{\mathbf{l}_z,\mathbf{r}_z} = ((-S') \oplus I_{n(h+1)}) M_{\mathbf{l}_z,\mathbf{r}_z} (S' \oplus I_{n(h+1)})$$

and, simultaneously,

$$M^T_{\mathbf{l}_q,\mathbf{q},\mathbf{r}_q} = (I_{n(k-h-1)} \oplus (-S'')) M_{\mathbf{l}_q,\mathbf{q},\mathbf{r}_q} (I_{n(k-h-1)} \oplus S''). \tag{3.4}$$

Since $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ satisfies the SIP, $rev(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ also satisfies the SIP. Thus, the blocks in both $M_{rev(\mathbf{l}_q,\mathbf{q},\mathbf{r}_q)}$ and $M_{\mathbf{l}_q,\mathbf{q},\mathbf{r}_q}$ are of the form 0, I_n, and $-A_i$ for some i's [21]. Since $M^T_{\mathbf{l}_q,\mathbf{q},\mathbf{r}_q} = revtr(M_{rev(\mathbf{l}_q,\mathbf{q},\mathbf{r}_q)})$ is the matrix obtained from $M_{rev(\mathbf{l}_q,\mathbf{q},\mathbf{r}_q)}$ by changing the signs of all the blocks $-A_i$, it is clear that for (3.4) to be satisfied, that is, in order to get $M^T_{\mathbf{l}_q,\mathbf{q},\mathbf{r}_q}$ from $M_{\mathbf{l}_q,\mathbf{q},\mathbf{r}_q}$ by changing the signs of some blocks, we should have $M_{rev(\mathbf{l}_q,\mathbf{q},\mathbf{r}_q)} = M_{\mathbf{l}_q,\mathbf{q},\mathbf{r}_q}$. By Lemma 4.2 in [4], this implies that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ is symmetric. Similarly, we get that $(\mathbf{l}_z, \mathbf{r}_z)$ is symmetric. □

A consequence of Lemma 3.8 and [4, Lemmas 5.1 and 5.5] is the following.

COROLLARY 3.9. *Let $L_P(\lambda)$ be a FPR of the form (2.2) depending on the coefficients of $P(\lambda)$. Let S be a fixed $nk \times nk$ block-signature matrix. If $SL_P(\lambda)$ is skew-symmetric for any skew-symmetric matrix polynomial $P(\lambda)$, then $L_P(\lambda)$ is block-symmetric for any $P(\lambda)$.*

We then have the following consequence of Corollary **??**.

COROLLARY 3.10. *Let $L_P(\lambda) = \lambda L_1 - L_0$ be a FPR of the form(2.2) depending on the coefficients of $P(\lambda)$. If $L_P(\lambda)$ is skew-symmetric for all skew-symmetric $P(\lambda)$, then no block I_n can appear in L_1 and L_0. On the other hand, if $L_P(\lambda)$ is block-symmetric and no block I_n appears in L_1 and L_0, then $L_P(\lambda)$ is skew-symmetric for every skew-symmetric $P(\lambda)$.*

We note however that $L_P(\lambda)$ being block-symmetric is not sufficient for $SL_P(\lambda)$ to be skew-symmetric. In fact, the positions of the $\pm I_n$ blocks in the coefficients of $L_P(\lambda)$ cannot be arbitrary. By [4, Corollary 5.6], $L_P(\lambda)$ can be expressed as a pencil of the form described in [4, Theorem 5.2]. A restriction on the index tuples in this pencil is considered here to ensure that $SL_P(\lambda)$ is skew-symmetric for some block-signature matrix S. Namely, when the additional property of being of type 1 is satisfied by the tuples $\mathbf{t}_w$ and $\mathbf{t}_{w'}$, the identity blocks in the coefficients of the corresponding block-symmetric FPR appear in "good" positions, that is, by multiplying the FPR by a convenient signature matrix S, their sign can be fixed so that the skew-symmetry holds. We note that the pencils in the standard basis of $\mathbb{DL}(P)$ are block-symmetric and have no identity blocks for any $P(\lambda)$. Hence, they belong to the subset of skew-symmetric FPR and, as we will show, they are in fact the only pencils with this property.

We next show that if $\mathbf{w}$ is an admissible tuple and $\mathbf{r}_w$ is the corresponding symmetric complement, then both $(\mathbf{w}, \mathbf{r}_w)$ and $\mathbf{r}_w$ are skew-symmetric complement-transpose related.

LEMMA 3.11. *Let $P(\lambda)$ be a skew-symmetric matrix polynomial of degree $k \geq 2$ of the form (1.1). Let $\mathbf{w}$ be an admissible tuple with index l relative to $\{0 : h\}$, $0 \leq h < k$, and $\mathbf{r}_w$ be the associated symmetric complement. Then*

$$M^T_{\mathbf{w},\mathbf{r}_w} = S_{H \ominus (k-Z)} M_{\mathbf{w},\mathbf{r}_w} S_{k-Z} \tag{3.5}$$

and

$$M^T_{\mathbf{r}_w} = S_{H' \ominus (k-Z)} M_{\mathbf{r}_w} S_{k-Z}, \tag{3.6}$$

where $H = \{k-h : k\}$, $H' = \{k-h+1 : k\}$, and Z is the set of end points of $(\mathbf{w}, \mathbf{r}_w)$.

Proof. Taking into account Lemma 3.5 (using the notation therein), we have

$$\begin{aligned} M^T_{\mathbf{w},\mathbf{r}_w} &= M^T_{csf(\mathbf{w},\mathbf{r}_w)} = revtr(M_{csf(\mathbf{w},\mathbf{r}_w)}) \\ &= revtr(M_{\mathbf{b}_m}) revtr(M_{\mathbf{b}_{m-1}}) \cdots revtr(M_{\mathbf{b}_1}) revtr(M_{\mathbf{b}_0}), \end{aligned}$$

where $m = (h-l)/2$. By (3.2), if $m > 0$

$$revtr(M_{\mathbf{b}_m}) = S_{k-h+2} M_{h-1:h} S_{k-h}.$$

Also, since $\mathbf{b}_{m-i} = (h-2i-1 : h-2i+1)$ for $i = 1 : m-1$, we get

$$revtr(M_{\mathbf{b}_{m-i}}) = S_{k-h+2i+2} M_{\mathbf{b}_{m-i}} S_{k-h+2i-1}, \quad i = 1 : m-1.$$

Finally,

$$revtr(M_{\mathbf{b}_0}) = \begin{cases} M_{\mathbf{b}_0} S_k, & \text{if } h = l = 0, \\ M_{\mathbf{b}_0} S_{k-1}, & \text{if } 0 = l < h, \\ M_{\mathbf{b}_0} S_{k-h:k}, & \text{if } 0 < l = h, \\ M_{\mathbf{b}_0} S_{k-l-1} S_{k-l+1:k}, & \text{if } 0 < l < h. \end{cases}$$

Using Proposition 2.2, we get

$$M^T_{\mathbf{w},\mathbf{r}_w} = \begin{cases} M_{\mathbf{w},\mathbf{r}_w} S_k, & \text{if } h = l = 0, \\ S_{k-h+2:_2 k} M_{\mathbf{w},\mathbf{r}_w} S_{k-h+1:_2 k-1} S_{k-h}, & \text{if } 0 = l < h, \\ M_{\mathbf{w},\mathbf{r}_w} S_{k-h:k}, & \text{if } 0 < l = h, \\ S_{k-h+2:_2 k-l} M_{\mathbf{w},\mathbf{r}_w} S_{k-h} S_{k-h+1:_2 k-l-1} S_{k-l+1:k}, & \text{if } 0 < l < h. \end{cases}$$

In all cases, (3.5) follows.

To prove (3.6), observe that

$$r_w = \begin{cases} \emptyset, & \text{if } h = l = 0, \\ (h-1, h-3, ..., 1), & \text{if } 0 = l < h, \\ (0 : h)_{rev_c}, & \text{if } 0 < l = h, \\ (h-1, h-3, ..., l+1, (0 : l)_{rev_c}), & \text{if } 0 < l < h. \end{cases}$$

If $h = 0$, the result is immediate.

If $0 = l < h$, then $M_{\mathbf{r}_w}^T = S_{k-h+2:_2k} M_{\mathbf{r}_w} S_{k-h+1:_2k-1}$. Note that $S_{k-Z} = S_{k-h} S_{k-h+1:_2k-1}$ and $S_{H'\ominus(k-Z)} = S_{k-h} S_{k-h+2:_2k}$. Since S_{k-h} commutes with $M_{\mathbf{r}_w}$, the result follows. The other two cases can be proven similarly. □

We now extend the claim in Lemma 3.11 for $(\mathbf{w}, \mathbf{r}_w)$ to tuples of the form $(rev(\mathbf{t}), \mathbf{w}, \mathbf{r}_w, \mathbf{t})$ by showing that they are skew-symmetric complement-transpose related.

Lemma 3.12. *Let $P(\lambda)$ be a skew-symmetric matrix polynomial of degree $k \geq 2$ of the form (1.1). Let $\mathbf{w}$ be an admissible tuple relative to $\{0 : h\}$, $0 \leq h < k$, and $\mathbf{r}_w$ be the symmetric complement of $\mathbf{w}$. Let $\mathbf{t}$, with indices from $\{0 : h-1\}$, be a right index tuple of type 1 relative to $rev(\mathbf{w})$. Then,*

$$M^T_{rev(\mathbf{t}),\mathbf{w},\mathbf{r}_w,\mathbf{t}} = S_{H\ominus(k-Z)} M_{rev(\mathbf{t}),\mathbf{w},\mathbf{r}_w,\mathbf{t}} S_{k-Z},$$

and

$$M^T_{rev(\mathbf{t}),\mathbf{r}_w,\mathbf{t}} = S_{H'\ominus(k-Z)} M_{rev(\mathbf{t}),\mathbf{r}_w,\mathbf{t}} S_{k-Z},$$

where $H = \{k-h : k\}$, $H' = \{k-h+1 : k\}$, and Z is the set of end points of $(\mathbf{w}, \mathbf{r}_w, \mathbf{t})$.

Proof. We prove the result by induction on the length of $\mathbf{t}$. If $\mathbf{t}$ is empty, the result follows from Lemma 3.11.

Assume that $\mathbf{t} = (\mathbf{t}', s)$, where s is a single index. Note that, by Remark 3.7, $rev(\mathbf{w}) \sim z(\mathbf{w}, \mathbf{r}_w)$. Thus, $\mathbf{t}$ is a right index tuple of type 1 relative to $z(\mathbf{w}, \mathbf{r}_w)$. Let $z(\mathbf{w}, \mathbf{r}_w, \mathbf{t}) = z(z(\mathbf{w}, \mathbf{r}_w), \mathbf{t})) = (a_m : b_m, ..., a_1 : b_1, a_0 : b_0)$. Since s is a right index of type 1 relative to $z(\mathbf{w}, \mathbf{r}_w, \mathbf{t}')$, we have $z(\mathbf{w}, \mathbf{r}_w, \mathbf{t}') = (a_m : b_m, ..., b_i : b_{i+1}, a_i : b_i - 1, \ldots, a_1 : b_1, a_0 : b_0)$, where $b_i = s = a_{i+1} - 1$, for some $i = 0 : m-1$. Note that $b_{i+1} \geq a_{i+1} > b_i$. We have

$$M^T_{rev(\mathbf{t}),\mathbf{w},\mathbf{r}_w,\mathbf{t}} = M_s^T M^T_{rev(\mathbf{t}'),\mathbf{w},\mathbf{r}_w,\mathbf{t}'} M_s^T.$$

Observe that $\mathbf{t}'$ is a right index tuple of type 1 relative to $rev(\mathbf{w})$ and, by Remark 3.4, $z(\mathbf{w}, \mathbf{r}_w, \mathbf{t}')$ and $(\mathbf{w}, \mathbf{r}_w, \mathbf{t}')$ have the same end points. By the inductive hypothesis,

$$M^T_{rev(\mathbf{t}'),\mathbf{w},\mathbf{r}_w,\mathbf{t}'} = S_{H\ominus(k-\widetilde{Z})} M_{rev(\mathbf{t}'),\mathbf{w},\mathbf{r}_w,\mathbf{t}'} S_{k-\widetilde{Z}},$$

where $\widetilde{Z} = \{b_m, \ldots, b_{i+1}, b_i - 1, b_{i-1}, ..., b_0\}$. Thus,

$$M^T_{rev(\mathbf{t}),\mathbf{w},\mathbf{r}_w,\mathbf{t}} = M_s^T S_{H\ominus(k-\widetilde{Z})} M_{rev(\mathbf{t}'),\mathbf{w},\mathbf{r}_w,\mathbf{t}'} S_{k-\widetilde{Z}} M_s^T.$$

Note that $M_s^T = S_{k-s+1} M_s S_{k-s} = S_{k-s} M_s S_{k-s+1}$. Since $s-1 = b_i - 1 \in \widetilde{Z}$ and $s \notin \widetilde{Z}$, as $b_{i+1} > b_i$, it follows that $S_{k-\widetilde{Z}} S_{k-s+1}$ commutes with M_s. Taking into account that $S_{H\ominus(k-\widetilde{Z})} = (I_{n(k-h-1)} \oplus -I_{n(h+1)}) S_{k-\widetilde{Z}}$, it also follows that $S_{H\ominus(k-\widetilde{Z})} S_{k-s+1}$ commutes with M_s. Thus,

$$M^T_{rev(\mathbf{t}),\mathbf{w},\mathbf{r}_w,\mathbf{t}} = S_{k-s+1} S_{k-s} S_{H\ominus(k-\widetilde{Z})} M_{rev(\mathbf{t}),\mathbf{w},\mathbf{r}_w,\mathbf{t}} S_{k-\widetilde{Z}} S_{k-s+1} S_{k-s}.$$

Considering Remark 3.4, we have $S_{k-\widetilde{Z}} S_{k-s+1} S_{k-s} = S_{k-Z}$ and $S_{k-s+1} S_{k-s} S_{H\ominus(k-\widetilde{Z})} = S_{H\ominus(k-Z)}$, where Z is as in the statement. Thus, the first claim follows. The second claim can be proven similarly. □

The next lemma allows us to extend Lemma 3.12 to tuples of negative indices.

Lemma 3.13. *Let* $\mathbf{s}$ *be a tuple with indices from* $\{0:k-1\}$, $k \geq 2$. *Let* $\mathbf{r} = -k+\mathbf{s}$ *and* S_1, S_2 *be block-signature matrices. Let* R *be the matrix (2.1). Then,* $M_{\mathbf{s}}^T = S_1 M_{\mathbf{s}} S_2$ *for any skew-symmetric matrix polynomial* $P(\lambda)$ *of degree* k *if and only if*

$$M_{\mathbf{r}}^T = (RS_1R)\, M_{\mathbf{r}}\, (RS_2R)$$

for any skew-symmetric matrix polynomial $P(\lambda)$ *of degree* k.

Proof. We prove the "only if" implication. Suppose that $P(\lambda)$ is skew-symmetric. Observe that for $i = -k+j$, with $j \in \{0 : k-1\}$, $M_i = RM_j(P')R$, where $P'(\lambda) = -\lambda^k P(1/\lambda) = -rev(P(\lambda))$. Thus,

$$M_{\mathbf{r}} = RM'_{\mathbf{s}}R, \tag{3.7}$$

where $M'_{\mathbf{s}}$ denotes $M_{s_1}(P') \cdots M_{s_l}(P')$, for $\mathbf{s} = (s_1, \ldots, s_l)$. Note that $P'(\lambda)$ is skew-symmetric since $P(\lambda)$ is. Thus, from (3.7) and taking into account the hypothesis, we have

$$\begin{aligned} M_{\mathbf{r}}^T &= (RM'_{\mathbf{s}}R)^T = R\,(S_1 M'_{\mathbf{s}} S_2)\,R \\ &= (RS_1R)\,(RM'_{\mathbf{s}}R)\,(RS_2R) \\ &= (RS_1R)\,M_{\mathbf{r}}\,(RS_2R)\,. \end{aligned}$$

The "if" implication can be proven with similar arguments. □

From the previous lemma we get the next result.

Lemma 3.14. *Let* $P(\lambda)$ *be a skew-symmetric matrix polynomial of degree* $k \geq 2$ *of the form (1.1). Let* $\mathbf{w}$ *be an admissible tuple relative to* $\{0 : k-h-1\}$, $0 \leq h < k$, *and* $\mathbf{r}_w$ *be the symmetric complement of* $\mathbf{w}$. *Let* $\mathbf{t}$ *be a right index tuple of type* 1 *relative to* $rev(\mathbf{w})$, *with indices from* $\{0 : k-h-2\}$. *Let* $\mathbf{z} = -k+\mathbf{w}$, $\mathbf{r}_z = -k+\mathbf{r}_w$ *and* $\mathbf{t}' = -k+\mathbf{t}$. *Then,* $(rev(\mathbf{t}'), \mathbf{z}, \mathbf{r}_z, \mathbf{t}')$ *satisfies the SIP,*

$$M^T_{rev(\mathbf{t}'),\mathbf{z},\mathbf{r}_z,\mathbf{t}'} = S_{\tilde{H}\ominus(1+Z)} M_{rev(\mathbf{t}'),\mathbf{z},\mathbf{r}_z,\mathbf{t}'} S_{1+Z},$$

and

$$M^T_{rev(\mathbf{t}'),\mathbf{r}_z,\mathbf{t}'} = S_{\tilde{H}'\ominus(1+Z)} M_{rev(\mathbf{t}'),\mathbf{r}_z,\mathbf{t}'} S_{1+Z},$$

where $\tilde{H} = \{1 : k-h\}$, $\tilde{H}' = \{1 : k-h-1\}$, *and* Z *is the set of end points of* $(\mathbf{w}, \mathbf{r}_w, \mathbf{t})$.

Proof. The SIP claim follows from Remark 2.17 in [4] and Lemma 3.6. The second part of the statement follows from Lemmas 3.12 and 3.13, taking into account that $RS_{k-Z}R = S_{1+Z}$, $RS_{H\ominus(k-Z)}R = S_{\tilde{H}\ominus(1+Z)}$, $RS_{H'\ominus(k-Z)}R = S_{\tilde{H}'\ominus(1+Z)}$, where $H = \{h+1 : k\}$ and $H' = \{h+2 : k\}$. □

The next theorem is the main result in this section and gives a characterization of a family of skew-symmetric strong linearizations of skew-symmetric matrix polynomials constructed from FPR.

Recall from Corollary **??** that, if $SL_P(\lambda)$ is skew-symmetric for any skew-symmetric $P(\lambda)$, where S is a block-signature matrix and $L_P(\lambda)$ is a FPR, then $L_P(\lambda)$ must be block-symmetric. Then, from [4, Corollary 5.6], $L_P(\lambda)$ must be of the form

$$\lambda M_{rev(\mathbf{t}_w),rev(\mathbf{t}_z),\mathbf{z},\mathbf{r}_z,\mathbf{t}_z,\mathbf{r}_w,\mathbf{t}_w} - M_{rev(\mathbf{t}_w),rev(\mathbf{t}_z),\mathbf{w},\mathbf{r}_z,\mathbf{t}_z,\mathbf{r}_w,\mathbf{t}_w} \tag{3.8}$$

for some admissible index tuples $\mathbf{w}$ and $k+\mathbf{z}$ relative to $\{0:h\}$ and $\{0:k-h-1\}$, respectively, and some tuples $\mathbf{t}_w$ and $k+\mathbf{t}_z$ with indices in $\{0 : h-1\}$ and $\{0 :$

$k-h-2\}$, respectively ($\mathbf{r}_w$ and $k+\mathbf{r}_z$ are the symmetric complements of $\mathbf{w}$ and $k+\mathbf{z}$, respectively).

In the next theorem we assume that $\mathbf{t}_w$ and $k+\mathbf{t}_z$ are index tuples of type 1 relative to $rev(\mathbf{w})$ and $rev(k+\mathbf{z})$. Notice that in this case, by Lemmas 3.6 and 3.14, the tuples $(rev(\mathbf{t}_w), w, \mathbf{r}_w, \mathbf{t}_w)$ and $(rev(\mathbf{t}_z), \mathbf{z}, \mathbf{r}_z, \mathbf{t}_z)$ satisfy the SIP, which implies that (3.8) is a FPR. By Lemma 3.19 in [4], we also deduce that the index tuples $\mathbf{t}_w$ and $k+\mathbf{t}_z$ satisfy the SIP and are $\mathbf{w}$-compatible and $(k+\mathbf{z})$-compatible, respectively. Thus, the family of FPR that we construct the skew-symmetric linearizations from is a subset of the set of block-symmetric FPR described in Corollary 5.6 in [4].

Theorem 3.15. *Let $k \geq 2$ and h be integers with $0 \leq h < k$. Let $\mathbf{w}$ and $\mathbf{w}'$ be admissible tuples relative to $\{0:h\}$ and $\{0:k-h-1\}$, respectively, and $\mathbf{r}_w$ and $\mathbf{r}_{w'}$ be the symmetric complements of $\mathbf{w}$ and $\mathbf{w}'$, respectively. Let $\mathbf{t}_w$ with indices from $\{0:h-1\}$ and $\mathbf{t}_{w'}$ with indices from $\{0:k-h-2\}$ be index tuples of type 1 relative to $rev(\mathbf{w})$ and $rev(\mathbf{w}')$, respectively. Let $\mathbf{z} = -k+\mathbf{w}'$, $\mathbf{r}_z = -k+\mathbf{r}_{w'}$, $\mathbf{t}_z = -k+\mathbf{t}_{w'}$. For a matrix polynomial $P(\lambda)$ of degree k, let $L_P(\lambda)$ be the block-symmetric FPR given in (3.8) associated with $P(\lambda)$.*

Then, up to multiplication by -1, there exists a unique block-signature matrix S such that $SL_P(\lambda)$ is skew-symmetric for any skew-symmetric matrix polynomial $P(\lambda)$ of degree k. Moreover, $S = S_{k-h}S_{k-Z}S_{1+Z'}$, where Z and Z' are the sets of end points of $(\mathbf{w}, \mathbf{r}_w, \mathbf{t}_w)$ and $(\mathbf{w}', \mathbf{r}_{w'}, \mathbf{t}_{w'})$, respectively. Additionally, if $L_P(\lambda)$ satisfies the nonsingularity conditions, then the pencil $SL_P(\lambda)$ is a skew-symmetric strong linearization of $P(\lambda)$.

Proof. Let $L_P(\lambda) = \lambda L_1 - L_0$, with $L_1 = M_{rev(\mathbf{t}_w), rev(\mathbf{t}_z), \mathbf{z}, \mathbf{r}_z, \mathbf{t}_z, \mathbf{r}_w, \mathbf{t}_w}$ and $L_0 = M_{rev(\mathbf{t}_w), rev(\mathbf{t}_z), \mathbf{w}, \mathbf{r}_z, \mathbf{t}_z, \mathbf{r}_w, \mathbf{t}_w}$. In order to show that $SL_P(\lambda)$ is skew-symmetric when $P(\lambda)$ is, where $S = S_{k-h}S_{k-Z}S_{1+Z'}$, we need to see that

$$SL_0^T S = -L_0 \tag{3.9}$$

and

$$SL_1^T S = -L_1. \tag{3.10}$$

We show (3.9). The proof of (3.10) is analogous. Let $H = \{k-h : k\}$ and $\widetilde{H}' = \{1 : k-h-1\}$. Taking into account Lemmas 3.12 and 3.14, we have

$$L_0^T = M^T_{rev(\mathbf{t}_w), \mathbf{w}, \mathbf{r}_w, \mathbf{t}_w} M^T_{rev(\mathbf{t}_w), \mathbf{r}_z, \mathbf{t}_z}$$

$$= \left(S_{H\ominus(k-Z)} M_{rev(\mathbf{t}_w), \mathbf{w}, \mathbf{r}_w, \mathbf{t}_w} S_{k-Z}\right)\left(S_{\widetilde{H}'\ominus(1+Z')} M_{rev(\mathbf{t}_z), \mathbf{r}_z, \mathbf{t}_z} S_{1+Z'}\right)$$

$$= S_{H\ominus(k-Z)} S_{\widetilde{H}'\ominus(1+Z')} S_{k-h} M_{rev(\mathbf{t}_w), \mathbf{w}, \mathbf{r}_w, \mathbf{t}_w} M_{rev(\mathbf{t}_z), \mathbf{r}_z, \mathbf{t}_z} S_{k-h} S_{k-Z} S_{1+Z'}$$

Note that $1+Z', \widetilde{H}'\ominus(1+Z') \subseteq \{1 : k-h\}$. Moreover, $k-h \in \widetilde{H}'\ominus(1+Z')$ since $k-h \in 1+Z'$ but $k-h \notin \widetilde{H}'$. Thus, the block-signature matrix $S_{k-h}S_{\widetilde{H}'\ominus(1+Z')}$ commutes with $M_{rev(\mathbf{t}_w), \mathbf{w}, \mathbf{r}_w, \mathbf{t}_w}$. Additionally, $S_{k-h}S_{k-Z}$ commutes with $M_{rev(\mathbf{t}_z), \mathbf{r}_z, \mathbf{t}_z}$.

Finally, note that, from Remark 2.3, $S_{H\ominus(k-Z)} = \left(I_{n(k-h-1)} \oplus -I_{n(h+1)}\right) S_{k-Z}$ and $S_{\widetilde{H}'\ominus(1+Z')} = (-I_{n(k-h-1)} \oplus I_{n(h+1)}) S_{1+Z'}$. Thus, it follows that

$$S_{H\ominus(k-Z)} S_{\widetilde{H}'\ominus(1+Z')} = -S_{k-Z} S_{1+Z'},$$

showing (3.9).

Now we show the uniqueness of S. Suppose that $SL_P(\lambda)$ and $S'L_P(\lambda)$ are skew-symmetric when $P(\lambda)$ is, where S' is a block-signature matrix. Consider the decompositions of S, $S = T_1U_1 = T_2U_2$, where T_1, U_1, T_2, U_2 have the forms $I_{n(k-h-1)} \oplus [\star]$, $[\star] \oplus I_{n(h+1)}$, $[\star] \oplus I_{nh}$ and $I_{n(k-h)} \oplus [\star]$, respectively, and the corresponding decompositions of S', $S' = T_1'U_1' = T_2'U_2'$. Since $SL_P(\lambda)$ is skew-symmetric, we have

$$\begin{aligned} L_0^T &= -SL_0S = -U_1T_1M_{rev(\mathbf{t}_w),\mathbf{w},\mathbf{r}_w,\mathbf{t}_w}M_{rev(\mathbf{t}_z),\mathbf{r}_z,\mathbf{t}_z}T_1U_1 \\ &= -(T_1M_{rev(\mathbf{t}_w),\mathbf{w},\mathbf{r}_w,\mathbf{t}_w}T_1)(U_1M_{rev(\mathbf{t}_z),\mathbf{r}_z,\mathbf{t}_z}U_1) \end{aligned}$$

and

$$L_1^T = -SL_1S = -(T_2M_{rev(\mathbf{t}_z),\mathbf{z},\mathbf{r}_z,\mathbf{t}_z}T_2)(U_2M_{rev(\mathbf{t}_w),\mathbf{r}_w,\mathbf{t}_w}U_2).$$

Taking into account the form of the matrices, it follows that

$$M^T_{rev(\mathbf{t}_w),\mathbf{w},\mathbf{r}_w,\mathbf{t}_w} = (I_{n(k-h-1)} \oplus -I_{n(h+1)})T_1M_{rev(\mathbf{t}_w),\mathbf{w},\mathbf{r}_w,\mathbf{t}_w}T_1, \tag{3.11}$$

and

$$M^T_{rev(\mathbf{t}_z),\mathbf{z},\mathbf{r}_z,\mathbf{t}_z} = (-I_{n(k-h)} \oplus I_{nh})T_2M_{rev(\mathbf{t}_z),\mathbf{z},\mathbf{r}_z,\mathbf{t}_z}T_2. \tag{3.12}$$

Analogously, since $S'L_P(\lambda)$ is skew-symmetric, conditions (3.11) and (3.12) hold with T_1 replaced by T_1' and T_2 replaced by T_2'. By Remark 2.8, Lemma 2.10 and Corollary 2.9, either $T_1 = T_1'$ or $T_1 = (I_{n(k-h-1)} \oplus -I_{n(h+1)})T_1'$. Also, by Remark 2.8, Lemmas 2.10 and 3.13 and Corollary 2.9, we have $T_2' = T_2$ or $T_2' = (-I_{n(k-h)} \oplus I_{nh})T_2$. Since the entries in position $(k-h, k-h)$ of T_1 and T_2, and of T_1' and T_2', coincide, we have that either $S' = S$ or $S' = -S$.

The last claim in the statement follows from Lemma 4.4 in [4]. □

Observe that the matrix S given by Theorem 3.15 does not depend on $P(\lambda)$ but just on the tuples $\mathbf{w}$, $\mathbf{r}_w$, $\mathbf{t}_w$, $\mathbf{w}'$, $\mathbf{r}_{w'}$, $\mathbf{t}_{w'}$.

If in Theorem 3.15 both admissible tuples $\mathbf{w}$ and $\mathbf{w}'$ have index 0, and 0 is neither in $\mathbf{t}_w$ nor in $\mathbf{t}_{w'}$, then $L_P(\lambda)$ satisfies the nonsingularity conditions. Thus, if k is odd, by choosing h even and $\mathbf{w}$, $\mathbf{w}'$, $\mathbf{t}_w$ and $\mathbf{t}_{w'}$ as just described, our theorem produces skew-symmetric linearizations.

When k is even, $P(\lambda)$ has even size n, and either A_k or A_0 is nonsingular, Theorem 3.15 also gives skew-symmetric strong linearizations of $P(\lambda)$ when $P(\lambda)$ is skew-symmetric. More precisely, if $\det(A_k) \neq 0$, by choosing h even, $\mathbf{w}$ of index 0, and $\mathbf{t}_w$ not containing 0, we get strong skew-symmetric linearizations, independently of A_0 being nonsingular or not. If $\det(A_k) = 0$, then our theorem produces strong skew-symmetric linearizations if and only if A_0 is nonsingular, as $\mathbf{w}'$ must be of index 0 and $\mathbf{w}$ must be of odd index.

If k is even and n is odd (note that in this case, the skew-symmetry of $P(\lambda)$ implies $det(A_i) = 0$ for $i = 0 : k$), our theorem does not give strong skew-symmetric linearizations of $P(\lambda)$ (as expected [7, Theorem 7.22]). In fact, in this case, since h and $k-h-1$ cannot be both even, either $-k$ is in $(\mathbf{l}_z, \mathbf{r}_z)$ or 0 is in $(\mathbf{l}_q, \mathbf{r}_q)$.

Note that, if $P(\lambda)$ is regular, k is even, and $det(A_k) = det(A_0) = 0$, our theorem does not give strong skew-symmetric linearizations of $P(\lambda)$, although strong skew-symmetric linearizations exist in this case [7, Theorem 7.22].

Example 3.16. *Let $P(\lambda)$ be a skew-symmetric matrix polynomial of degree $k = 5$ as in (1.1). Here we give all the distinct skew-symmetric pencils associated with $P(\lambda)$*

given by Theorem 3.15. If the pencil satisfies the nonsingularity conditions, then it is a strong linearization of $P(\lambda)$.

$\mathbf{w}$	$\mathbf{w}'$	$\mathbf{t}_w$	$\mathbf{t}_{w'}$	S
(0)	(3:4, 1:2, 0)	$\emptyset$	$\emptyset$	$S_2S_{4:5}$
(0)	(3:4, 1:2, 0)	$\emptyset$	(0)	$S_{1:2}S_{4:5}$
(0)	(3:4, 1:2, 0)	$\emptyset$	(2)	$S_{3:5}$
(0)	(3:4, 1:2, 0)	$\emptyset$	(2, 0)	$S_1S_{3:5}$
(0)	(3:4, 1:2, 0)	$\emptyset$	(2, 0, 1)	$S_{2:5}$
(0)	(3:4, 1:2, 0)	$\emptyset$	(2,0,1,0)	$-I$
(0:1)	(2:3, 0:1)	$\emptyset$	$\emptyset$	$S_1S_{3:5}$
(0:1)	(2:3, 0:1)	$\emptyset$	(1)	$S_{2:5}$
(0:1)	(2:3, 0:1)	$\emptyset$	(1,0)	$-I$
(1:2,0)	(1:2,0)	$\emptyset$	$\emptyset$	$S_{2:4}$
(1:2,0)	(1:2,0)	(0)	$\emptyset$	$S_{2:5}$
(1:2,0)	(1:2,0)	$\emptyset$	(0)	$S_{1:4}$
(1:2,0)	(1:2, 0)	(0)	(0)	$-I$
(2:3, 0:1)	(0:1)	$\emptyset$	$\emptyset$	$S_{1:2}S_3S_5$
(2:3, 0:1)	(0:1)	(1)	$\emptyset$	$S_{1:4}$
(2:3, 0:1)	(0:1)	(1,0)	$\emptyset$	$-I$
(3:4, 1:2, 0)	(0)	$\emptyset$	$\emptyset$	$S_{1:2}S_4$
(3:4, 1:2, 0)	(0)	(0)	$\emptyset$	$S_{1:2}S_{4:5}$
(3:4, 1:2, 0)	(0)	(2)	$\emptyset$	$S_{1:3}$
(3:4, 1:2, 0)	(0)	(2, 0)	$\emptyset$	$S_{1:3}S_5$
(3:4, 1:2, 0)	(0)	(2, 0, 1)	$\emptyset$	$S_{1:4}$
(3:4, 1:2, 0)	(0)	(2,0,1,0)	$\emptyset$	$-I$

Note that if the matrix S given by Theorem 3.15 equals $-I_{nk}$, then the pencil $L_P(\lambda)$ is skew-symmetric when the matrix polynomial $P(\lambda)$ is. Also observe that we get $S = -I_{nk}$ when the tuples $\mathbf{t}_w$ and $\mathbf{t}_{w'}$ have the largest possible number of indices. In fact, the corresponding pencils are precisely the pencils in the standard basis for $\mathbb{DL}(P)$ introduced in [12]. In Corollary 3.20, we prove that these are the only FPR that are skew-symmetric when $P(\lambda)$ is.

Next we give explicitly the pencil corresponding to one of the cases in the previous table, which is not in $\mathbb{DL}(P)$. Notice that all the pencils are obtained from the block-symmetric pencils presented in [4, Corollary 5.6], corresponding to the same tuples, by changing the sign of some blocks.

Let $\mathbf{w} = (0)$, $\mathbf{w}' = (3:4, 1:2, 0)$, $\mathbf{t}_w = \emptyset$, and $\mathbf{t}_{w'} = (2, 0, 1)$. Then,

$$SL_P(\lambda) = \lambda \begin{bmatrix} 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & -A_5 & -A_4 \\ 0 & 0 & -A_5 & -A_4 & -A_3 \\ 0 & -A_5 & -A_4 & -A_3 & -A_2 \\ -I & -A_4 & -A_3 & -A_2 & -A_1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & I & 0 \\ 0 & 0 & -A_5 & -A_4 & 0 \\ 0 & -A_5 & -A_4 & -A_3 & 0 \\ -I & -A_4 & -A_3 & -A_2 & 0 \\ 0 & 0 & 0 & 0 & A_0 \end{bmatrix}.$$

A natural question is if we could extend the family of skew-symmetric linearizations given in Theorem 3.15 by suppressing the condition on $\mathbf{t}_w$ and/or $\mathbf{t}_{w'}$ of being index tuples of type 1. The analysis of some examples show that, in general, this is not the case. We include one such example.

EXAMPLE 3.17. *Let $P(\lambda)$ be a skew-symmetric matrix polynomial of degree k. Let $L_P(\lambda)$ be an FPR as in (3.8) with $\mathbf{w} = (4:5, 2:3, 0:1)$ and $\mathbf{t}_w = (3, 2)$. Note* *that* $\mathbf{t}_w$ *is not of type 1 relative to* $rev(\mathbf{w})$*. Then, we have* $(rev(\mathbf{t}_w), \mathbf{w}, \mathbf{r}_w, \mathbf{t}_w) \sim (2:5, 2:4, 0:3, 2, 0)$ *and* $M_{rev(\mathbf{t}_w),\mathbf{w},\mathbf{r}_w,\mathbf{t}_w} = I_{n(k-6)} \oplus H$*, where*

$$H = \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & I & 0 \\ -A_4 & -A_3 & -A_2 & I & 0 & 0 \\ -A_3 & -A_2 & -A_1 & 0 & 0 & -A_0 \\ -A_2 & I & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -A_0 & 0 & 0 & 0 \end{bmatrix}.$$

Assume that there exists S *such that* $SL_P(\lambda)$ *is skew-symmetric and let* $S = S' \oplus S''$ *where* $S'' \in M_{6n}$*. Then* $H^T = -S''HS''$*, which implies* $S''(1,1)S''(5,5) = S''(2,2)S''(4,4) = -I_n$*, and* $S''(1,1)S''(2,2) = S''(1,1)S''(4,4) = I_n$*. This leads to* $S''(4,4) = -S''(4,4)$*, a contradiction.*

We finish this section by describing the tuples $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ and $(\mathbf{l}_z, \mathbf{z}, \mathbf{r}_z)$ for which the associated FPR is skew-symmetric when $P(\lambda)$ is. Recall from Corollary 3.9 that for a FPR to be skew-symmetric, no block in the matrix coefficients can be the identity matrix. The next lemma characterizes the tuples that produce such matrices.

Lemma 3.18. *Let* $P(\lambda)$ *be a matrix polynomial of degree* $k \geq 2$ *of the form (1.1). Let* $\mathbf{t}$ *be an index tuple satisfying the SIP with indices from* $\{0:h\}$*,* $0 \leq h < k$*, and such that* h *is the largest index in* $\mathbf{t}$*. Then,* $M_{\mathbf{t}} = I_{k-h-1} \oplus H$ *for some* $(h+1) \times (h+1)$ *matrix* H*. Moreover, all blocks in* H *are either* 0 *or* $-A_i$ *if and only if* $\mathbf{t} \sim (0:h+1)_{rev_c}$*.*

Proof. Note that, for $0 \leq i \leq h$, $M_{0:i} = I_{k-i-1} \oplus T$, where

$$T = \left[\begin{array}{c|c} -A_i & \\ -A_{i-1} & \\ \vdots & I_{ni} \\ -A_1 & \\ \hline -A_0 & 0 \end{array}\right]. \tag{3.13}$$

Taking this into account, it can be proven, by induction on h, that

$$M_{(0:h+1)_{rev_c}} = I_{k-h-1} \oplus H,$$

where

$$H = \begin{bmatrix} -A_h & -A_{h-1} & \cdots & -A_1 & -A_0 \\ -A_{h-1} & -A_{h-2} & \cdots & -A_0 & 0 \\ -A_{h-2} & -A_{h-3} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -A_0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Thus, one implication follows.

Now we prove the converse by contradiction. Assume that $\mathbf{t}$ is an index tuple satisfying the SIP with indices from $\{0:h\}$, where h is the largest index in $\mathbf{t}$. Let $csf(\mathbf{t}) = (\mathbf{b}_m, ..., \mathbf{b}_1, \mathbf{b}_0)$, $m \geq 0$, where $\mathbf{b}_i = (a_i : b_i)$, with $a_i \leq b_i$, $i = 1:m$, and $b_i > b_{i-1}$ for $i = 2:m$. Moreover, since $\mathbf{t}$ is not equivalent to $(0:h+1)_{rev_c}$, one of the following conditions holds: i) $\mathbf{b}_0 = (0:l+1)_{rev_c}$, $m \geq 1$, $b_1 > l$ and $b_1 > l+1$ if $a_1 = 0$, or ii) $\mathbf{b}_0 = (a_0 : b_0)$ with $a_0 > 0$ and $b_1 > b_0$ if $m > 0$.

Let $M_{\mathbf{t}} = I_{k-h-1} \oplus H$. We prove, by induction on m, that H contains at least one block equal to I_n. We have $M_{\mathbf{b}_i} = I_{k-b_i-1} \oplus T_i$, $i = 1, ..., m$, where

$$T_i = \left[\begin{array}{c|c|c} -A_{b_i} & & \\ -A_{b_i-1} & & \\ \vdots & I_{n(b_i-a_i+1)} & 0 \\ -A_{a_i} & & \\ \hline I_n & 0 & 0 \\ \hline 0 & 0 & I_{n(a_i-1)} \end{array}\right], \quad \text{if } a_i \neq 0,$$

and T_i has the form (3.13), with i replaced with b_i, if $a_i = 0$.

Suppose that i) holds. If $b_1 > l+1$, then the first row of $M_{\mathbf{b}_1} M_{(0:l+1)_{rev_c}}$ is $[-A_{b_1}\ I_n\ 0]$. If $b_1 = l+1$, then $a_1 \neq 0$ and the $(k-a_1+1)$th row of $M_{\mathbf{b}_1} M_{(0:l+1)_{rev_c}}$ is $[I_n\ 0 \cdots 0]$.

If ii) holds then T_0 contains the block I_n in position $(1,2)$.

Thus, if either i) holds and $m = 1$ or ii) holds and $m = 0$, the result follows.

Now suppose that either i) holds and $m > 1$ or ii) holds and $m > 0$. Let $M_{\mathbf{b}_i,...,\mathbf{b}_0} = I_{k-b_i-1} \oplus W_i$. By the induction hypothesis, the matrix W_i contains a block I_n, say in position (r,j). Note that T_{i+1} contains a block I_n in columns $2 : b_i+1$. Then, $T_{i+1}(I_{b_{i+1}-b_i} \oplus W_i)$ contains a block equal to the identity in the jth column, which implies that H contains some block equal to I_n, proving the claim. □

The next lemma is a technical result useful to determine necessary and sufficient conditions for a FPR as in (2.2) to be skew-symmetric when $P(\lambda)$ is.

Lemma 3.19. *Let $\mathbf{q}$ be a permutation of $\{0 : h\}$ and $\mathbf{l}_q, \mathbf{r}_q$ be tuples with indices from $\{0 : h-1\}$. Suppose that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) \sim (0 : h+1)_{rev_c}$. Then, $(\mathbf{l}_q, \mathbf{r}_q) \sim (0 : h)_{rev_c}$.*

Proof. Let $csf(\mathbf{q}) = (\mathbf{b}_m, ..., \mathbf{b}_1, \mathbf{b}_0)$, where $\mathbf{b}_i = (a_i : b_i)$, with $a_i = b_{i-1}+1$, are the strings of $csf(\mathbf{q})$. The tuple $(0 : h+1)_{rev_c}$ is equivalent to a tuple of the form $(\mathbf{l}'_q, \mathbf{q}, \mathbf{r}'_q)$, with $\mathbf{l}'_q = (\mathbf{l}_m, ..., \mathbf{l}_2, \mathbf{l}_1)$, where $\mathbf{l}_i = (0 : a_i - 1)$, and $\mathbf{r}'_q = (\mathbf{r}_{m-1}, ..., \mathbf{r}_1, \mathbf{r}_0, (0 : h-m)_{rev_c})$, where $\mathbf{r}_i = (b_i+1 : h-m+i)$. By Lemma 2.14 in [4], $(\mathbf{l}_q, \mathbf{r}_q) \sim (\mathbf{l}'_q, \mathbf{r}'_q)$. Since $(\mathbf{l}'_q, \mathbf{r}'_q) \sim (\mathbf{l}_m, \mathbf{r}_{m-1}, ..., \mathbf{l}_2, \mathbf{r}_1, \mathbf{l}_1, \mathbf{r}_0, (0 : h-m)_{rev_c})$ and $(\mathbf{l}_i, \mathbf{r}_{i-1}) \sim (0 : h-m+i-1)$, the result follows. □

We now describe the FPR that are skew-symmetric when the associated matrix polynomial $P(\lambda)$ is. These pencils form precisely the standard basis for $\mathbb{DL}(P)$. Note the expression for these pencils given in the proof of [4, Lemma 5.7]. When the nonsingularity conditions are satisfied, they are strong linearizations of $P(\lambda)$.

Corollary 3.20. *Let $P(\lambda)$ be a skew-symmetric matrix polynomial of degree $k \geq 2$ of the form (1.1) and let $L_P(\lambda)$ be a FPR of the form (2.2). Then $L_P(\lambda)$ is skew-symmetric if and only if $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) \sim (0 : h+1)_{rev_c}$ and $k + (\mathbf{l}_z, \mathbf{z}, \mathbf{r}_z) \sim (0 : k-h)_{rev_c}$.*

Proof. Suppose that $L_P(\lambda)$ is skew-symmetric. Taking into account the forms of the matrices $M_{\mathbf{l}_q,\mathbf{q},\mathbf{r}_q} M_{\mathbf{l}_z,\mathbf{r}_z}$, and $M_{\mathbf{l}_z,\mathbf{z},\mathbf{r}_z} M_{\mathbf{l}_q,\mathbf{r}_q}$, it follows from Corollary 3.9 that the principal submatrices of $M_{\mathbf{l}_q,\mathbf{q},\mathbf{r}_q}$ and $M_{\mathbf{l}_z,\mathbf{z},\mathbf{r}_z}$ lying on the block rows $k-h : k$ and $1 : k-h$, respectively, contain no block equal to I_n. Now the claim follows from Lemma 3.18.

To prove the converse, suppose that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) \sim (0 : h+1)_{rev_c}$ and $k + (\mathbf{l}_z, \mathbf{z}, \mathbf{r}_z) \sim (0 : k-h)_{rev_c}$. By Lemma 3.19, we have

$$L_P(\lambda) = \lambda M_{-k+\mathbf{w}', -k+\mathbf{r}_{w'}, \mathbf{r}_w} - M_{\mathbf{w}, \mathbf{r}_w, -k+\mathbf{r}_{w'}}.$$

where $\mathbf{w} = (0 : h)$, $\mathbf{r}_w = (0 : h)_{rev_c}$, $\mathbf{w}' = (0 : k-h-1)$ and $\mathbf{r}_{w'} = (0 : k-h-1)_{rev_c}$. By Theorem 3.15, $-L_P(\lambda)$ is skew-symmetric (the matrix S given by the theorem is $-I_{nk}$.), which implies that $L_P(\lambda)$ is also skew-symmetric. □

We observe that from Corollary 3.20 and its proof, it follows that the FPR that are skew-symmetric linearizations of a skew-symmetric matrix polynomial are included in the family given by Theorem 3.15. Moreover, these pencils are precisely those in the standard basis for $\mathbb{DL}(P)$ (see the proof of [4, Lemma 5.7]).

4. T-alternating strong linearizations. In this section we study T-alternating strong linearizations of T-alternating matrix polynomials obtained from FPR.

It is known [20] that not all T-alternating matrix polynomials $P(\lambda)$ have T-alternating strong linearizations of different parity. Moreover, not all T-alternating matrix polynomials $P(\lambda)$ of even degree have T-alternating strong linearizations with the same parity as $P(\lambda)$. However, any T-alternating matrix polynomial $P(\lambda)$ (regular or singular) of odd degree has a T-alternating strong linearization with the same parity as $P(\lambda)$ [20, Theorem 5.4].

Here we first study the existence of T-even (respectively, T-odd) strong linearizations of a T-even (respectively, T-odd) matrix polynomial $P(\lambda)$ in the set of FPR and show that no FPR $L_P(\lambda)$ preserves the T-alternating structure of a matrix polynomial. Then we consider pencils of the form $SL_P(\lambda)$, where $L_P(\lambda)$ is a FPR and S is a block-signature matrix, and give some necessary conditions for $SL_P(\lambda)$ to be T-alternating with the same parity as $P(\lambda)$, in particular, that $L_P(\lambda)$ must be block-symmetric. Finally, we describe a class of block-symmetric FPR such that, for any pencil $L_P(\lambda)$ in this class and some block-signature matrix S (not depending on $P(\lambda)$), $SL_P(\lambda)$ is a T-alternating strong linearization with the same parity as the T-alternating matrix polynomial $P(\lambda)$, as long as $L_P(\lambda)$ satisfies the nonsingularity conditions. Notice that our procedure will produce T-alternating pencils even when the nonsingularity conditions are not satisfied. But, in those cases, these pencils may not be strong linearizations of the matrix polynomial. In particular, the nonsingularity conditions of the pencils we obtain do not hold when the T-alternating polynomial has no T-alternating strong linearizations.

In the next result we consider the problem of the existence of T-alternating FPR associated with T-alternating matrix polynomials.

THEOREM 4.1. *No FPR $L_P(\lambda)$, depending on the coefficients of the matrix polynomial $P(\lambda)$ of the form (1.1) with degree $k \geq 2$, is always T-even or always T-odd for any T-even (respectively, T-odd) matrix polynomial.*

Proof. Assume that $P(\lambda)$, as in (1.1), is T-even (a similar argument leads to the same conclusion in the T-odd case). Let $L_P(\lambda) = \lambda L_1 - L_0$ be a FPR of the form (2.2), where $L_0 = M_{\mathbf{l}_q,\mathbf{q},\mathbf{r}_q} M_{\mathbf{l}_z,\mathbf{r}_z}$ and $L_1 = M_{\mathbf{l}_z,\mathbf{z},\mathbf{r}_z} M_{\mathbf{l}_q,\mathbf{r}_q}$. If $L_P(\lambda)$ is T-odd, then

$$L_1^T = L_1, \quad \text{and} \quad L_0^T = -L_0.$$

But the last equality cannot hold because L_0 contains the block $-A_0$ and $A_0^T = A_0$. Now suppose that $L_P(\lambda)$ is T-even. Then,

$$L_1^T = -L_1 \quad \text{and} \quad L_0^T = L_0. \tag{4.1}$$

We have $\mathbf{q} = (0)$, as otherwise the matrix L_0 would contain the block $-A_1$, which cannot happen because of the second equality in (4.1) and the fact that $A_1^T = -A_1$. On the other hand, if $\mathbf{q} = (0)$ the matrix L_1 contains the block A_2 (as $k \geq 2$), which cannot happen because of the first equality in (4.1) and the fact that $A_2^T = A_2$. □

The next lemma gives necessary conditions for $SL_P(\lambda)$ to be T-alternating when $P(\lambda)$ is, where $L_P(\lambda)$ is a FPR and S is a block-signature matrix. Its proof is omitted as it is similar to the proof of Lemma 3.8.

LEMMA 4.2. *Let $L_P(\lambda)$ be a FPR of the form (2.2) of degree $k \geq 2$, depending on the coefficients of $P(\lambda)$. Let S be a fixed $nk \times nk$ block-signature matrix. If $SL_P(\lambda)$ is T-even (resp. T-odd) for any T-even (resp. T-odd) matrix polynomial $P(\lambda)$, then $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$, $(\mathbf{l}_q, \mathbf{r}_q)$, $(\mathbf{l}_z, \mathbf{z}, \mathbf{r}_z)$, and $(\mathbf{l}_z, \mathbf{r}_z)$ are symmetric tuples.*

A consequence of Lemma 4.2 and [4, Lemmas 5.1 and 5.5] is the following.

COROLLARY 4.3. *Let $L_P(\lambda)$ be a FPR of the form (2.2) depending on the coefficients of $P(\lambda)$. Let S be a fixed $nk \times nk$ block-signature matrix. If $SL_P(\lambda)$ is T-even (resp. T-odd) for any T-even (resp. T-odd) matrix polynomial $P(\lambda)$, then $L_P(\lambda)$ is block-symmetric for any $P(\lambda)$.*

REMARK 4.4. *Suppose that $SL_P(\lambda)$ is a T-even (respectively, T-odd) pencil for any T-even (respectively, T-odd) matrix polynomial $P(\lambda)$, where $L_P(\lambda)$ is a FPR depending on the coefficients of $P(\lambda)$, which must be block-symmetric. From [4, Corollary 5.6], and Corollary* **??** *we can conclude that, when k is even and both coefficients A_0 and A_k of $P(\lambda)$ are singular, $L_P(\lambda)$ does not satisfy the nonsingularity conditions, since h and $k-h-1$ cannot be both even and, therefore, $-k$ is in $(\mathbf{l}_z, \mathbf{r}_z)$ or 0 is in $(\mathbf{l}_q, \mathbf{r}_q)$.*

4.1. T-alternating products for admissible tuples. In this section we study the relationship between the matrices $M_{\mathbf{s}}^T$ and $M_{\mathbf{s}}$, when $P(\lambda)$ is a T-alternating matrix polynomial and $\mathbf{s}$ is either $(\mathbf{w}, \mathbf{r}_w)$ or $\mathbf{r}_w$, where $\mathbf{w}$ is an admissible tuple relative to $\{0 : h\}$, $0 \leq h < k$, and $\mathbf{r}_w$ is the symmetric complement of $\mathbf{w}$. We consider separately the T-even and T-odd cases. The results in this section will be given for tuples of nonnegative integers. The next lemma allows us to state parallel results for tuples of negative integers.

LEMMA 4.5. *Let $\mathbf{s}$ be a tuple with indices from $\{0 : k-1\}$, $k \geq 2$. Let $\mathbf{r} = -k + \mathbf{s}$ and S_1, S_2 be block-signature matrices. Let R be the matrix (2.1). Then, $M_{\mathbf{s}}^T = S_1 M_{\mathbf{s}} S_2$ for any T-even (resp. T-odd) matrix polynomial $P(\lambda)$ of degree k if and only if*

$$M_{\mathbf{r}}^T = (RS_1R)\, M_{\mathbf{r}}\, (RS_2R)$$

for any T-even (resp. T-odd) matrix polynomial $P(\lambda)$, if k is even, and for any T-odd (resp. T-even) matrix polynomial $P(\lambda)$, if k is odd.

Proof. We prove the claim not in parentheses. The proof of the other claim is similar. We show the "if" implication. The "only if" implication can be proven using similar arguments. Suppose that $P(\lambda)$ is T-even in order to show that $M_{\mathbf{s}}^T = S_1 M_{\mathbf{s}} S_2$. Observe that for $i = -k + j$, with $j \in \{0 : k-1\}$, $M_i(P') = RM_jR$, where $P'(\lambda) = -rev(P(\lambda)) = -\lambda^k P(1/\lambda)$. Then, we have

$$M_{\mathbf{s}} = RM'_{\mathbf{r}}R, \tag{4.2}$$

where $M'_{\mathbf{r}}$ denotes $M_{r_1}(P') \cdots M_{r_l}(P')$, for $\mathbf{r} = (r_1, \ldots, r_l)$. Note that $P'(\lambda)$ is T-even if k is even, and is T-odd if k is odd. Suppose that k is even. Then, by hypothesis, we have

$$(M'_{\mathbf{r}})^T = (RS_1R)M'_{\mathbf{r}}(RS_2R). \tag{4.3}$$

From (4.2) and (4.3), we have

$$\begin{aligned} M_{\mathbf{s}}^T &= (RM'_{\mathbf{r}}R)^T = S_1 RM'_{\mathbf{r}}RS_2 \\ &= S_1 M_{\mathbf{s}} S_2. \end{aligned}$$

The case in which k is odd can be proven similarly. □

Let $\mathbf{s} = (\mathbf{w}, \mathbf{r}_w)$ or $\mathbf{s} = \mathbf{r}_w$, where $\mathbf{w}$ is an admissible tuple of index l relative to $\{0 : h\}$, $0 \leq h < k$, and $\mathbf{r}_w$ is the symmetric complement of $\mathbf{w}$. In the next subsections we show that, in both the T-even and the T-odd cases, $\mathbf{s}$ is either direct-transpose related or complement-transpose related. Moreover, $(\mathbf{w}, \mathbf{r}_w)$ is direct-transpose related if and only if $\mathbf{r}_w$ is complement-transpose related. We also give block-signature matrices S and S' such that $M_{\mathbf{s}}^T = S' M_{\mathbf{s}} S$. The other possible choices of S and S' can be obtained by using Lemma 2.10, taking into account Remark 2.8, which characterizes the connected components of $(\mathbf{w}, \mathbf{r}_w)$ and $\mathbf{r}_w$ needed for the application of the lemma.

4.1.1. The T-even case. Here we assume that $P(\lambda)$ is T-even. In this case, we have $M_i^T = M_i$ if i is even and $M_i^T = S_{k-i} M_i S_{k-i+1} = S_{k-i+1} M_i S_{k-i}$ if i is odd. The main result here shows that if $\mathbf{w}$ is an admissible tuple and $\mathbf{r}_w$ is the corresponding symmetric complement then, depending on the parity of the index of $\mathbf{w}$, either $(\mathbf{w}, \mathbf{r}_w)$ is T-even direct-transpose related and $\mathbf{r}_w$ is T-even complement-transpose related, or $(\mathbf{w}, \mathbf{r}_w)$ is T-even complement-transpose related and $\mathbf{r}_w$ is T-even direct-transpose related.

The next proposition is used in the proof of Lemma 4.6.

PROPOSITION 4.6. *Let $P(\lambda)$ be a T-even matrix polynomial of degree $k \geq 2$. Let $\mathbf{s} = (0 : l+1)_{rev_c}$ for some $0 \leq l < k$. Then,*

$$M_{\mathbf{s}}^T = \begin{cases} S_J M_{\mathbf{s}} S_{H \ominus J} & \text{if } l \text{ is odd,} \\ S_{J'} M_{\mathbf{s}} S_{J'} & \text{if } l \text{ is even,} \end{cases} \tag{4.4}$$

or, equivalently,

$$M_{\mathbf{s}}^T = \begin{cases} S_{H \ominus J} M_{\mathbf{s}} S_J & \text{if } l \text{ is odd,} \\ S_{H \ominus J'} M_{\mathbf{s}} S_{H \ominus J'} & \text{if } l \text{ is even,} \end{cases} \tag{4.5}$$

where $J = \{k-l :_2 k-1\}$, $J' = \{k-l+1 :_2 k-1\}$ and $H = \{k-l : k\}$.

Proof. Note that, for each l, the expressions for $M_{\mathbf{s}}^T$ in (4.4) and (4.5) are equal, as the one in (4.5) is obtained from that in (4.4) by multiplying it on the left and on the right by the matrix $I_{k-l-1} \oplus -I_{l+1}$, which commutes with $M_{\mathbf{s}}$. We prove the expressions in (4.4) by induction on l. We use Proposition 2.2. For $l = 0$ we have $\mathbf{s} = (0)$. Since $M_0^T = M_0$, the claim follows in this case. It is easy to see that $M_{0:1,0}^T = S_{k-1} M_{0:1,0} S_k$, proving the claim for $l = 1$. Assume now that l is even. For l odd the proof is similar. By the inductive hypothesis and taking into account that $\mathbf{s}$ is symmetric, we have

$$\begin{aligned} M_{(0:l+1)_{rev_c}}^T &= revtr(M_{0:l}) M_{(0:l)_{rev_c}}^T \\ &= \left(S_{k-l+1:_2k-1} M_{0:l} S_{k-l+2:_2k}\right)\left(S_{k-l+2:_2k} M_{(0:l)_{rev_c}} S_{k-l+1:_2k-1}\right) \\ &= S_{k-l+1:_2k-1} M_{(0:l+1)_{rev_c}} S_{k-l+1:_2k-1}. \end{aligned}$$

□

LEMMA 4.7. *Let $P(\lambda)$ be a T-even matrix polynomial of degree $k \geq 2$. Let $\mathbf{w}$ be an admissible tuple of index l relative to $\{0 : h\}$, with $0 \leq l \leq h < k$. Let $\mathbf{r}_w$ be the symmetric complement of $\mathbf{w}$.*

- *If l is even, then*

$$M_{\mathbf{w},\mathbf{r}_w}^T = S_J M_{\mathbf{w},\mathbf{r}_w} S_J, \quad \textit{and} \quad M_{\mathbf{r}_w}^T = S_J M_{\mathbf{r}_w} S_{H \ominus J}, \tag{4.6}$$

where

$$J = \begin{cases} \{k-h+1 :_4 k-l-3, k-h+4 :_4 k-l, k-l+1 :_2 k-1\} & \text{if } h-l \equiv 0 \mod 4, \\ \{k-h+1 :_4 k-l-1, k-h+4 :_4 k-l-2, k-l+2 :_2 k\} & \text{if } h-l \equiv 2 \mod 4, \end{cases}$$

and $H = \{k-h+1 : k\}$.

- *If* l *is odd, then*

$$M^T_{\mathbf{w},\mathbf{r}_w} = S_J M_{\mathbf{w},\mathbf{r}_w} S_{H \ominus J}, \quad \text{and} \quad M^T_{\mathbf{r}_w} = S_J M_{\mathbf{r}_w} S_J, \tag{4.7}$$

where

$$J = \begin{cases} \{k-h :_4 k-l-4, k-h+3 :_4 k-l-1, k-l :_2 k-1\} & \text{if } h-l \equiv 0 \mod 4, \\ \{k-h+2 :_4 k-l-4, k-h+1 :_4 k-l-1, k-l :_2 k-1\} & \text{if } h-l \equiv 2 \mod 4, \end{cases}$$

and $H = \{k-h : k\}$.

Proof. We assume that l is even. Note that in this case h is even as well. The proof when l is odd is similar. The proof is by induction on the number $h-l$. We first prove the claim for $M_{\mathbf{w},\mathbf{r}_w}$. If $h-l=0$, the claim follows from Proposition 4.5. Suppose that $h-l=2$. Taking into account Proposition 4.5, we get

$$\begin{aligned} M^T_{\mathbf{w},\mathbf{r}_w} &= \left(M^T_{l+1}M_{l+2}\right) M^T_{(0:l+1)_{rev_c}} M^T_{l+1} \\ &= (S_{k-l-1}M_{l+1}S_{k-l}M_{l+2}) \left(S_{k-l:_2k}M_{(0:l+1)_{rev_c}}S_{k-l:_2k}\right)(S_{k-l}M_{l+1}S_{k-l-1}) \\ &= S_{k-l-1}S_{k-l+2:_2k}M_{l+1:l+2}M_{(0:l+1)_{rev_c}}S_{k-l+2:_2k}M_{l+1}S_{k-l-1} \\ &= S_{k-l-1}S_{k-l+2:_2k}M_{\mathbf{w},\mathbf{r}_w}S_{k-l+2:_2k}S_{k-l-1}, \end{aligned}$$

which proves the first claim in (4.6) when $h-l=2$. Now suppose that $h-l \geq 4$. We have

$$(\mathbf{w},\mathbf{r}_w) \sim (h-1:h, \mathbf{w}', \mathbf{r}_{w'}, h-1),$$

where $\mathbf{w}' = (h-3:h-2, \ldots, l+1:l+2, (0:l+1)_{rev_c})$ and $\mathbf{r}_{w'}$ is the symmetric complement of $\mathbf{w}'$. Using the induction hypothesis and the fact that $(\mathbf{w},\mathbf{r}_w)$ is symmetric, we get

$$\begin{aligned} M^T_{\mathbf{w},\mathbf{r}_w} &= \left(M^T_{h-1}M_h\right) M^T_{\mathbf{w}',\mathbf{r}_{w'}} M^T_{h-1} \\ &= (S_{k-h+1}M_{h-1}S_{k-h+2}M_h)\, M^T_{\mathbf{w}',\mathbf{r}_{w'}}\, (S_{k-h+1}M_{h-1}S_{k-h+2}) \\ &= (S_{k-h+1}M_{h-1:h})\, S_{k-h+2}S'M_{\mathbf{w}',\mathbf{r}_{w'}}S'\, (S_{k-h+1}M_{h-1}S_{k-h+2}) \\ &= (S_{k-h+1}S'M_{h-1:h})\, S_{k-h+3:k}S_{k-h+2:k}M_{\mathbf{w}',\mathbf{r}_{w'}}S'\, (S_{k-h+1}M_{h-1}S_{k-h+2}) \\ &= (S_{k-h+1}S'S_{k-h+3:k}M_{h-1:h})\, M_{\mathbf{w}',\mathbf{r}_{w'}}S'S_{k-h+2:k}\, (S_{k-h+1}M_{h-1}S_{k-h+2}) \\ &= (S_{k-h+1}S'S_{k-h+3:k})\left(M_{h-1:h}M_{\mathbf{w}',\mathbf{r}_{w'}}M_{h-1}\right) S'S_{k-h+2:k}S_{k-h+2}S_{k-h+1}, \end{aligned}$$

where $S' = S_{k-h+3:_4k-l-3}S_{k-h+6:_4k-l}S_{k-l+1:_2k-1}$ if $h-l \equiv 2, (\mod 4)$, and $S' = S_{k-h+3:_4k-l-1}S_{k-h+6:_4k-l-2}S_{k-l+2:_2k}$ if $h-l \equiv 0 \mod 4$. It can be easily seen that $S_{k-h+1}S'S_{k-h+3:k} = S_J$, which proves the first claim in (4.6).

Now we prove the claim for $M_{\mathbf{r}_w}$. Let $\widetilde{J} = \{k-l+1 :_2 k-1\}$ and $\widetilde{H} = \{k-l+1 : k\}$. Suppose that $h-l=0$. Then, using Proposition 4.5,

$$M^T_{r_w} = S_{\widetilde{J}}M_{(0:l)_{rev_c}}S_{\widetilde{H}\ominus\widetilde{J}} = S_{k-h+1:_2k-1}M_{\mathbf{r}_w}S_{k-h+2:_2k} = S_JM_{\mathbf{r}_w}S_{H\ominus J}.$$

Suppose that $h-l=2$. Again, using Proposition 4.5, we have

$$\begin{aligned} M_{r_w}^T &= \left(S_{\widetilde{H}\ominus\widetilde{J}}M_{(0:l)_{rev_c}}S_{\widetilde{J}}\right)(S_{k-l-1}M_{l+1}S_{k-l}) = S_{\widetilde{H}\ominus\widetilde{J}}S_{k-l-1}M_{(0:l)_{rev_c}}M_{l+1}S_{\widetilde{J}}S_{k-l} \\ &= S_{k-l+2:_2k}S_{k-l-1}M_{\mathbf{r}_w}S_{k-l+1:_2k-1}S_{k-l} = S_J M_{\mathbf{r}_w}S_{H\ominus J}. \end{aligned}$$

Note that $S_{k-l+1:k}$ commutes with $M_{\mathbf{r}_w}$. Now suppose that $h-l\geq 4$. We have $\mathbf{r}_w \sim (\mathbf{r}_{w'}, h-1)$, where $\mathbf{r}_{w'} = ((0:l)_{rev_c}, l+1, ..., h-3)$. Using the induction hypothesis and the fact that $\mathbf{r}_w$ is symmetric, we get

$$\begin{aligned} M_{\mathbf{r}_w}^T &= \left(S_{J'}M_{\mathbf{r}_{w'}}S_{H'\ominus J'}\right)(S_{k-h+2}M_{h-1}S_{k-h+1}) \\ &= S_{J'}S_{k-h+2}M_{\mathbf{r}_{w'}}M_{h-1}S_{H'\ominus J'}S_{k-h+1} \\ &= S_{J'}S_{k-h+2}S_{k-h+1:k}M_{\mathbf{r}_w}S_{k-h+1:k}S_{H'\ominus J'}S_{k-h+1} \\ &= S_J M_{\mathbf{r}_w}S_{H\ominus J}, \end{aligned}$$

where $J' = \{k-h+3 :_4 k-l-3, k-h+6 :_4 k-l, k-l+1 :_2 k-1\}$ if $h-l\equiv 2$ (mod 4), $J' = \{k-h+3 :_4 k-l-1, k-h+6 :_4 k-l-2, k-l+2 :_2 k\}$ if $h-l\equiv 0$ (mod 4), and $H' = \{k-h+3 : k\}$. $\square$

4.1.2. The T-odd case. We now state a parallel result to Lemma 4.6 for the case in which the matrix polynomial $P(\lambda)$ is T-odd. Its proof is omitted as it is similar to the proof of Lemma 4.6. Note that, when $P(\lambda)$ is T-odd, we have $M_i^T = M_i$ if i is odd and $M_i^T = S_{k-i}M_iS_{k-i+1} = S_{k-i+1}M_iS_{k-i}$ if i is even.

Lemma 4.8. *Let $P(\lambda)$ be a T-odd matrix polynomial of degree $k\geq 2$. Let $\mathbf{w}$ be an admissible tuple of index l relative to $\{0:h\}$, with $0\leq l\leq h<k$. Let $\mathbf{r}_w$ be the symmetric complement of $\mathbf{w}$.*

- *If l is even, then*

$$M_{\mathbf{w},\mathbf{r}_w}^T = S_J M_{\mathbf{w},\mathbf{r}_w}S_{H\ominus J}, \quad \text{and} \quad M_{\mathbf{r}_w}^T = S_J M_{\mathbf{r}_w}S_J, \tag{4.8}$$

where

$$J = \begin{cases} \{k-h :_4 k-l-4, k-h+3 :_4 k-l-1, k-l :_2 k\} & \text{if } h-l\equiv 0 \mod 4, \\ \{k-h+2 :_4 k-l-4, k-h+1 :_4 k-l-1, k-l :_2 k\} & \text{if } h-l\equiv 2 \mod 4, \end{cases}$$

and $H = \{k-h : k\}$.

- *If l is odd, then*

$$M_{\mathbf{w},\mathbf{r}_w}^T = S_J M_{\mathbf{w},\mathbf{r}_w}S_J, \quad \text{and} \quad M_{\mathbf{r}_w}^T = S_J M_{\mathbf{r}_w}S_{H\ominus J}, \tag{4.9}$$

where

$$J = \begin{cases} \{k-h+1 :_4 k-l-3, k-h+4 :_4 k-l, k-l+1 :_2 k\} & \text{if } h-l\equiv 0 \mod 4, \\ \{k-h+1 :_4 k-l-1, k-h+4 :_4 k-l-2, k-l+2 :_2 k-1\} & \text{if } h-l\equiv 2 \mod 4, \end{cases}$$

and $H = \{k-h+1 : k\}$.

4.2. T-even strong linearizations: auxiliary results. We now derive some necessary and sufficient conditions on a FPR $L_P(\lambda)$ so that $SL_P(\lambda)$ is T-even for some block-signature matrix S, when the matrix polynomial $P(\lambda)$ is.

In the next lemma we show that, if $L_P(\lambda)$ is a FPR of the form (2.2) and $SL_P(\lambda)$ is T-even, then $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ and $(\mathbf{l}_z, \mathbf{r}_z)$ are T-even direct-transpose related, while $(\mathbf{l}_z, \mathbf{z}, \mathbf{r}_z)$ and $(\mathbf{l}_q, \mathbf{r}_q)$ are T-even complement-transpose related. Note that, because of Corollary **??** and [4, Corollary 5.6], we can assume that $L_P(\lambda)$ has the form considered next.

Lemma 4.9. *Let $P(\lambda)$ be a T-even matrix polynomial of degree $k \geq 2$ of the form (1.1). Let h be an integer such that $0 \leq h < k$. Let $\mathbf{w}$ and $\mathbf{w}'$ be admissible tuples relative to $\{0 : h\}$ and $\{0 : k-h-1\}$, respectively, and $\mathbf{r}_w$ and $\mathbf{r}_{w'}$ be the symmetric complements of $\mathbf{w}$ and $\mathbf{w}'$, respectively. Let $\mathbf{t}_w$ and $\mathbf{t}_{w'}$ be tuples with indices from $\{0 : h-1\}$ and $\{0 : k-h-2\}$, respectively, such that $(rev(\mathbf{t}_w), \mathbf{w}, \mathbf{r}_w, \mathbf{t}_w)$ and $(rev(\mathbf{t}_{w'}), \mathbf{w}', \mathbf{r}_{w'}, \mathbf{t}_{w'})$ satisfy the SIP. Let $\mathbf{z} = -k + \mathbf{w}'$, $\mathbf{r}_z = -k + \mathbf{r}_{w'}$ and $\mathbf{t}_z = -k + \mathbf{t}_{w'}$. Let $L_P(\lambda)$ be the FPR given by (3.8). Let S be a block-signature matrix and consider the following decompositions of S*

$$S = T_1 U_1 = T_2 U_2,$$

where T_1, U_1, T_2, U_2 have the forms $I_{n(k-h-1)} \oplus [\star]$, $[\star] \oplus I_{n(h+1)}$, $[\star] \oplus I_{nh}$ and $I_{n(k-h)} \oplus [\star]$, respectively. Then, $SL_P(\lambda)$ is T-even if and only if

$$M^T_{rev(\mathbf{t}_w),\mathbf{w},\mathbf{r}_w,\mathbf{t}_w} = T_1 M_{rev(\mathbf{t}_w),\mathbf{w},\mathbf{r}_w,\mathbf{t}_w} T_1, \tag{4.10}$$
$$M^T_{rev(\mathbf{t}_w),\mathbf{r}_w,\mathbf{t}_w} = (I_{n(k-h)} \oplus -I_{nh}) U_2 M_{rev(\mathbf{t}_w),\mathbf{r}_w,\mathbf{t}_w} U_2 \tag{4.11}$$

and

$$M^T_{rev(\mathbf{t}_z),\mathbf{z},\mathbf{r}_z,\mathbf{t}_z} = (-I_{n(k-h)} \oplus I_{nh}) T_2 M_{rev(\mathbf{t}_z),\mathbf{z},\mathbf{r}_z,\mathbf{t}_z} T_2, \tag{4.12}$$
$$M^T_{rev(\mathbf{t}_z),\mathbf{r}_z,\mathbf{t}_z} = U_1 M_{rev(\mathbf{t}_z),\mathbf{r}_z,\mathbf{t}_z} U_1. \tag{4.13}$$

Proof. Let $L_P(\lambda) = \lambda L_1 - L_0$, with $L_1 = M_{rev(\mathbf{t}_w),rev(\mathbf{t}_z),\mathbf{z},\mathbf{r}_z,\mathbf{t}_z,\mathbf{r}_w,\mathbf{t}_w}$ and $L_0 = M_{rev(\mathbf{t}_w),rev(\mathbf{t}_z),\mathbf{w},\mathbf{r}_z,\mathbf{t}_z,\mathbf{r}_w,\mathbf{t}_w}$. Suppose that $SL_P(\lambda)$ is T-even. Then

$$\begin{aligned} L_0^T &= SL_0S = U_1 T_1 M_{rev(\mathbf{t}_w),\mathbf{w},\mathbf{r}_w,\mathbf{t}_w} M_{rev(\mathbf{t}_z),\mathbf{r}_z,\mathbf{t}_z} T_1 U_1 \\ &= (T_1 M_{rev(\mathbf{t}_w),\mathbf{w},\mathbf{r}_w,\mathbf{t}_w} T_1)(U_1 M_{rev(\mathbf{t}_z),\mathbf{r}_z,\mathbf{t}_z} U_1). \end{aligned}$$

Analogously,

$$L_1^T = -SL_1S = -(T_2 M_{rev(\mathbf{t}_z),\mathbf{z},\mathbf{r}_z,\mathbf{t}_z} T_2)(U_2 M_{rev(\mathbf{t}_w),\mathbf{r}_w,\mathbf{t}_w} U_2).$$

Thus, taking into account the form of the matrices, the claim follows. The converse follows by reversing the arguments. ◻

Remark 4.10. *Since $U_1T_1 = T_2U_2$ and taking into account the form of the matrices U_i,T_i, $i = 1, 2$, it follows that the block-signature matrices T_1 and U_2 (resp. T_2 and U_1) are equal except possibly in the block-entry in position $(k-h, k-h)$. Thus, conditions (4.11) and (4.13) are equivalent to those obtained from them by replacing U_1 and U_2 by T_2 and T_1, respectively. Note also that the block-entries in position $(k-h, k-h)$ in T_1 and in T_2 coincide.*

The next lemma gives a necessary condition on h for $SL_P(\lambda)$ to be T-even when $P(\lambda)$ is or, taking into account Lemma 4.8, for conditions (4.10)-(4.13) to hold.

Lemma 4.11. *Let $(rev(\mathbf{t}_z), \mathbf{z}, \mathbf{r}_z, \mathbf{t}_z)$ and $(rev(\mathbf{t}_w), \mathbf{w}, \mathbf{r}_w, \mathbf{t}_w)$ be index tuples as in Lemma 4.8. If there exists a block-signature matrix S such that the pencil $SL_P(\lambda)$, where $L_P(\lambda)$ is of the form (3.8), is T-even for any T-even matrix polynomial $P(\lambda)$ of degree k, then h is even.*

Proof. By Lemma 4.8, $M_{rev(\mathbf{t}_w),\mathbf{w},\mathbf{r}_w,\mathbf{t}_w}$ is T-even direct-transpose related. Thus, no block A_i with i odd can be on the main diagonal of $M_{rev(\mathbf{t}_w),\mathbf{w},\mathbf{r}_w,\mathbf{t}_w}$. We will show that A_h appears in position $(k-h, k-h)$ in $M_{rev(\mathbf{t}_w),\mathbf{w},\mathbf{r}_w,\mathbf{t}_w}$, which implies the result. If $h = 0$ then $(rev(\mathbf{t}_w), \mathbf{w}, \mathbf{r}_w, \mathbf{t}_w) = (0)$ and the claim follows. Now suppose that $h > 0$. Because $(rev(\mathbf{t}_w), \mathbf{w}, \mathbf{r}_w, \mathbf{t}_w)$ satisfies the SIP, it is equivalent to a tuple in column standard form, which has the form $(a : h, \mathbf{t})$ for some index $0 \leq a \leq h$ and some tuple $\mathbf{t}$ with indices from $\{0 : h-1\}$. Note that $\mathbf{t}$ is nonempty as $(rev(\mathbf{t}_w), \mathbf{w}, \mathbf{r}_w, \mathbf{t}_w)$ is not a simple tuple. A calculation shows that

$$M_{a:h} = \left[\begin{array}{c|c|ccc|c} I_{n(k-h-1)} & 0 & 0 & \cdots & & 0 \\ \hline 0 & -A_h & & & & 0 \\ \vdots & -A_{h-1} & & & & \vdots \\ \vdots & \vdots & & I & & \vdots \\ 0 & -A_a & & & & 0 \\ \hline 0 & I_n & 0 & \cdots & 0 & 0 \\ \hline 0 & 0 & 0 & \cdots & 0 & I_{n(a-1)} \end{array}\right],$$

if $a \neq 0$, and

$$M_{a:h} = \left[\begin{array}{c|c|c} I_{n(k-h-1)} & 0 & 0 \\ \hline 0 & -A_h & \\ & -A_{h-1} & \\ \vdots & \vdots & I \\ 0 & -A_1 & \\ \hline 0 & -A_0 & 0 \end{array}\right],$$

if $a = 0$. On the other hand, $M_{\mathbf{t}}$ has the form

$$\begin{bmatrix} I_{n(k-h)} & 0 \\ 0 & \star \end{bmatrix}.$$

Therefore, the matrix $M_{rev(\mathbf{t}_w), \mathbf{w}, \mathbf{r}_w, \mathbf{t}_w} = M_{a:h} M_{\mathbf{t}}$ has the form

$$\left[\begin{array}{c|cc|c} I_{n(k-h-1)} & 0 & 0 & 0 \\ \hline 0 & -A_h & \star & 0 \\ 0 & -A_{h-1} & \star & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & -A_a & \star & 0 \\ 0 & I_n & \star & 0 \\ \hline 0 & 0 & 0 & \star \end{array}\right]$$

if $a \neq 0$, and the form

$$\left[\begin{array}{c|cc} I_{n(k-h-1)} & 0 & 0 \\ \hline 0 & -A_h & \star \\ 0 & -A_{h-1} & \star \\ \vdots & \vdots & \vdots \\ 0 & -A_1 & \star \\ 0 & -A_0 & \star \end{array}\right],$$

if $a = 0$. Thus, the claim follows. $\square$

4.3. T-odd strong linearizations: auxiliary results. In this section we present an analysis similar to the one in the previous section, considering now that the matrix polynomial $P(\lambda)$ is T-odd.

We start by stating an analog of Lemma 4.8 in the T-odd case. In this case, though, if $L_P(\lambda)$ is a FPR of the form (2.2), we get that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ and $(\mathbf{l}_z, \mathbf{r}_z)$ are T-odd complement-transpose related, while $(\mathbf{l}_z, \mathbf{z}, \mathbf{r}_z)$ and $(\mathbf{l}_q, \mathbf{r}_q)$ are T-odd direct-transpose related. We omit the proof as it is analogous to the one of Lemma 4.8.

LEMMA 4.12. *Let $P(\lambda)$ be a T-odd matrix polynomial of degree $k \geq 2$ of the form (1.1). Let h be an integer such that $0 \leq h < k$. Let $\mathbf{w}$ and $\mathbf{w}'$ be admissible tuples relative to $\{0 : h\}$ and $\{0 : k-h-1\}$, respectively, and $\mathbf{r}_w$ and $\mathbf{r}_{w'}$ be the symmetric complements of $\mathbf{w}$ and $\mathbf{w}'$, respectively. Let $\mathbf{t}_w$ and $\mathbf{t}_{w'}$ be tuples with indices from $\{0 : h-1\}$ and $\{0 : k-h-2\}$, respectively, such that $(rev(\mathbf{t}_w), \mathbf{w}, \mathbf{r}_w, \mathbf{t}_w)$ and $(rev(\mathbf{t}_{w'}), \mathbf{w}', \mathbf{r}_{w'}, \mathbf{t}_{w'})$ satisfy the SIP. Let $\mathbf{z} = -k + \mathbf{w}'$, $\mathbf{r}_z = -k + \mathbf{r}_{w'}$ and $\mathbf{t}_z = -k + \mathbf{t}_{w'}$. Let $L_P(\lambda)$ be the FPR of the form (3.8) and S be a block-signature matrix. Consider the decompositions of S*

$$S = T_1 U_1 = T_2 U_2$$

where T_1, U_1, T_2, U_2 have the forms $I_{n(k-h-1)} \oplus [\star]$, $[\star] \oplus I_{n(h+1)}$, $[\star] \oplus I_{nh}$ and $I_{n(k-h)} \oplus [\star]$, respectively. Then, $SL_P(\lambda)$ is T-odd if and only if

$$M^T_{rev(\mathbf{t}_w),\mathbf{w},\mathbf{r}_w,\mathbf{t}_w} = (I_{n(k-h-1)} \oplus -I_{n(h+1)}) T_1 M_{rev(\mathbf{t}_w),\mathbf{w},\mathbf{r}_w,\mathbf{t}_w} T_1, \tag{4.14}$$
$$M^T_{rev(\mathbf{t}_w),\mathbf{r}_w,\mathbf{t}_w} = U_2 M_{rev(\mathbf{t}_w),\mathbf{r}_w,\mathbf{t}_w} U_2 \tag{4.15}$$

and

$$M^T_{rev(\mathbf{t}_z),\mathbf{z},\mathbf{r}_z,\mathbf{t}_z} = T_2 M_{rev(\mathbf{t}_z),\mathbf{z},\mathbf{r}_z,\mathbf{t}_z} T_2, \tag{4.16}$$
$$M^T_{rev(\mathbf{t}_z),\mathbf{r}_z,\mathbf{t}_z} = (-I_{n(k-h-1)} \oplus I_{n(h+1)}) U_1 M_{rev(\mathbf{t}_z),\mathbf{r}_z,\mathbf{t}_z} U_1. \tag{4.17}$$

An observation analogous to Remark 4.9 can be made in the T-odd case.

The next lemma gives a necessary condition on h for $SL_P(\lambda)$ to be T-odd when $P(\lambda)$ is. We omit its proof as it is similar to the one of Lemma 4.10.

LEMMA 4.13. *Let $(rev(\mathbf{t}_z), \mathbf{z}, \mathbf{r}_z, \mathbf{t}_z)$ and $(rev(\mathbf{t}_w), \mathbf{w}, \mathbf{r}_w, \mathbf{t}_w)$ be index tuples as in Lemma 4.11. If there exists a block-signature matrix S such that the pencil $SL_P(\lambda)$, where $L_P(\lambda)$ is of the form (3.8), is T-odd for any T-odd matrix polynomial $P(\lambda)$ of degree k, then h is even.*

4.4. Construction of T-alternating linearizations from FPR. In this section we give a family of strong linearizations from the set of FPR which are T-alternating when the matrix polynomial $P(\lambda)$ is, with the same parity as $P(\lambda)$. These linearizations are of the form $SL_P(\lambda)$, where $L_P(\lambda)$ is a FPR and S is a block-signature matrix. As follows from Theorem 4.14, this block-signature matrix is the matrix given by the next algorithm and is denoted by S^+ in the T-even case and by S^{++} in the T-odd case. In each case it only depends on the admissible tuples used in the construction of the FPR $L_P(\lambda)$.

ALGORITHM 4.14. *(Construction of the matrix S) Let $k \geq 2$ and h be even such that $0 \leq h < k$. Let $\mathbf{w}$ be an admissible tuple of index l relative to $\{0 : h\}$ and $\mathbf{w}'$ be an admissible tuple of index l' relative to $\{0 : k-h-1\}$.*

1. *Let Q_1^+ be the matrix given by S_J in (4.6) and Q_1^{++} be the matrix given by S_J in (4.8).*

2. Let $Q_2^+ = RV_2^+R$ and $Q_2^{++} = RV_2^{++}R$, where V_2^+ and V_2^{++} are constructed as follows:

- *V_2^+ is the matrix S_J in (4.7) if k is even and in (4.8) if k is odd, with $\mathbf{w}$ being $\mathbf{w}'$ (that is, with h and l replaced by $k-h-1$ and l', resp.),*
- *V_2^{++} is the matrix S_J in (4.6) if k is odd and in (4.9) if k is even, with $\mathbf{w}$ being $\mathbf{w}'$.*

3. Let

- *$S^+ = Q_2^+[1:k-h] \oplus \epsilon^+ Q_1^+[k-h+1:k]$, where $\epsilon^+ \in \{1,-1\}$ is such that $Q_2^+[k-h] = \epsilon^+ Q_1^+[k-h]$, and*
- *$S^{++} = Q_2^{++}[1:k-h] \oplus \epsilon^{++} Q_1^{++}[k-h+1:k]$, where $\epsilon^{++} \in \{1,-1\}$ is such that $Q_2^{++}[k-h] = \epsilon^{++} Q_1^{++}[k-h]$.*

By considering a subfamily of pencils $L_P(\lambda)$ in the family of FPR, in the next theorem we give a necessary and sufficient condition for the existence of S such that $SL_P(\lambda)$ is T-alternating with the same parity as the T-alternating matrix polynomial $P(\lambda)$. Observe that the matrix S given by Theorem 4.14 does not depend on $P(\lambda)$.

THEOREM 4.15. *Let h be an integer such that $0 \leq h < k$, $k \geq 2$. Let $\mathbf{w}$ and $\mathbf{w}'$ be admissible tuples relative to $\{0:h\}$ and $\{0:k-h-1\}$, respectively, and $\mathbf{r}_w$ and $\mathbf{r}_{w'}$ be the symmetric complements of $\mathbf{w}$ and $\mathbf{w}'$, respectively. Let $\mathbf{t}_w$ and $\mathbf{t}_{w'}$ be tuples with indices from $\{0:h-1\}$ and $\{0:k-h-2\}$, respectively, such that $(rev(\mathbf{t}_w), \mathbf{w}, \mathbf{r}_w, \mathbf{t}_w)$ and $(rev(\mathbf{t}_{w'}), \mathbf{w}', \mathbf{r}_{w'}, \mathbf{t}_{w'})$ satisfy the SIP. Let $\mathbf{z} = -k + \mathbf{w}'$, $\mathbf{r}_z = -k + \mathbf{r}_{w'}$ and $\mathbf{t}_z = -k + \mathbf{t}_{w'}$. For a matrix polynomial $P(\lambda)$ of degree k, let $L_P(\lambda)$ be the block-symmetric FPR given in (3.8) associated with $P(\lambda)$. If there exists a block-signature matrix S such that $SL_P(\lambda)$ is T-even (resp. T-odd) for any T-even (resp. T-odd) matrix polynomial $P(\lambda)$, then h is even.*

Additionally, if h is even and $\mathbf{t}_z$ and $\mathbf{t}_w$ are the empty tuple, then, up to multiplication by -1, there exists a unique block-signature matrix S such that $SL_P(\lambda)$ is T-even (resp. T-odd) for any T-even (resp. T-odd) matrix polynomial $P(\lambda)$ of degree k, which is the matrix S^+ (resp. S^{++}) given by Algorithm 4.13. Moreover, if $L_P(\lambda)$ satisfies the nonsingularity conditions, then the pencil $SL_P(\lambda)$ is a strong linearization of $P(\lambda)$.

Proof. The necessity of h being even follows from Lemmas 4.10 and 4.12. Now suppose that h is even, $\mathbf{t}_z$ and $\mathbf{t}_w$ are the empty tuple and S is the matrix given by Algorithm 4.13. Let $S = T_1U_1 = T_2U_2$, with T_1 and U_1 of the forms $I_{n(k-h-1)} \oplus [\star]$ and $[\star] \oplus I_{n(h+1)}$, respectively, and T_2 and U_2 of the forms $[\star] \oplus I_{nh}$ and $I_{n(k-h-1)} \oplus [\star]$, respectively, as in Lemmas 4.8 and 4.11. By Lemmas 4.6, 4.7 and 4.4, conditions (4.10)-(4.13) hold if $P(\lambda)$ is T-even, and conditions (4.14)-(4.17) hold if $P(\lambda)$ is T-odd. Thus, from Lemma 4.8 (resp. Lemma 4.11), it follows that $SL_P(\lambda)$ is T-even (resp. T-odd) when $P(\lambda)$ is.

We now show the uniqueness claim. Suppose that $SL_P(\lambda)$ and $S'L_P(\lambda)$ are T-even when $P(\lambda)$ is, where S' is a block-signature matrix. Consider the decompositions of S given above and the corresponding decompositions for S', $S' = T_1'U_1' = T_2'U_2'$. Then, condition (4.10) is satisfied with T_1 and also with T_1 replaced with T_1'. By Remark 2.8 and Lemma 2.10, either $T_1 = T_1'$ or $T_1 = (I_{n(k-h-1)} \oplus -I_{n(h+1)})T_1'$. Similarly, condition (4.12) holds with T_2 and also with T_2 replaced with T_2'. By Remark 2.8 and Lemmas 4.4 and 2.10, we have $T_2' = T_2$ or $T_2' = (-I_{n(k-h)} \oplus I_{nh})T_2$. Since the entries in position $(k-h, k-h)$ of T_1 and T_2, and of T_1' and T_2', coincide, we have that either $S' = S$ or $S' = -S$. A similar argument applies if $P(\lambda)$ is T-odd.

The last claim in the statement follows from Lemma 4.4 in [4].$\square$

If in Theorem 4.14 the pencil $L_P(\lambda)$ satisfies the nonsingularity conditions, then

$SL_P(\lambda)$ is a strong linearization of $P(\lambda)$. In particular, if both admissible tuples $\mathbf{w}$ and $\mathbf{w}'$ have index 0, then $L_P(\lambda)$ satisfies the nonsingularity conditions. These choices of $\mathbf{w}$ and $\mathbf{w}'$ are possible when k is odd, as h and $k-h-1$ are even. If k is even and $\det(A_k) \neq 0$, as h is even, by choosing $\mathbf{w}$ of index 0, we also get T-alternating strong linearizations, independently of A_0 being nonsingular or not. This is not true though if $\det(A_k) = 0$.

EXAMPLE 4.16. *Let $k = 5$. The next table gives the family of T-even pencils of the form $SL_P(\lambda)$ associated with a T-even matrix polynomial $P(\lambda)$ of degree k, as described in Theorem 4.14. Whenever the index of $\mathbf{w}$ (resp. $\mathbf{w}'$) is not zero, we assume that A_0 (resp. A_k) is invertible, so that the corresponding pencil satisfies the nonsingularity conditions and is, therefore, a strong linearization of $P(\lambda)$.*

As in the skew-symmetric case, we only present one pencil explicitly. However, in all cases we give the index tuples and the block-signature matrices that describe the pencils. Note that all these pencils are obtained from the block-symmetric FPR given in [4, Corollary 5.6], corresponding to the same tuples, by changing the sign of some blocks.

$\mathbf{w}$	$\mathbf{w}'$	Q_1	Q_2	S
(0)	(3:4,1:2,0)	I_n	$S_{1:2}S_5$	$S_{1:2}S_5$
(0)	(3:4,0:2)	I_n	$S_1S_{3:4}$	$S_1S_{3:4}$
(0)	(0:4)	I_n	$S_{1:_25}$	$S_{1:_25}$
(1:2,0)	(1:2,0)	S_4	$S_{1:2}$	$S_{1:2}S_4$
(1:2,0)	(0:2)	S_4	S_1S_3	$S_{1:_25}$
(3:4,1:2,0)	(0)	S_2S_5	S_1	$S_1S_{3:4}$
(0:2)	(1:2,0)	S_4	$S_{1:2}$	$S_{1:2}S_4$
(0:2)	(0:2)	S_4	S_1S_3	$S_{1:_25}$
(3:4,0:2)	(0)	S_2S_5	S_1	$S_1S_{3:4}$
(0:4)	(0)	S_2S_4	S_1	$S_{1:_25}$

When $\mathbf{w} = (0)$ *and* $\mathbf{w}' = (3:4, 1:2, 0)$ *we obtain the pencil*

$$\lambda \begin{bmatrix} 0 & 0 & 0 & -I & 0 \\ 0 & -A_5 & 0 & -A_4 & 0 \\ 0 & 0 & 0 & 0 & I \\ I & A_4 & 0 & A_3 & A_2 \\ 0 & 0 & -I & -A_2 & -A_1 \end{bmatrix} - \begin{bmatrix} 0 & -I & 0 & 0 & 0 \\ -I & -A_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & I & A_2 & 0 \\ 0 & 0 & 0 & 0 & A_0 \end{bmatrix}$$

A natural question to be posed is if we could obtain T-alternating linearizations from FPR as in (3.8) when $\mathbf{t}_w$ and/or $\mathbf{t}_{w'}$ are not empty. Although some restrictions are needed, as follows from the example below, the answer to this question is affirmative. Note that the tuples $(\mathbf{w}, \mathbf{r}_w)$ and $(\mathbf{w}', \mathbf{r}_{w'})$ in the table above are equivalent to tuples of the forms $(rev(\mathbf{t}_{w^*}), \mathbf{w}^*, \mathbf{r}_{w^*}, \mathbf{t}_{w^*})$ and $(rev(\mathbf{t}_{w'^*}), \mathbf{w}'^*, \mathbf{r}_{w'^*}, \mathbf{t}_{w'^*})$, respectively, where $\mathbf{w}^*$ and $\mathbf{w}'^*$ are admissible tuples of index 0. In fact, the FPR in the previous table can be rewritten alternatively using these last kind of tuples as follows.

$\mathbf{w}^*$	$\mathbf{w}'^*$	$\mathbf{t}_{w^*}$	$\mathbf{t}_{w'^*}$
(0)	(3:4,1:2,0)	$\emptyset$	$\emptyset$
(0)	(3:4,1:2,0)	$\emptyset$	(0)
(0)	(3:4,1:2,0)	$\emptyset$	$(2, 0:1, 0)$
(1:2,0)	(1:2,0)	$\emptyset$	$\emptyset$
(1:2,0)	(1:2,0)	$\emptyset$	(0)
(1:2,0)	(1:2,0)	(0)	$\emptyset$
(1:2,0)	(1:2,0)	(0)	(0)
(3:4,1:2,0)	(0)	$\emptyset$	$\emptyset$
(3:4,1:2,0)	(0)	(0)	$\emptyset$
(3:4,1:2,0)	(0)	$(2, 0:1, 0)$	$\emptyset$

We finish this section with an example in which $\mathbf{t}_w$ is nonempty and there does not exist S such that $SL_P(\lambda)$ is T-alternating with the same parity as $P(\lambda)$.

EXAMPLE 4.17. *Let $P(\lambda)$ be a T-even matrix polynomial of degree $k = 5$. Let $\mathbf{w} = (3:4, 1:2, 0)$, $\mathbf{t}_w = (2)$, $\mathbf{z} = -5 + (0)$, $\mathbf{t}_z = \emptyset$ and $L_P(\lambda)$ be the FPR (3.8). A calculation shows that*

$$L_P(\lambda) = \lambda \begin{bmatrix} -A_4 & -A_3 & -A_2 & I & 0 \\ -A_3 & -A_2 & -A_1 & 0 & I \\ -A_2 & -A_1 & -A_0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} A_5 & 0 & 0 & 0 & 0 \\ 0 & -A_3 & -A_2 & I & 0 \\ 0 & -A_2 & -A_1 & 0 & I \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \end{bmatrix}.$$

It is clear that there is no block-signature matrix S such that $SL_P(\lambda)$ is T-even because of the structure of the matrix coefficient of the first degree term of $L_P(\lambda)$.

5. Conclusions. In this paper we constructed a family of skew-symmetric (respectively, T-alternating) linearizations of an $n \times n$ skew-symmetric (respectively, T-alternating) matrix polynomial $P(\lambda)$. The linearizations in this family are of the form $SL_P(\lambda)$, where $L_P(\lambda)$ is a FPR and S is a block-signature matrix, that is, a direct sum of blocks of the form I_n and $-I_n$, where I_n denotes the $n \times n$ identity matrix. When $P(\lambda)$ and the pencil $L_P(\lambda)$ satisfy what we call the nonsingularity conditions, then $L_P(\lambda)$, and therefore $SL_P(\lambda)$, is a strong linearization of $P(\lambda)$. We also showed that $L_P(\lambda)$ must be of the form (3.8) for certain tuples $\mathbf{w}$, $\mathbf{z}$, $\mathbf{r}_w$, $\mathbf{r}_z$, $\mathbf{t}_w$,$\mathbf{t}_z$.

In the skew-symmetric case, the linearizations obtained satisfy the restriction that the tuples $\mathbf{t}_w$ and $\mathbf{t}_z$ are of type 1 relative to the tuples $\mathbf{w}$ and $\mathbf{z}$, respectively. The analysis of some examples suggest that it would not be possible to consider tuples which are not of this type, though proving it remains an open problem.

Regarding the T-alternating case, we showed that if, for some block-signature matrix S, $SL_P(\lambda)$ is T-even (resp. T-odd) when the matrix polynomial $P(\lambda)$ is, the number of elements in $\mathbf{w}$ must be odd. Moreover, we showed that the converse holds when the tuples $\mathbf{t}_z$ and $\mathbf{t}_w$ are empty. The case in which $\mathbf{t}_z$ and $\mathbf{t}_w$ are nonempty remains to be studied.

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REFERENCES

[1] E. N. ANTONIOU AND S. VOLOGIANNIDIS, *A new family of companion forms of polynomial matrices*, Electron. J. Linear Algebra, 11 (2004), pp. 78–87.

[2] E. N. Antoniou and S. Vologiannidis, *Linearizations of polynomial matrices with symmetries and their applications*, Electron. J. Linear Algebra, 15 (2006), pp. 107–114.
[3] M.I. Bueno, F. De Teran, and F. M. Dopico, *Recovery of eigenvectors and minimal bases of matrix polynomials from generalized Fiedler linearizations*, SIAM. J. Matrix. Anal. Appl. 32(2011), pp. 463-483.
[4] M.I. Bueno, K. Curlett and S. Furtado, *Structured strong linearizations from Fiedler pencils with repetition I*, preprint.
[5] M.I. Bueno and S. Furtado, *Palindromic linearizations of a matrix polynomial of odd degree obtained from Fiedler pencils with repetition*, Electr. Journal of Linear Algebra, 23 (2012), 562-577.
[6] F. De Terán, F. M. Dopico, and D. S. Mackey, *Fiedler companion linearizations and the recovery of minimal indices*, SIAM J. Matrix Anal. Appl., 31 (2010), pp. 2181–2204.
[7] F. De Terán, F. M. Dopico, and D. S. Mackey, *Spectral equivalence of matrix polynomials and the index sum theorem*, MIMS EPring; 2013.47. Manchester Institute of Mathematical Sciences. School of Mathematics. The University of Manchester (2013).
[8] F. De Terán, F. M. Dopico, and D. S. Mackey, *Structured linearizations for palindromic matrix polynomials of odd degree*, J.Comp. and Appl.Math, 236 (2011), pp. 1464–1480.
[9] M. Fiedler *A note on companion matrices*, Linear Algebra Appl. 372 (2003), pp. 325-331.
[10] F. R. Gantmacher, *The theory of matrices*, Chelsea, New York, 1959.
[11] I. Gohberg, P. Lancaster, and L. Rodman, *Matrix Polynomials*, Academic Press, New York, 1982.
[12] N. J. Higham, D. S. Mackey, N. Mackey, and F. Tisseur, *Symmetric linearizations for matrix polynomials*, SIAM Journal on Matrix Analysis and Applications, 29 (2006), pp. 143-159.
[13] A. Hilliges, C. Mehl, and V. Mehrmann, *On the solution of palindromic eigenvalue problems*, in Proceedings of the 4th European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS), Jyväskylä, Finland, 2004. CD-ROM.
[14] M. Hofer, N. Finger, J. Schöberl, S. Zaglmayr, U. Langer, and R. Lerch, *Finite element simulation of wave propagation in periodic piezoelectric SAW structures*, IEEE Transactions on UFFC, 53 (2006), pp. 1192–1201.
[15] I. C. F. Ipsen, *Accurate eigenvalues for fast trains*, SIAM News, 37(9), Nov. 2004, pp. 1–2.
[16] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann, *Vector spaces of linearizations for matrix polynomials*, SIAM J. Matrix Anal. Appl., 28 (2006), pp. 971–1004.
[17] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann, *Structured polynomial eigenvalue problems: Good vibrations from good linearizations*, SIAM J. Matrix Anal. Appl., 28 (2006), pp. 1029–1051.
[18] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann, *Smith forms of palindromic matrix polynomials*, Electr. Journal of Linear Algebra, 22(2011), pp. 53–91.
[19] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann, *Skew-symmetric matrix polynomials and their Smith forms*, Linear Algebra Appl. 438 (2013), pp. 4625–4653.
[20] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann, *Jordan structures of alternating matrix polynomials*, Linear Algebra Appl. 432(2010), pp. 867–891.
[21] S. Vologiannidis and E. N. Antoniou, *A permuted factors approach for the linearization of polynomial matrices*, Math. Control Signals Syst., 22 (2011), pp. 317–342.
[22] S. Zaglmayr *Eigenvalue problems in SAW-filter simulations*, Diplomarbeit, Inst. of Comp. Mathematics, J. Kepler Univ. Linz, Austria, 2002.