A SIMPLIFIED APPROACH TO FIEDLER-LIKE PENCILS VIA STRONG BLOCK MINIMAL BASES PENCILS.

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Abstract. The standard way of solving the polynomial eigenvalue problem associated with a matrix polynomial is to embed the matrix coefficients of the polynomial into a matrix pencil, transforming the problem into an equivalent generalized eigenvalue problem. Such pencils are known as linearizations. Many of the families of linearizations for matrix polynomials available in the literature are extensions of the so-called family of Fiedler pencils. These families are known as generalized Fiedler pencils, Fiedler pencils with repetition and generalized Fiedler pencils with repetition, or Fiedler-like pencils for simplicity. The goal of this work is to unify the Fiedler-like pencils approach with the more recent one based on strong block minimal bases pencils introduced in [F. M. Dopico, P. W. Lawrence, J. Pérez and P. Van Dooren, *Block Kronecker linearizations of matrix polynomials and their backward errors*, MIMS EPrint 2016.34]. To this end, we introduce a family of pencils that we have named extended block Kronecker pencils, whose members are, under some generic nonsingularity conditions, strong block minimal bases pencils, and show that, with the exception of the non-proper generalized Fiedler pencils, all Fiedler-like pencils belong to this family modulo permutations. As a consequence of this result, we obtain a much simpler theory for Fiedler-like pencils than the one available so far. Moreover, we expect this simplification to allow for further developments in the theory of Fiedler-like pencils such as global or local backward error analyses and eigenvalue conditioning analyses of polynomial eigenvalue problems solved via Fiedler-like linearizations.

Key words. Fiedler pencils, generalized Fiedler pencils, Fiedler pencils with repetition, generalized Fiedler pencils with repetition, matrix polynomials, strong linearizations, block minimal bases pencils, block Kronecker pencils, extended block Kronecker pencils, minimal basis, dual minimal bases

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1. Introduction.

Matrix polynomials and their associated *polynomial eigenvalue problems* appear in many areas of applied mathematics, and they have received in the last years considerable attention. For example, they are ubiquitous in a wide range of problems in engineering, mechanics, control theory, computer-aided graphic design, etc. For detailed discussions of different applications of matrix polynomials, we refer the reader to the classical references [26, 31, 47], the modern surveys [4, Chapter 12] and [41, 48] (and the references therein), and [36, 37, 38]. For those readers not familiar with the theory of matrix polynomials and polynomial eigenvalue problems, those topics are briefly reviewed in Section 2.

The standard way of solving the polynomial eigenvalue problem associated with a matrix polynomial is to *linearize* the polynomial into a matrix pencil (i.e., matrix polynomials of grade 1), known as linearization [16, 25, 26]. The linearization process transforms the polynomial eigenvalue problem into an equivalent generalized eigenvalue problem, which,

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then, can be solved using eigensolvers such as the QZ algorithm or the staircase algorithm, in the case of singular matrix polynomials [42, 49, 50]. Ideally, to make a set of linearizations desirable for numerical applications, it should satisfy the following list of properties:

(i) the linearizations should be *strong linearizations*, regardless whether the matrix polynomial is regular or singular;
(ii) the linearizations should be easily constructible from the coefficients of the matrix polynomials (ideally, without any matrix operation other than scalar multiplication);
(iii) eigenvectors of regular matrix polynomials should be easily recovered from those of the linearizations;
(iv) minimal bases of singular matrix polynomials should be easily recovered from those of the linearizations;
(v) there should exist simple relations between the minimal indices of singular matrix polynomials and the minimal indices of the linearizations, and such relations should be robust under perturbations;
(vi) guarantee global backward stability of polynomial eigenvalue problems solved via linearization.

Additionally, some authors like to add to the above list the following property:

(vii) the linearizations should present one-sided factorizations (such as those used in [27]), which are useful for performing local (i.e., for each particular computed eigenpair) backward error and eigenvalue conditioning analyses of regular polynomial eigenvalue problems solved by linearizations [28, 30].

Furthermore, matrix polynomials that appear in applications usually present algebraic structures, which are reflected in their spectra (see [16, Section 7.2] or [38], for example). If the spectrum of such a polynomial is computed without taking into account the algebraic structure of the polynomial, the rounding errors inherent to numerical computations may destroy qualitative properties of these spectra. Thus, to the mentioned list of desirable properties that a set of linearizations should satisfy, the following property should be added:

(viii) the linearizations of a matrix polynomial $P(\lambda)$ should preserve any algebraic structure that $P(\lambda)$ might posses [38],

and property (vi) should be replaced in the structured case by

(vi-b) guaranteed structured and global backward stability of structured polynomial eigenvalue problems solved via structure-preserving linearizations [19].

In practice, the linearizations used to solve polynomial eigenvalue problems are the well known *Frobenius companion forms*. These pencils satisfy properties (i)–(vii) (see [16, 17, 50], for example). However, Frobenius companion forms do not preserve the algebraic structure that might be present in the matrix polynomials they are associated with, that is, they do not satisfy property (viii). This drawback of the Frobenius companion forms has motivated an intense activity on the theory of linearizations of matrix polynomials, with the goal of finding linearizations satisfying properties (i)–(vii) that, additionally, retain whatever structure the matrix polynomial might possess. See, for example, [2, 5, 8, 9, 10, 14, 21, 29, 34, 38, 44, 45, 51], which is a small sample of recent references in this area.

Although not the first approach (see [32, 33]), the first systematic approach for constructing structure-preserving linearizations was based on pencils belonging to the vector space $\mathbb{DL}(P)$. This vector space was introduced in [37] and further analyzed in [28, 29, 30, 38, 43]. The pencils in this vector space are easily constructible from the matrix coefficients of the matrix polynomial, and almost all of them are strong linearizations when the polynomial is regular. However, none of these pencils is a strong linearization when the polynomial is singular [12] (i.e., these linearizations do not satisfy property (i)), which "questions the utility of such space also when the polynomials are regular but very close to be singular" [8]. Further-

more, none of these pencils, when constructed for a symbolic arbitrary matrix polynomial is always a strong linearization of all regular matrix polynomials.

The second systematic approach for constructing structure-preserving linearizations is based on different extensions of *Fiedler pencils*. Fiedler pencils were introduced in [23] for monic scalar polynomials, and then generalized to regular matrix polynomials in [2], to square singular matrix polynomials in [13], and to rectangular matrix polynomials in [15]. These pencils were baptized as *Fiedler companion pencils* (or Fiedler companion linearizations) in [13]. Later on, the definition of Fiedler companion pencils was extended to include the families of *generalized Fiedler pencils*, *Fiedler pencils with repetition* and *generalized Fiedler pencils with repetition* [2, 8, 51]. These families were used for constructing large families of structure-preserving linearizations. Moreover, they unified the $\mathbb{DL}(P)$ approach with the Fiedler pencils approach, because the standard basis of $\mathbb{DL}(P)$ consists of Fiedler pencils with repetition [5, 51]. We notice that these extensions of the family of Fiedler pencils are only defined for square matrix polynomials. For this reason, in this paper, we restrict our study to the square case. Furthermore, the definition of Fiedler pencils has been extended, also, to allow the construction of linearizations for matrix polynomials that are expressed in other non-monomial bases without any conversion to the monomials [40, 44].

The family of Fiedler companion linearizations is a remarkable set of pencils, since, as it has been proven in [13, 15, 17], it satisfies properties (i)–(v) and property (vii). However, the proofs of these properties are extremely involved, stemming from the difficulties in the implicit way Fiedler pencils are defined, either in terms of a product of matrices for square polynomials or as the output of a symbolic algorithm for rectangular ones. Moreover, the proofs of the extensions of some of these results to the different generalizations of Fiedler pencils mentioned in the previous paragraph are much more involved, and they only work for square matrix polynomials [6, 7].

An interesting recent advance in the theory and analysis of linearizations of matrix polynomials has been the introduction of the family of *block minimal bases pencils* and its subfamily of *block Kronecker pencils* [18]. These families have been shown to be a fertile source of linearizations satisfying properties (i)-(vii) [18], and also (vi-b) and (viii) in the structured matrix polynomial case [19, 34, 45]. Additionally, these families have found unexpected applications in some theoretical and numerical problems: (a) they have been used for constructing linearizations of matrix polynomials expressed in bases other than the monomials [34, 45]; (b) it has been possible to generalize these families in order to construct ℓ-ifications of any degree ℓ [20]; (c) they have been used for constructing linearizations of rational matrices [1]; and (d) the structure of some block minimal bases pencils has been used (d1) to run efficiently Krylov methods [3] and two-sided Krylov methods [35] for solving certain nonlinear eigenvalue problems; and (d2) to run efficiently the Ehrilich-Aberth method for stable and efficient computation of roots of polynomials that are the sum of two polynomials expressed in two different bases or the computation of roots of certain rational equations [45, 46].

In addition, the block minimal bases pencils framework has been used for performing a rigorous global (structured and unstructured) backward error analysis of polynomial eigenvalue problems solved by Fiedler companion linearizations [13, 18, 19]. The reason why the theory developed for block minimal bases pencils works for Fiedler pencils as well is that, up to permutations of rows and columns, Fiedler pencils are block Kronecker pencils [18, Theorem 4.5]. The subtle point of the backward error analysis in [18] lies in the fact that a perturbation of a linearization in the block Kronecker pencil family (like the permuted Fiedler pencils) is, after some manipulations, a block minimal bases pencil, which is a linearization of a certain matrix polynomial. Hence, it was key for performing a backward error analysis of polynomial eigenvalue problems solved by using Fiedler pencils to embed the Fiedler pencil

family into the block minimal bases pencils family, which provides a larger and more flexible set of linearizations that are robust under certain perturbations. Proving these results by using the original implicit definition of Fiedler pencils as products of matrices seems to us a very difficult task.

The goal of this work is to extend the embedding result obtained for Fielder pencils in [18] to all the other families of Fiedler-like pencils (proper generalized Fiedler pencils, Fiedler pencils with repetition and generalized Fiedler pencils with repetition). More explicitly, our main result is that the pencils in all these families belong, up to permutations, to what we call the family of *extended block Kronecker pencils*, which, under some generic nonsingularity conditions, are block minimal bases pencils. This result provides a simplified theory for all these Fiedler-like pencils families, since, as opposed to their implicit definition in terms of matrix products, extended block Kronecker pencils are explicitly defined in terms of their block entries. This result also allows the application of all the tools and machinery developed for block minimal bases pencils in [18] to Fielder-like pencils, too. For example, the robustness of the block minimal bases pencils family could be used to try to determine whether or not solving a polynomial eigenvalue problem via a Fiedler-like linearization is backward stable from the point of view of the polynomial, which is still an open problem (except for the particular case of Fiedler pencils), or to perform local residual backward error and eigenvalue conditioning analyses.

We want to remark that the family of extended block Kronecker pencils has been introduced independently in [22] under the name of block-Kronecker ansatz spaces motivated, as in our case, by the results in [18]. However, the goal of [22] is different than ours. While the goal of [22] is to construct a new source of strong linearizations of matrix polynomials over the real numbers and to establish connections of the "block-Kronecker ansatz spaces" with other ansatz spaces of potential strong linearizations previously available in the literature [37], our goal is to provide a unified and simplified approach to all the families of Fiedler-like pencils in any field via the more general concept of strong block minimal bases pencils. We emphasize again in this context that all the extended block Kronecker pencils and, so, all pencils in [22] that are strong linearizations are particular cases of the (strong) block minimal bases pencils introduced in [18] which seem to be a key unifying concept for studying linearizations of matrix polynomials.

The paper is structured as follows. In Section 2, we introduce the definitions and notation used throughout the paper, and present some basic results needed in other sections. In Section 3, we review the framework of (strong) block minimal bases pencils and the family of block Kronecker pencils. We introduce the family of extended block Kronecker pencils that will be used to express all the Fiedler-like pencils into the framework of block minimal bases pencils. In this section, we also state, informally, our main results, and we illustrate them with some illuminating examples. In Section 4, we review the families of (square) Fiedler pencils, generalized Fiedler pencils, Fiedler pencils with repetition and generalized Fiedler pencils with repetition, introduce the index tuple notation, which is needed to define and work in an effective way with the Fiedler-like pencils families when they are defined in terms of matrix products, and present some auxiliary results used in the subsequent sections. In Section 5, we introduce some technical lemmas regarding matrix pencils that are block-permutationally equivalent to extended block Kronecker pencils. These technical results are used to prove the main theorems in the paper but are interesting by themselves. In Sections 6, 7 and 8, we present and prove our main results. We show that, up to permutations, proper generalized Fiedler pencils, Fiedler pencils with repetition and generalized Fiedler pencils with repetition are extended block Kronecker pencils. In particular, we show that proper generalized Fiedler pencils are block Kronecker pencils modulo permutations. Finally, in Section 9, we

summarize our results.

2. Notation, definitions, and basic results. Throughout the paper, given two integers a and b, we denote

$$a:b := \begin{cases} a, a+1, \ldots, b, & \text{if } a \leq b, \\ \emptyset, & \text{if } a > b. \end{cases}$$

We will use $\mathbb{F}$ to denote an arbitrary field, $\mathbb{F}[\lambda]$ to denote the ring of polynomials with coefficients from the field $\mathbb{F}$, and $\mathbb{F}(\lambda)$ to denote the field of rational functions over $\mathbb{F}$. The algebraic closure of the field $\mathbb{F}$ is denoted by $\overline{\mathbb{F}}$. The set of $m \times n$ matrices with entries in $\mathbb{F}[\lambda]$ is denoted by $\mathbb{F}[\lambda]^{m\times n}$. Any $P(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ is called an $m\times n$ *matrix polynomial*, or, just a matrix polynomial when its size is clear from the context or is not relevant. Moreover, when $m = 1$ (resp. $n = 1$), we refer to $P(\lambda)$ as a *row vector polynomial* (resp. *column vector polynomial*). A matrix polynomial $P(\lambda)$ is said to be *regular* if it is square and the scalar polynomial $\det P(\lambda)$ is not identically equal to the zero polynomial, and *singular* otherwise. If $P(\lambda)$ is regular and $\det P(\lambda) \in \mathbb{F}$, then $P(\lambda)$ is said to be *unimodular*.

A matrix polynomial $P(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ is said to have *grade* k if it can be expressed in the form

$$P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i, \quad \text{with} \quad A_0, \ldots, A_k \in \mathbb{F}^{m\times n}, \tag{2.1}$$

where any of the coefficients, including A_k, can be zero. We call the degree of $P(\lambda)$ the maximum integer d such that $A_d \neq 0$. The degree of a matrix polynomial $P(\lambda)$ is denoted by $\deg(P(\lambda))$. Note, in addition, that the degree of $P(\lambda)$ is fixed while the grade of a matrix polynomial, which is always larger than or equal to its degree, is a choice. A matrix polynomial of grade 1 is called a *matrix pencil*, or, sometimes for simplicity, a pencil.

For any $k \geq \deg(P(\lambda))$, the *k-reversal matrix polynomial* of $P(\lambda)$ is $\operatorname{rev}_k P(\lambda) := \lambda^k P(\lambda^{-1})$. When the degree of a matrix polynomial $P(\lambda)$ is clear from the context, we write $\operatorname{rev} P(\lambda)$ to denote the $\deg(P(\lambda))$-reversal of $P(\lambda)$.

The *complete eigenstructure* of a regular matrix polynomial consists of its finite and infinite elementary divisors (spectral structure), and for a singular matrix polynomial it consists of its finite and infinite elementary divisors together with its right and left minimal indices (spectral structure and singular structure). The singular structure of matrix polynomials is reviewed later in this section. For more detailed definitions of the spectral structure of matrix polynomials, we refer the reader to [16, Section 2]. The problem of computing the complete eigenstructure of a matrix polynomial is called the *complete polynomial eigenvalue problem*.

The standard approach to solving a complete polynomial eigenvalue problem is via linearizations. A matrix pencil $L(\lambda)$ is a *linearization* of a matrix polynomial $P(\lambda)$ of grade k if, for some $s \geq 0$, there exist two unimodular matrix polynomials $U(\lambda)$ and $V(\lambda)$ such that

$$U(\lambda)L(\lambda)V(\lambda) = \begin{bmatrix} I_s & 0 \\ 0 & P(\lambda) \end{bmatrix},$$

where I_s denotes the identity matrix of size s. Additionally, a linearization $L(\lambda)$ of $P(\lambda)$ is a *strong linearization* if $\operatorname{rev}_1 L(\lambda)$ is a linearization of $\operatorname{rev}_k P(\lambda)$. We recall that the key property of any strong linearization $L(\lambda)$ of $P(\lambda)$ is that $P(\lambda)$ and $L(\lambda)$ have the same finite and infinite elementary divisors, and the same numbers of right and left minimal indices.

Minimal bases and minimal indices play an important role in the developments of this work, so we briefly review them here (for a more complete description of minimal bases and

their properties, see Forney [24]). Any subspace $\mathcal{W}$ of $\mathbb{F}(\lambda)^n$ has bases consisting entirely of vector polynomials. The *order* of a vector polynomial basis of $\mathcal{W}$ is defined as the sum of the degrees of its vectors. Among all of the possible polynomial bases of $\mathcal{W}$, those with least order are called *minimal bases* of $\mathcal{W}$. In general, there are many minimal bases of $\mathcal{W}$, but the ordered list of degrees of the vector polynomials in any of its minimal bases is always the same. This list of degrees is called the list of *minimal indices* of $\mathcal{W}$.

When an $m \times n$ matrix polynomial $P(\lambda)$ is singular, it has nontrivial right and/or left *rational null spaces*:

$$\mathcal{N}_r(P) := \{x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1} : \ P(\lambda)x(\lambda) = 0\},$$
$$\mathcal{N}_\ell(P) := \{y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times m} : \ y(\lambda)^T P(\lambda) = 0\}.$$

The left (resp. right) minimal bases and minimal indices of $P(\lambda)$ are defined as those of the rational subspace $\mathcal{N}_\ell(P)$ (resp. $\mathcal{N}_r(P)$).

In order to give a useful characterization of the minimal bases of a rational subspace, we introduce the concept of *highest row-degree coefficient matrix*. Here and thereafter, by the i*th row degree* of a matrix polynomial $Q(\lambda)$ we denote the degree of the ith row of $Q(\lambda)$.

DEFINITION 2.1. *Let $Q(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ be a matrix polynomial with row degrees $d_1, d_2, \ldots, d_m$. The* highest row-degree coefficient matrix *of $Q(\lambda)$, denoted Q_h, is the $m \times n$ constant matrix whose ith entry in the jth row is the coefficient of λ^{d_j} in the (j, i)th entry of $Q(\lambda)$, for $i = 1 : m$ and $j = 1 : n$.*

In this paper, we arrange minimal bases as the rows of matrix polynomials. Then, we say that an $m \times n$ matrix polynomial, with $m \leq n$, is a minimal basis if its rows form a minimal basis for the rational space they span.

THEOREM 2.2. [18, Theorem 2.2] and [24] *A matrix polynomial $Q(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ is a minimal basis if and only if $Q(\lambda_0) \in \overline{\mathbb{F}}^{m\times n}$ has full row rank for all $\lambda_0 \in \overline{\mathbb{F}}$ and the highest row degree coefficient matrix Q_h of $Q(\lambda)$ has full rank.*

Next, we introduce the definition of *dual minimal bases*, a concept that plays a key role in the construction of the matrix pencils in Section 3.

DEFINITION 2.3. [24] *Matrix polynomials $K(\lambda) \in \mathbb{F}[\lambda]^{m_1\times q}$ and $N(\lambda) \in \mathbb{F}[\lambda]^{m_2\times q}$ are called* dual minimal bases *if $m_1 + m_2 = q$, and $K(\lambda)$ and $N(\lambda)$ are minimal bases satisfying $K(\lambda)N(\lambda)^T = 0$. We say that $K(\lambda)$ is a dual minimal basis to $N(\lambda)$, or vice versa.*

We introduce in Example 2.1 a simple pair of dual minimal bases that plays an important role in this paper. Here and throughout the paper we omit occasionally some, or all, of the zero entries of a matrix.

EXAMPLE 2.4. *Consider the following matrix polynomials:*

$$L_s(\lambda) := \begin{bmatrix} -1 & \lambda & & \\ & -1 & \lambda & \\ & & \ddots & \ddots \\ & & & -1 \quad \lambda \end{bmatrix} \in \mathbb{F}[\lambda]^{s\times(s+1)}, \tag{2.2}$$

and

$$\Lambda_s(\lambda) := \begin{bmatrix} \lambda^s & \lambda^{s-1} & \cdots & \lambda & 1 \end{bmatrix} \in \mathbb{F}[\lambda]^{1\times(s+1)}. \tag{2.3}$$

It follows easily from Theorem 2.2 that the matrix polynomials $L_s(\lambda)$ and $\Lambda_s(\lambda)$ are both minimal bases. Also, since $L_s(\lambda)\Lambda_s(\lambda)^T = 0$, we conclude that $L_s(\lambda)$ and $\Lambda_s(\lambda)$ are dual minimal bases. Additionally, it follows from basic properties of the Kronecker product that $L_s(\lambda) \otimes I_n$ and $\Lambda_s(\lambda) \otimes I_n$ are also dual minimal bases.

REMARK 2.5. *The vector polynomial* $\Lambda_s(\lambda)$ *in* (2.3) *is very well known in the theory of linearizations of matrix polynomials, and plays an essential role, for instance, in [18, 19, 37, 38]. However, notice that in those references* $\Lambda_s(\lambda)$ *is defined as a column vector, while in this work, for convenience, we define it as a row vector.*

In Section 3, we will work with minimal bases dual to $\Lambda_s(\lambda)\otimes I_n$ other than $L_s(\lambda)\otimes I_n$. This motivates the following definition.

DEFINITION 2.6. *Let* $n, s \in \mathbb{N}$*, and let* $\Lambda_s(\lambda)$ *be the matrix polynomial in* (2.3)*. A matrix pencil* $K(\lambda) \in \mathbb{F}[\lambda]^{sn\times(s+1)n}$ *is called a* (s,n)-wing pencil *or, simply, a wing pencil, if* $K(\lambda)(\Lambda_s(\lambda)^T \otimes I_n) = 0$*.*

We characterize in Theorem 2.5 all the (s,n)-wing pencils that are dual minimal bases to the matrix polynomial $\Lambda_s(\lambda)\otimes I_n$. A characterization of the wing pencils can also be found in [22, Lemma 3], a work that was produced independently and simultaneously to our work.

THEOREM 2.7. *Let* $L_s(\lambda)$ *and* $\Lambda_s(\lambda)$ *be the matrix polynomials in* (2.2) *and* (2.3)*, respectively. Then, a* (s,n)*-wing pencil* $K(\lambda) \in \mathbb{F}[\lambda]^{sn\times(s+1)n}$ *is a dual minimal basis to* $\Lambda_s(\lambda)\otimes I_n$ *if and only if* $K(\lambda) = B(L_s(\lambda)\otimes I_n)$ *for some nonsingular matrix* $B \in \mathbb{F}^{sn\times sn}$*.*

Proof. Assume that $K(\lambda) = B(L_s(\lambda)\otimes I_n)$ for some nonsingular matrix $B \in \mathbb{F}^{sn\times sn}$. Since the condition $K(\lambda)(\Lambda_s(\lambda)^T \otimes I_n) = 0$ holds, the pencil $K(\lambda)$ is a (s,n)-wing pencil. Thus, we only need to prove that $K(\lambda)$ is a minimal basis. To prove this, we will use Theorem 2.2. First, we have to show that $K(\lambda_0)$ has full row rank for any $\lambda_0 \in \overline{\mathbb{F}}$. Indeed, let $\lambda_0 \in \overline{\mathbb{F}}$. Since $L_s(\lambda)\otimes I_n$ is a minimal basis, we know that $L_s(\lambda_0)\otimes I_n$ has full row rank. Since the product of a nonsingular matrix by a full row rank matrix is a full row rank matrix, we obtain that $K(\lambda_0) = B(L_s(\lambda_0)\otimes I_n)$ has full row rank. Now, we prove that the highest row degree coefficient matrix of $K(\lambda)$ has full row rank as well. Writting $L_s(\lambda) = \lambda F + E$, it is not difficult to check that the highest row degree coefficient matrix of $K(\lambda)$ is given by $B(F\otimes I_n)$. The matrix B is nonsingular and $F\otimes I_n$ has full row rank, therefore the matrix $B(F\otimes I_n)$ has full row rank. From Theorem 2.2, we conclude that $K(\lambda)$ is a minimal basis. Thus, $K(\lambda)$ and $\Lambda_s(\lambda)\otimes I_n$ are dual minimal bases.

Assume now that $K(\lambda)$ and $\Lambda_s(\lambda)\otimes I_n$ are dual minimal bases. We will show that $K(\lambda) = B(L_s(\lambda)\otimes I_n)$ for some nonsingular matrix B. To this purpose, let us partition the pencil $K(\lambda)$ as a $s\times(s+1)$ block-pencil with blocks $\lambda[K_1]_{ij} + [K_0]_{ij} \in \mathbb{F}[\lambda]^{n\times n}$, for $i = 1:s$ and $j = 1:s+1$. Let

$$\begin{bmatrix}\lambda[K_1]_{i1} + [K_0]_{i1} & \lambda[K_1]_{i2} + [K_0]_{i2} & \cdots & \lambda[K_1]_{i,s+1} + [K_0]_{i,s+1}\end{bmatrix}$$

denote the ith block-row of $K(\lambda)$. From $K(\lambda)(\Lambda_s(\lambda)^T \otimes I_n) = 0$, we get

$$\begin{aligned}0 =& \sum_{j=0}^{s} \lambda^j \left(\lambda[K_1]_{i,s+1-j} + [K_0]_{i,s+1-j}\right)\\ =& \lambda^{s+1}[K_1]_{i1} + \sum_{j=1}^{s} \lambda^j \left([K_0]_{i,s+1-j} + [K_1]_{i,s+2-j}\right) + [K_0]_{i,s+1},\end{aligned}$$

or equivalently,

$$[K_1]_{i1} = 0, \quad [K_1]_{ij} = -[K_0]_{i,j-1}, \quad j = 2:s+1, \quad [K_0]_{i,s+1} = 0,$$

which shows that $K(\lambda)$ is of the form

$$K(\lambda) = -\begin{bmatrix}[K_0]_{11} & \cdots & [K_0]_{1s}\\ \vdots & \ddots & \vdots\\ [K_0]_{s1} & \cdots & [K_0]_{ss}\end{bmatrix}\begin{bmatrix}-I_n & \lambda I_n & & \\ & \ddots & \ddots & \\ & & -I_n & \lambda I_n\end{bmatrix} =: B(L_s(\lambda)\otimes I_n).$$

To complete the proof, we only need to show that B is nonsingular. The proof proceeds by contradiction. Assume that B is singular, i.e, there exists a nonzero vector $x \in \mathbb{F}^{sn\times 1}$ such that $x^TB = 0$. Let $\lambda_0 \in \overline{\mathbb{F}}$. Then, we have $x^TK(\lambda_0) = x^TB(L_s(\lambda_0) \otimes I_n) = 0$. Thus, $K(\lambda_0)$ does not have full row rank. But this contradicts the fact that $K(\lambda)$ is a minimal basis. Therefore, the matrix B is nonsingular. □

REMARK 2.8. *Since $K(\lambda)(\Lambda_s(\lambda) \otimes I_n) = 0$ is a necessary condition for $K(\lambda)$ to be a dual minimal basis to $\Lambda_s(\lambda) \otimes I_n$, we note that the set of (s, n)-wing pencils of the form $B(L_s(\lambda) \otimes I_n)$, with B nonsingular, corresponds to the set of minimal bases dual to $\Lambda_s(\lambda) \otimes I_n$.*

The following result can be checked in a straightforward way.

PROPOSITION 2.9. *A pencil $B(\lambda) \in \mathbb{F}[\lambda]^{(q+1)n\times(p+1)n}$ satisfies $B(\lambda)(\Lambda_p(\lambda)^T \otimes I_n) = 0$ if and only it is of the form*

$$B(\lambda) = \begin{bmatrix} B_{11} & -\lambda B_{11} + B_{12} & -\lambda B_{12} + B_{13} & \cdots & -\lambda B_{1p} \\ B_{21} & -\lambda B_{21} + B_{22} & -\lambda B_{22} + B_{23} & \cdots & -\lambda B_{2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{q+1,1} & -\lambda B_{q+1,1} + B_{q+1,2} & -\lambda B_{q+1,2} + B_{q+1,3} & \cdots & -\lambda B_{q+1,p} \end{bmatrix}, \tag{2.4}$$

for some matrices $B_{ij} \in \mathbb{F}^{n\times n}$, with $i = 1 : q + 1$ and $j = 1 : p$. Any pencil as in (2.4), with $q + 1 \neq p$, will be called generalized wing pencil.

Note that the previous result implies that any (s, n)-wing pencil is of the form (2.4), as extracted from the proof of Theorem 2.5.

As an immediate corollary of Theorem 2.5, we obtain the following result for minimal basis (s, n)-wing pencils, whose simple proof is omitted.

COROLLARY 2.10. *Let $M \in \mathbb{F}^{sn\times sn}$ be a nonsingular matrix. If $K(\lambda) \in \mathbb{F}[\lambda]^{ns\times n(s+1)}$ is a minimal basis (s, n)-wing pencil, then $MK(\lambda)$ is a minimal basis (s, n)-wing pencil.*

The next theorem is the last auxiliary result about minimal basis wing pencils that we present in this section.

THEOREM 2.11. *Let $n, s_1, s_2 \in \mathbb{N}$, and let $K(\lambda)$ and $L(\lambda)$ be, respectively, a minimal basis (s_1, n)-wing pencil and a minimal basis (s_2, n)-wing pencil. If the pencils $K(\lambda)$ and $L(\lambda)$ are partitioned as follows*

$$K(\lambda) = \begin{bmatrix} K_1(\lambda) & k(\lambda) \end{bmatrix} \quad \text{and} \quad L(\lambda) = \begin{bmatrix} \ell(\lambda) & L'(\lambda) \end{bmatrix},$$

where $K_1(\lambda) \in \mathbb{F}[\lambda]^{s_1n\times s_1n}$ and $L'(\lambda) \in \mathbb{F}[\lambda]^{s_2n\times s_2n}$, then the pencil

$$S(\lambda) := \begin{bmatrix} K_1(\lambda) & k(\lambda) & 0 \\ 0 & \ell(\lambda) & L'(\lambda) \end{bmatrix}$$

is a minimal basis $(s_1 + s_2, n)$-wing pencil.

Proof. Set $s := s_1 + s_2$. By Theorem 2.5, the pencils $K(\lambda)$ and $L(\lambda)$ can be written, respectively, as $K(\lambda) = B(L_{s_1}(\lambda) \otimes I_n)$ and $L(\lambda) = C(L_{s_2}(\lambda) \otimes I_n)$, for some nonsingular matrices $B \in \mathbb{F}^{s_1n\times s_1n}$ and $C \in \mathbb{F}^{s_2n\times s_2n}$. Then, notice that the pencil $S(\lambda)$ can be written as

$$S(\lambda) = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} (L_s(\lambda) \otimes I_n).$$

Thus, the desired result follows applying Theorem 2.5 to the pencil $S(\lambda)$. □

In future sections we will work with block-row and block-column permutations of block-partitioned pencils, which will involve the use of block-permutation matrices. Next, we introduce the notation used in this paper for block-permutation matrices.

DEFINITION 2.12. *Let $k, n \in \mathbb{N}$. Let $\mathbf{c} = (c_1, c_2, \ldots, c_k)$ be a permutation of the set $\{1 : k\}$. Then, we call the block-permutation matrix associated with $(\mathbf{c}, n)$, and denote it by $\Pi_{\mathbf{c}}^n$, the $k \times k$ block-matrix whose (c_i, i)th block-entry is I_n, for $i = 1 : k$, and having 0_n in every other block-entry. In particular, we denote by $\mathbf{id} := (1 : k)$ the identity permutation.*

A particular block-permutation matrix that will be very useful in the proofs of our main results is the block version of the *sip matrix* (standard involutory permutation). Such $sn \times sn$ block-permutation matrix is denoted by $R_{s,n}$ and is defined as follows

$$R_{s,n} := \begin{bmatrix} 0 & \cdots & I_n \\ \vdots & \iddots & \vdots \\ I_n & \cdots & 0 \end{bmatrix} \in \mathbb{F}^{sn \times sn}. \tag{2.5}$$

REMARK 2.13. *When the scalar n is clear in the context, we will write $\Pi_{\mathbf{c}}$ and R_s instead of $\Pi_{\mathbf{c}}^n$ and $R_{s,n}$, respectively, to simplify the notation.*

Finally, we introduce the concept of *block-transposition* of a block-matrix. Let $H = [H_{ij}]_{i=1:p,j=1:q}$ be a $p \times q$ block-matrix with block-entries $H_{ij} \in \mathbb{F}^{n \times n}$. We define the block-transpose $H^{\mathcal{B}}$ of H as the $q \times p$ block-matrix having the block H_{ji} in the block-position (i, j), for $1 \leq i \leq q$ and $1 \leq j \leq p$.

3. The block minimal bases pencils framework for Fiedler-like pencils. In [18], Fiedler pencils were expressed via some block-rows and block-columns permutations as block Kronecker pencils, a particular type of block minimal bases pencils. In this section we recall the definition of block minimal bases pencils and introduce the family of extended block Kronecker pencils, which contains the family of block Kronecker pencils. The pencils in this new family are block minimal bases pencils under some generic nonsingularity conditions and allow us to express all Fiedler-like pencils into the block minimal bases framework.

3.1. Block minimal bases pencils. We discuss in this section the family of block minimal bases pencils, recently introduced in [18], and its main properties.

DEFINITION 3.1. [18] *A matrix pencil*

$$C(\lambda) = \left[\begin{array}{c|c} M(\lambda) & K_2(\lambda)^T \\ \hline K_1(\lambda) & 0 \end{array}\right] \tag{3.1}$$

is called a block minimal bases pencil *if $K_1(\lambda)$ and $K_2(\lambda)$ are minimal bases. If, in addition, the row degrees of $K_1(\lambda)$ are all equal to 1, the row degrees of $K_2(\lambda)$ are all equal to 1, the row degrees of a minimal basis dual to $K_1(\lambda)$ are all equal, and the row degrees of a minimal basis dual to $K_2(\lambda)$ are all equal, then $C(\lambda)$ is called a* strong block minimal bases pencil.

Theorem 3.2, which will be used in the proofs of some of our results, shows that every block minimal bases pencil is a linearization of at least one matrix polynomial.

THEOREM 3.2. [18, Theorem 3.3] *Let $K_1(\lambda)$ and $N_1(\lambda)$ be a pair of dual minimal bases, and let $K_2(\lambda)$ and $N_2(\lambda)$ be another pair of dual minimal bases. Let $C(\lambda)$ be a block minimal bases pencil as in (3.1). Consider the matrix polynomial*

$$Q(\lambda) := N_2(\lambda) M(\lambda) N_1(\lambda)^T. \tag{3.2}$$

Then:

(i) *$C(\lambda)$ is a linearization of $Q(\lambda)$.*
(ii) *If $C(\lambda)$ is a strong block minimal bases pencil, then $C(\lambda)$ is a strong linearization of $Q(\lambda)$, considered as a polynomial with grade $1 + \deg(N_1(\lambda)) + \deg(N_2(\lambda))$.*

3.2. Block Kronecker pencils. Next, we recall the subfamily of block minimal bases pencils called block Kronecker pencils, which was introduced in [18] and has proven to be fruitful in providing a simple block-structure characterization of Fiedler pencils up to permutation of rows and columns.

DEFINITION 3.3. [18] *Let $L_s(\lambda)$ be the matrix pencil defined in* (2.2) *and let $M(\lambda)$ be an arbitrary pencil. Then any matrix pencil of the form*

$$C(\lambda) = \left[\begin{array}{c|c} M(\lambda) & L_q(\lambda)^T \otimes I_m \\ \hline L_p(\lambda) \otimes I_n & 0 \end{array}\right] \begin{array}{l} \} \, (q+1)m \\ \} \, pn \end{array} \quad , \tag{3.3}$$

$$\underbrace{}_{(p+1)n} \quad \underbrace{}_{qm}$$

is called a (p, n, q, m)-block Kronecker pencil or, simply, a block Kronecker pencil.

Fiedler pencils are, modulo permutations, block Kronecker pencils (we recall this result in Theorem 6.3). In Section 7, we will show that this result is also true for the most important subfamily of generalized Fiedler pencils (i.e., proper generalized Fiedler pencils). However, we will see that the family of block Kronecker pencils is not large enough to include the other families of Fiedler-like pencils (Fiedler pencils with repetition and generalized Fiedler pencils with repetition). To provide a block minimal bases pencils framework large enough to include all families of Fiedler-like pencils is the goal of the following section, where we introduce the family of *extended block Kronecker pencils*. These pencils are strong block minimal bases pencils under some generic nonsingularity conditions.

3.3. Extended block Kronecker pencils and the antidiagonal sum condition. Here we introduce a new family of pencils named *extended block Kronecker pencils*, such that most of its members are strong block minimal bases pencils. As we will show, this family of pencils provides a unified approach to all Fielder-like pencils (Fiedler pencils, generalized Fiedler pencils, Fiedler pencils with repetition and generalized Fiedler pencils with repetition) in the sense that every Fiedler-like pencil is, up to permutations, an extended block Kronecker pencil. This family was simultaneously introduced in [22, Definition 1, Theorem 4]. In this section we point out the connections between the theory that we develop here and the results that were obtained independently in [22].

DEFINITION 3.4. *Let $K_1(\lambda)$ and $K_2(\lambda)$ be, respectively, a (p, n)-wing pencil and a (q, m)-wing pencil (recall Definition 2.4), and let $M(\lambda)$ be an arbitrary pencil. A matrix pencil*

$$C(\lambda) = \left[\begin{array}{c|c} M(\lambda) & K_2(\lambda)^T \\ \hline K_1(\lambda) & 0 \end{array}\right] \begin{array}{l} \} \, (q+1)m \\ \} \, pn \end{array} \quad , \tag{3.4}$$

$$\underbrace{}_{(p+1)n} \quad \underbrace{}_{qm}$$

*is called an extended (p, n, q, m)-*block Kronecker pencil*, or, simply, an* extended block Kronecker pencil*. Moreover, the block $M(\lambda)$ is called the* body *of $C(\lambda)$.*

REMARK 3.5. *In Definition 3.4, we allow the border cases $p = 0$ and $q = 0$, i.e., pencils of the form*

$$C(\lambda) = \left[\begin{array}{c|c} M(\lambda) & K(\lambda)^T \end{array}\right] \quad \textit{or} \quad C(\lambda) = \left[\begin{array}{c} M(\lambda) \\ \hline K(\lambda) \end{array}\right],$$

where $K(\lambda)$ is a wing pencil, are also considered extended block Kronecker pencils.

REMARK 3.6. *In view of Proposition 2.6 we notice that the extended block Kronecker pencil* (3.4) *can be factorized as*

$$\begin{bmatrix} I_{(q+1)m} & 0 \\ 0 & A \end{bmatrix} \left[\begin{array}{c|c} \lambda M_1 + M_0 & L_q(\lambda)^T \otimes I_m \\ \hline L_p(\lambda) \otimes I_n & 0 \end{array}\right] \begin{bmatrix} I_{(p+1)n} & 0 \\ 0 & B \end{bmatrix}, \tag{3.5}$$

for some matrices $A \in \mathbb{F}^{np\times np}$ and $B \in \mathbb{F}^{mq\times mq}$. Taking $A = I_{np}$ and $B = I_{mq}$ in (3.5) *the extended block Kronecker pencil reduces to a block Kronecker pencil. Thus, we obtain that the family of (p, n, q, m)-block Kronecker pencils is included in the family of extended (p, n, q, m)-block Kronecker pencils. The factorization (3.5) is equivalent to (18) in [22].*

REMARK 3.7. *When $n = m$, a pencil may be partitioned in more than one way as an extended (p, n, q, n)-block Kronecker pencil. For example, let A_0, A_1, A_2, A_3 be arbitrary $n \times n$ matrices. The pencil*

$$C(\lambda) = \begin{bmatrix} \lambda A_3 + A_2 & A_1 & A_0 \\ A_1 & -\lambda A_1 + A_0 & -\lambda A_0 \\ A_0 & -\lambda A_0 & 0 \end{bmatrix}$$

can be seen as an extended $(1, n, 1, n)$-block Kronecker pencil since it can be partitioned as follows

$$\left[\begin{array}{cc|c} \lambda A_3 + A_2 & A_1 & A_0 \\ A_1 & -\lambda A_1 + A_0 & -\lambda A_0 \\ \hline A_0 & -\lambda A_0 & 0 \end{array}\right].$$

But notice that we could also partition $C(\lambda)$ in the next two alternative ways

$$\left[\begin{array}{c|cc} \lambda A_3 + A_2 & A_1 & A_0 \\ A_1 & -\lambda A_1 + A_0 & -\lambda A_0 \\ A_0 & -\lambda A_0 & 0 \end{array}\right], \quad \left[\begin{array}{ccc} \lambda A_3 + A_2 & A_1 & A_0 \\ \hline A_1 & -\lambda A_1 + A_0 & -\lambda A_0 \\ A_0 & -\lambda A_0 & 0 \end{array}\right].$$

Thus, the pencil $C(\lambda)$ can also be seen as an extended $(0, n, 2, n)$-block Kronecker pencil or as an extended $(2, n, 0, n)$-block Kronecker pencil. Some consequences of this ambiguity are studied in Theorem 3.9 and Remark 3.4. The phenomenon presented in this example is called Superpartition Principle in [22].

As an immediate corollary of Theorem 3.2, we obtain that, under some generic nonsingularity conditions, extended block Kronecker pencils are strong linearizations of some matrix polynomials. The following theorem corresponds with Theorem 6 in [22].

THEOREM 3.8. *Let $C(\lambda)$ be an extended block Kronecker pencil as in* (3.4)*, with body $M(\lambda)$. Let*

$$Q(\lambda) := (\Lambda_q(\lambda) \otimes I_m)M(\lambda)(\Lambda_p(\lambda)^T \otimes I_n) \in \mathbb{F}[\lambda]^{m\times n}, \tag{3.6}$$

viewed as a matrix polynomial with grade $p + q + 1$. If A and B are nonsingular, then $C(\lambda)$ is a strong block minimal bases pencil and, so, a strong linearization of $Q(\lambda)$.

Theorem 3.5 shows that most of the extended block Kronecker pencils as in (3.5) are strong linearizations of the associated matrix polynomial $Q(\lambda)$ since the nonsingularity conditions on A and B are generic in $\mathbb{F}^{np\times np} \times \mathbb{F}^{qm\times qm}$.

Now, we address the inverse problem, that is, given a matrix polynomial $P(\lambda)$, how to construct extended block Kronecker pencils that are strong linearizations of $P(\lambda)$. With this goal in mind, we introduce next some useful concepts.

DEFINITION 3.9. *Let $M(\lambda) = \lambda M_1 + M_0 \in \mathbb{F}[\lambda]^{(q+1)m\times(p+1)n}$ be a matrix pencil and set $k := p + q + 1$. Let us denote by $[M_0]_{ij}$ and $[M_1]_{ij}$ the (i,j)th block-entries of M_0 and M_1, respectively, when M_1 and M_0 are partitioned as $(q+1)\times(p+1)$ block-matrices with blocks of size $m \times n$. We call the* antidiagonal sum of $M(\lambda)$ related to $s \in \{0:k\}$ *the matrix*

$$\mathrm{AS}(M,s) := \sum_{i+j=k+2-s} [M_1]_{ij} + \sum_{i+j=k+1-s} [M_0]_{ij}.$$

Additionally, given a matrix polynomial $P(\lambda) = \sum_{i=0}^{k} A_i\lambda^i \in \mathbb{F}[\lambda]^{m\times n}$, we say that $M(\lambda)$ satisfies the antidiagonal sum condition (AS condition) *for $P(\lambda)$ if*

$$\mathrm{AS}(M,s) = A_s, \quad s = 0:k. \tag{3.7}$$

The main result of this section is Theorem 3.7, where we give sufficient conditions on the body of an extended block Kronecker pencil as in (3.5) with A, B nonsingular, to be a strong linearization of a given matrix polynomial. It is not hard to see that this result also follows from Theorems 4 and 6 in [22].

THEOREM 3.10. *Let $P(\lambda) = \sum_{i=0}^{k} A_i\lambda^i \in \mathbb{F}[\lambda]^{m\times n}$, and let $C(\lambda)$ be an extended (p,n,q,m)-block Kronecker pencil as in* (3.5) *with $p+q+1 = k$ and A, B nonsingular. If the body $M(\lambda) \in \mathbb{F}[\lambda]^{(q+1)m\times(p+1)n}$ of $C(\lambda)$ satisfies the AS condition for $P(\lambda)$, then $C(\lambda)$ is a strong linearization of $P(\lambda)$.*

Proof. By Theorem 3.5, the pencil $C(\lambda)$ is a strong linearization of (3.6). Some simple algebraic manipulations show that the AS condition implies $Q(\lambda) = P(\lambda)$. □

Our main focus in future sections will be on block-pencils whose blocks are square. In this case, surprisingly, given an extended block Kronecker pencil $C(\lambda)$ whose body satisfies the AS condition for a given matrix polynomial $P(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$, the whole pencil $C(\lambda)$ satisfies also the AS condition but for a shifted version of $P(\lambda)$, as we show next.

THEOREM 3.11. *Let $P(\lambda) = \sum_{i=0}^{k} A_i\lambda^i \in \mathbb{F}[\lambda]^{n\times n}$ and let $C(\lambda) \in \mathbb{F}[\lambda]^{kn\times kn}$ be an extended block Kronecker pencil as in* (3.4) *with $k = p+q+1$ whose body satisfies the AS condition for $P(\lambda)$. Then, the pencil $C(\lambda)$ satisfies the AS condition for $\lambda^{k-1}P(\lambda)$, that is,*

$$\mathrm{AS}(C,s) = B_s, \quad s = 0:2k-1,$$

where $B_s = 0$, for $s = 0:k-2$, and $B_s = A_{s-k+1}$, for $s = k-1:2k-1$.

Proof. Let us write $C(\lambda) = \widetilde{M}(\lambda) + \widetilde{K}_1(\lambda) + \widetilde{K}_2(\lambda)^T$, where

$$\widetilde{M}(\lambda) := \begin{bmatrix} M(\lambda) & 0 \\ 0 & 0 \end{bmatrix}, \quad \widetilde{K}_2(\lambda)^T := \begin{bmatrix} 0 & K_2(\lambda)^T \\ 0 & 0 \end{bmatrix}, \quad \widetilde{K}_1(\lambda) := \begin{bmatrix} 0 & 0 \\ K_1(\lambda) & 0 \end{bmatrix},$$

Then, due to the linearity of the antidiagonal sum $\mathrm{AS}(\cdot,s)$, we get

$$\mathrm{AS}(C,s) = \mathrm{AS}(\widetilde{M},s) + \mathrm{AS}(\widetilde{K}_2^T,s) + AS(\widetilde{K}_1,s), \quad s = 0:2k-1.$$

Additionally, because of the block-structure of the wing pencils explained in Proposition 2.6, it is clear that $\mathrm{AS}(\widetilde{K}_1,s) = 0$ and $\mathrm{AS}(\widetilde{K}_2^T,s) = 0$, for all s. Thus, $\mathrm{AS}(C,s) = \mathrm{AS}(\widetilde{M},s)$ and the result follows taking into account that $M(\lambda)$ satisfies the AS condition for $P(\lambda)$. □

We pointed out in Remark 3.3 that a given pencil $L(\lambda)$ may be partitioned in more than one way as an extended (p,n,q,n)-block Kronecker pencil. The next result shows that the AS condition of the body is preserved under different representations of $L(\lambda)$ as an extended block Kronecker pencil. This result corresponds to Theorem 22 in [22].

THEOREM 3.12. *Let $P(\lambda)=\sum_{i=0}^{k}A_i\lambda^i\in\mathbb{F}[\lambda]^{n\times n}$ and let $C(\lambda)\in\mathbb{F}[\lambda]^{kn\times kn}$ be an extended block Kronecker pencil as in* (3.4) *with $k=p+q+1$. If the body of $C(\lambda)$ satisfies the AS condition for $P(\lambda)$, then the body of any other partition of $C(\lambda)$ as an extended block Kronecker pencil satisfies the AS condition for $P(\lambda)$ as well.*

Proof. The result is an immediate consequence of Theorem 3.8. Assume that the extended block Kronecker pencil $C(\lambda)$ can be partitioned as in (3.4) and also as follows

$$C(\lambda)=\left[\begin{array}{c|c}\widehat{M}(\lambda) & \widehat{K}_2(\lambda)^T\\ \hline \widehat{K}_1(\lambda) & 0\end{array}\right],$$

where $\widehat{K}_1(\lambda)\in\mathbb{F}[\lambda]^{n\widehat{p}\times n(\widehat{p}+1)}$ and $\widehat{K}_2(\lambda)\in\mathbb{F}[\lambda]^{n\widehat{q}\times n(\widehat{q}+1)}$ are wing pencils, $\widehat{M}(\lambda)\in\mathbb{F}[\lambda]^{n(\widehat{q}+1)\times n(\widehat{p}+1)}$, and $\widehat{p}+\widehat{q}+1=k$. Let

$$\widetilde{M}(\lambda):=\left[\begin{array}{cc}\widehat{M}(\lambda) & 0\\ 0 & 0\end{array}\right].$$

By Theorem 3.8 and the block-structure of wing pencils explained in Proposition 2.6, we obtain

$$\mathrm{AS}(C,s)=\mathrm{AS}(\widetilde{M},s)=\left\{\begin{array}{cc}0, & \text{for } s=0:k-2\\ A_{s+1-k}, & \text{for } s=k-1:2k-1,\end{array}\right. \tag{3.8}$$

which shows that $\widehat{M}(\lambda)$ satisfies the AS condition for $P(\lambda)$. □

REMARK 3.13. *A pencil $C(\lambda)$ may be partitioned as an extended block Kronecker pencil that is a strong linearization of a given matrix polynomial $P(\lambda)$ in more than one way only if $P(\lambda)$ is regular. Otherwise, the application of [18, Theorem 3.7] to the different partitions would give contradictory values for the minimal indices of $C(\lambda)$.*

Lemma 3.10 shows that, if a pencil satisfying the AS condition for some $P(\lambda)$ is perturbed by the addition of a generalized wing pencil, the AS condition is preserved.

LEMMA 3.14. *Let $P(\lambda)=\sum_{i=0}^{k}A_i\lambda^i\in\mathbb{F}[\lambda]^{n\times n}$, let $p+q+1=k$, let $M(\lambda)\in\mathbb{F}[\lambda]^{(q+1)n\times(p+1)n}$ be a matrix pencil satisfying the AS condition for $P(\lambda)$, and let $B(\lambda)\in\mathbb{F}[\lambda]^{(q+1)n\times(p+1)n}$ be a generalized wing pencil. Then, the matrix pencil $M(\lambda)+B(\lambda)$ satisfies the AS condition for $P(\lambda)$.*

Proof. Let $\widetilde{M}(\lambda):=M(\lambda)+B(\lambda)$. The block-structure of generalized wing pencils explained in Proposition 2.6 implies that $\mathrm{AS}(B,s)=0$, for $s=0:k$. Therefore, by the linearity of the antidiagonal sum, we get $\mathrm{AS}(\widetilde{M},s)=\mathrm{AS}(M,s)+\mathrm{AS}(B,s)=\mathrm{AS}(M,s)=A_s$. Thus, the pencil $\widetilde{M}(\lambda)$ satisfies the AS condition for $P(\lambda)$. □

3.4. Illustrations and informal statements of the main results for Fiedler-like pencils. The goal of this section is twofold: first, to provide some examples that illustrate the main results of this paper, and, second, to state informally these results in Theorems 3.11 and 3.12. For the impatient reader, the precise statements of Theorems 3.11 and 3.12 can be found in Theorems 6.3, 7.1 and 8.1, however they require the index tuple notation that will be introduced in Section 4.

Let us start by considering the Fiedler pencil

$$F_{\mathbf{q}}(\lambda)=\begin{bmatrix}\lambda A_6+A_5 & -I_n & 0 & 0 & 0 & 0\\ A_4 & \lambda I_n & A_3 & -I_n & 0 & 0\\ -I_n & 0 & \lambda I_n & 0 & 0 & 0\\ 0 & 0 & A_2 & \lambda I_n & A_1 & -I_n\\ 0 & 0 & -I_n & 0 & \lambda I_n & 0\\ 0 & 0 & 0 & 0 & A_0 & \lambda I_n\end{bmatrix}$$

associated with a matrix polynomial $P(\lambda) = \sum_{i=0}^{6} A_i \lambda^i \in \mathbb{F}[\lambda]^{n\times n}$ (this Fiedler pencil will be formally introduced in Example 4.3). Then, it is not difficult to see that we can transform $F_{\mathbf{q}}(\lambda)$ into a block Kronecker pencil via block-row and block-column permutations, given by permutation matrices $\Pi^n_{\boldsymbol{\ell}_1}$ and $\Pi^n_{\mathbf{r}_1}$, to obtain

$$(\Pi^n_{\boldsymbol{\ell}_1})^{\mathcal{B}} F_{\mathbf{q}}(\lambda)\Pi^n_{\mathbf{r}_1} = \left[\begin{array}{ccc|ccc} \lambda A_6 + A_5 & 0 & 0 & -I_n & 0 & 0 \\ A_4 & A_3 & 0 & \lambda I_n & -I_n & 0 \\ 0 & A_2 & A_1 & 0 & \lambda I_n & -I_n \\ 0 & 0 & A_0 & 0 & 0 & \lambda I_n \\ \hline -I_n & \lambda I_n & 0 & 0 & 0 & 0 \\ 0 & -I_n & \lambda I_n & 0 & 0 & 0 \end{array}\right].$$

By Theorem 3.5, the above block Kronecker pencil is a strong linearization of

$$\begin{bmatrix} \lambda^3 I_n & \lambda^2 I_n & \lambda I_n & I_n \end{bmatrix} \begin{bmatrix} \lambda A_6 + A_5 & 0 & 0 \\ A_4 & A_3 & 0 \\ 0 & A_2 & A_1 \\ 0 & 0 & A_0 \end{bmatrix} \begin{bmatrix} \lambda^2 I_n \\ \lambda I_n \\ I_n \end{bmatrix} = P(\lambda).$$

Thus, this is an example of a Fiedler pencil associated with a matrix polynomial $P(\lambda)$ which is block-permutationally equivalent to a block Kronecker pencil. Note that $(\Pi^n_{\boldsymbol{\ell}_1})^{\mathcal{B}} F_{\mathbf{q}}(\lambda)\Pi^n_{\mathbf{r}_1}$ is a strong linearization of $P(\lambda)$. This result has been shown to be true for any Fiedler pencil [18, Theorem 4.5] (see, also, Theorem 6.3). Let us consider, now, the following proper generalized Fiedler pencil (see Definition 4.18)

$$K_{\mathbf{q},\mathbf{z}}(\lambda) = \begin{bmatrix} -I_n & \lambda A_6 & 0 & 0 & 0 & 0 \\ \lambda I_n & \lambda A_5 + A_4 & -I_n & 0 & 0 & 0 \\ 0 & A_3 & \lambda I_n & A_2 & -I_n & 0 \\ 0 & -I_n & 0 & \lambda I_n & 0 & 0 \\ 0 & 0 & 0 & -I_n & 0 & \lambda I_n \\ 0 & 0 & 0 & 0 & \lambda I_n & \lambda A_1 + A_0 \end{bmatrix},$$

associated also with the polynomial $P(\lambda)$ (this pencil will be formally introduced in Example 4.4). We can transform $K_{\mathbf{q},\mathbf{z}}(\lambda)$ into a block Kronecker pencil via block-row and block-column permutations, given by block-permutation matrices $\Pi^n_{\boldsymbol{\ell}_2}$ and $\Pi^n_{\mathbf{r}_2}$, to obtain

$$(\Pi^n_{\boldsymbol{\ell}_2})^{\mathcal{B}} K_{\mathbf{q},\mathbf{z}}(\lambda)\Pi^n_{\mathbf{r}_2} = \left[\begin{array}{ccc|ccc} \lambda A_6 & 0 & 0 & -I_n & 0 & 0 \\ \lambda A_5 + A_4 & 0 & 0 & \lambda I_n & -I_n & 0 \\ A_3 & A_2 & 0 & 0 & \lambda I_n & -I_n \\ 0 & 0 & \lambda A_1 + A_0 & 0 & 0 & \lambda I_n \\ \hline -I_n & \lambda I_n & 0 & 0 & 0 & 0 \\ 0 & -I_n & \lambda I_n & 0 & 0 & 0 \end{array}\right],$$

which, by Theorem 3.5, is a strong linearization of

$$\begin{bmatrix} \lambda^3 I_n & \lambda^2 I_n & \lambda I_n & I_n \end{bmatrix} \begin{bmatrix} \lambda A_6 & 0 & 0 \\ \lambda A_5 + A_4 & 0 & 0 \\ A_3 & A_2 & 0 \\ 0 & 0 & \lambda A_1 + A_0 \end{bmatrix} \begin{bmatrix} \lambda^2 I_n \\ \lambda I_n \\ I_n \end{bmatrix} = P(\lambda).$$

Therefore, this is an example of a proper generalized Fiedler pencil associated with a matrix polynomial $P(\lambda)$ which is block-permutationally equivalent to a block Kronecker pencil that

is a strong linearization of $P(\lambda)$. This result turns out to be true for all (proper) generalized Fiedler pencils (see Section 7). Since Fiedler pencils are particular examples of proper generalized Fiedler pencils, we can state the following theorem.

THEOREM 3.15. *Let $L(\lambda)$ be a proper generalized Fiedler pencil associated with a matrix polynomial $P(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$. Then, up to permutations, $L(\lambda)$ is a block Kronecker pencil which is a strong linearization of $P(\lambda)$.*

Let us consider, finally, the following generalized Fiedler pencil with repetition (in particular, it is a Fiedler pencil with repetition)

$$F_P(\lambda) = \begin{bmatrix} 0 & 0 & 0 & 0 & -A_6 & \lambda A_6 \\ 0 & 0 & 0 & -A_6 & \lambda A_6 - A_5 & \lambda A_5 \\ 0 & 0 & -A_6 & \lambda A_6 - A_5 & \lambda A_5 - A_4 & \lambda A_4 \\ 0 & -A_6 & \lambda A_6 - A_5 & \lambda A_5 - A_4 & \lambda A_4 - A_3 & \lambda A_3 \\ -A_6 & \lambda A_6 - A_5 & \lambda A_5 - A_4 & \lambda A_4 - A_3 & \lambda A_3 - A_2 & \lambda A_2 \\ \lambda A_6 & \lambda A_5 & \lambda A_4 & \lambda A_3 & \lambda A_2 & \lambda A_1 + A_0 \end{bmatrix}$$

associated with the matrix polynomial $P(\lambda) = \sum_{i=0}^{6} A_i\lambda^i \in \mathbb{F}[\lambda]^{n\times n}$. It is clear that there are no block-row and block-column permutations that transform the above pencil into a block Kronecker pencil. However, it is not difficult to show that there exist block-permutation matrices, denoted by $\Pi^n_{\boldsymbol{\ell}_3}$ and $\Pi^n_{\mathbf{r}_3}$, such that

$$(\Pi^n_{\boldsymbol{\ell}_3})^{\mathcal{B}} F_P(\lambda)\Pi^n_{\mathbf{r}_3} = \left[\begin{array}{ccc|ccc} \lambda A_6 - A_5 & \lambda A_5 - A_4 & \lambda A_4 & -A_6 & 0 & 0 \\ \lambda A_5 - A_4 & \lambda A_4 - A_3 & \lambda A_3 & \lambda A_6 - A_5 & -A_6 & 0 \\ \lambda A_4 - A_3 & \lambda A_3 - A_2 & \lambda A_2 & \lambda A_5 - A_4 & \lambda A_6 - A_5 & -A_6 \\ \lambda A_3 & \lambda A_2 & \lambda A_1 + A_0 & \lambda A_4 & \lambda A_5 & \lambda A_6 \\ \hline -A_6 & \lambda A_6 - A_5 & \lambda A_5 & 0 & 0 & 0 \\ 0 & -A_6 & \lambda A_6 & 0 & 0 & 0 \end{array}\right]$$

$$=: \left[\begin{array}{c|c} M(\lambda) & K_2(\lambda)^T \\ \hline K_1(\lambda) & 0 \end{array}\right].$$

Additionally, notice that

$$K_1(\lambda) = \begin{bmatrix} -A_6 & \lambda A_6 - A_5 & \lambda A_5 \\ 0 & -A_6 & \lambda A_6 \end{bmatrix} = \begin{bmatrix} A_6 & A_5 \\ 0 & A_6 \end{bmatrix} \begin{bmatrix} -I_n & \lambda I_n & 0 \\ 0 & -I_n & \lambda I_n \end{bmatrix}$$

and

$$K_2(\lambda) = \begin{bmatrix} -A_6 & 0 & 0 \\ \lambda A_6 - A_5 & -A_6 & 0 \\ \lambda A_5 - A_4 & \lambda A_6 - A_5 & -A_6 \\ \lambda A_4 & \lambda A_5 & \lambda A_6 \end{bmatrix}^T = \left(\begin{bmatrix} -I_n & 0 & 0 \\ \lambda I_n & -I_n & 0 \\ 0 & \lambda I_n & -I_n \\ 0 & 0 & \lambda I_n \end{bmatrix} \begin{bmatrix} A_6 & 0 & 0 \\ A_5 & A_6 & 0 \\ A_4 & A_5 & A_6 \end{bmatrix} \right)^T.$$

Therefore, if the leading coefficient A_6 of $P(\lambda)$ is nonsingular, by Theorem 2.5, the pencils $K_1(\lambda)$ and $K_2(\lambda)$ are both minimal basis wing pencils. Thus, the pencil $(\Pi^n_{\boldsymbol{\ell}_3})^{\mathcal{B}} L_P(\lambda)\Pi^n_{\mathbf{r}_3}$ is an extended block Kronecker pencil which, by Theorem 3.5, is a strong linearization of

$$\begin{bmatrix} \lambda^3 I_n & \lambda^2 I_n & \lambda I_n & I_n \end{bmatrix} \begin{bmatrix} \lambda A_6 - A_5 & \lambda A_5 - A_4 & \lambda A_4 \\ \lambda A_5 - A_4 & \lambda A_4 - A_3 & \lambda A_3 \\ \lambda A_4 - A_3 & \lambda A_3 - A_2 & \lambda A_2 \\ \lambda A_3 & \lambda A_2 & \lambda A_1 + A_0 \end{bmatrix} \begin{bmatrix} \lambda^2 I_n \\ \lambda I_n \\ I_n \end{bmatrix} = P(\lambda).$$

We would like to point out that, while $F_P(\lambda)$ is block-symmetric, the extended block Kronecker pencil it is block permutationally equivalent to is not. We attend to this problem in

[11] by showing that, every block-symmetric GFP or GFPR is, up to permutations, a block-symmetric minimal block bases pencil which is a strong linearization of $P(\lambda)$.

We conclude that the pencil $L_P(\lambda)$ is not permutationally equivalent to a block Kronecker $n \times n$ pencil, and it is only permutationally equivalent to a block minimal bases pencil under some nonsingularity conditions (A_6 nonsingular). In the latter case, though, $L_P(\lambda)$ is an extended block Kronecker pencil, which is a strong linearization of $P(\lambda)$. This result can be generalized as follows.

THEOREM 3.16. *Let $L(\lambda)$ be a generalized Fiedler pencil with repetition associated with a matrix polynomial $P(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$. Then, up to permutations, $L(\lambda)$ is an extended block Kronecker pencil which, under generic nonsingularity conditions, is a strong linearization of $P(\lambda)$.*

The rest of the paper is dedicated to introduce the needed notation and auxiliary technical results to define all the families of Fiedler-like pencils, to state in a precise way Theorems 3.11 and 3.12, and to prove these results.

4. Index tuples, elementary matrices and Fiedler-like pencils families. In this section, we introduce the notation needed to define and to work in an effective way with the families of (square) Fiedler pencils (FP), generalized Fiedler pencils (GFP), Fiedler pencils with repetition (FPR), and generalized Fiedler pencils with repetition (GFPR). Many of our results are still valid for FP associated with rectangular polynomials, however, since GFP and GFPR (and, therefore FPR) are not defined in the rectangular case, and we use known results about FP to prove our main results for GFPR, we only consider FP for square matrix polynomials. We also present some results that will be used to prove our main theorems.

4.1. Index tuples, elementary matrices and matrix assignments. The square Fiedler pencils as well as the generalized Fiedler pencils, Fiedler pencils with repetition and generalized Fiedler pencils with repetition were defined originally by expressing their matrix coefficients as products of the so-called *elementary matrices*, whose definition we recall next.

DEFINITION 4.1. [8] *Let $k \geq 2$ be an integer and let B be an arbitrary $n \times n$ matrix. We call* elementary matrices *the following block-matrices partitioned into $k \times k$ blocks of size $n \times n$:*

$$M_0(B) := \left[\begin{array}{c|c} I_{(k-1)n} & 0 \\ \hline 0 & B \end{array}\right], \quad M_{-k}(B) := \left[\begin{array}{c|c} B & 0 \\ \hline 0 & I_{(k-1)n} \end{array}\right],$$

$$M_i(B) := \left[\begin{array}{c|cc|c} I_{(k-i-1)n} & 0 & 0 & 0 \\ \hline 0 & B & I_n & 0 \\ 0 & I_n & 0 & 0 \\ \hline 0 & 0 & 0 & I_{(i-1)n} \end{array}\right], \quad i = 1 : k-1, \tag{4.1}$$

$$M_{-i}(B) := \left[\begin{array}{c|cc|c} I_{(k-i-1)n} & 0 & 0 & 0 \\ \hline 0 & 0 & I_n & 0 \\ 0 & I_n & B & 0 \\ \hline 0 & 0 & 0 & I_{(i-1)n} \end{array}\right] \quad i = 1 : k-1,$$

and

$$M_{-0}(B) = M_0(B)^{-1} \quad \textit{and} \quad M_k(B) = M_{-k}(B)^{-1}.$$

Notice that both 0 *and* -0 *are used with different meanings as subindices and that* I_0 *should be understood as an empty matrix.*

REMARK 4.2. *We note that, for* $i = 1 : k-1$*, the matrices* $M_i(B)$ *and* $M_{-i}(B)$ *are nonsingular for any* B*. Moreover,* $(M_i(B))^{-1} = M_{-i}(-B)$*. On the other hand, the matrices* $M_0(B)$ *and* $M_{-k}(B)$ *are nonsingular if and only if* B *is nonsingular.*

Given a matrix polynomial $P(\lambda) = \sum_{i=0}^{k} A_i\lambda^i \in \mathbb{F}[\lambda]^{n\times n}$, we consider the following abbreviated notation:

$$M_i^P := M_i(-A_i), \quad i = 0 : k-1,$$

and

$$M_{-i}^P := M_{-i}(A_i), \quad i = 1 : k.$$

When the polynomial $P(\lambda)$ is understood in the context, we will simply write M_i and M_{-i}, instead of M_i^P and M_{-i}^P to simplify the notation.

REMARK 4.3. *It is easy to check that the commutativity relation*

$$M_i(B_1)M_j(B_2) = M_j(B_2)M_i(B_1) \tag{4.2}$$

holds for any $n \times n$ *matrices* B_1 *and* B_2 *if* $||i| - |j|| \neq 1$ *and* $|i| \neq |j|$.

As mentioned earlier, the matrix coefficients of Fiedler-like pencils are defined as products of elementary matrices. In order to describe these products and their properties it is convenient to use index tuples, which are introduced in the following definition.

DEFINITION 4.4. [8, Definition 3.1] *We call an* index tuple *a finite ordered sequence of integer numbers. Each of these integers is called an* index *of the tuple.*

If $\mathbf{t} = (t_1, \ldots, t_r)$ is an index tuple, we denote $-\mathbf{t} := (-t_1, \ldots, -t_r)$, and, when a is an integer, we denote $a + \mathbf{t} := (a + t_1, a + t_2, \ldots, a + t_r)$. We call the *reversal index tuple* of $\mathbf{t}$ the index tuple $\mathrm{rev}(\mathbf{t}) := (t_r, \ldots, t_2, t_1)$.

When an index tuple $\mathbf{t}$ is of the form $(a : b)$ for some integers a and b, we call $\mathbf{t}$ a *string*. Additionally, given index tuples $\mathbf{t}_1, \ldots, \mathbf{t}_s$, we denote by $(\mathbf{t}_1, \ldots, \mathbf{t}_s)$ the index tuple obtained by concatenating the indices in the index tuples $\mathbf{t}_1, \ldots, \mathbf{t}_s$ in the indicated order.

Given an index tuple $\mathbf{t} = (t_1, t_2, \ldots, t_r)$, we call an $n \times n$ *matrix assignment for* $\mathbf{t}$ any ordered collection $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ of arbitrary $n \times n$ matrices [8, Definition 4.1]. If the size n of the matrices is not important in the context, we simply say that $\mathcal{X}$ is a matrix assignment for $\mathbf{t}$. In addition, if the indices of $\mathbf{t}$ are in $\{-k : k-1\}$, we define

$$M_{\mathbf{t}}(\mathcal{X}) := M_{t_1}(X_1)M_{t_2}(X_2)\cdots M_{t_r}(X_r). \tag{4.3}$$

Given a matrix polynomial $P(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$, we say that $\mathcal{X} = (X_1, \ldots X_r)$ is the trivial matrix assignment for $\mathbf{t} = (t_1, \ldots, t_r)$ associated with $P(\lambda)$ if $M_{t_j}(X_j) = M_{t_j}^P$ for $j = 1 : r$. Moreover, we define

$$M_{\mathbf{t}}^P := M_{t_1}^P M_{t_2}^P \cdots M_{t_r}^P. \tag{4.4}$$

When $P(\lambda)$ is clear from the context, we just write $M_{\mathbf{t}}$ instead of $M_{\mathbf{t}}^P$. If $\mathbf{t}$ is the empty tuple, then $M_{\mathbf{t}}^P$ and $M_{\mathbf{t}}(\mathcal{X})$ are defined to be the identity matrix I_{kn}.

Given a sequence $\mathcal{X} = (X_1, \ldots, X_r)$ of matrices, we denote $\mathrm{rev}(\mathcal{X}) := (X_r, X_{r-1}, \ldots, X_1)$.

If a tuple $\mathbf{t}$ is expressed as a concatenation of tuples $\mathbf{t}_1, \ldots, \mathbf{t}_s$ (i.e., $\mathbf{t} = (\mathbf{t}_1, \ldots, \mathbf{t}_s)$), and a matrix assignment $\mathcal{X}$ for $\mathbf{t}$ as a concatenation of matrix assignments $\mathcal{X}_1, \ldots, \mathcal{X}_s$ (i.e., $\mathcal{X} = (\mathcal{X}_1, \ldots, \mathcal{X}_s)$), where the number of indices in $\mathbf{t}_i$ is equal to the number of matrices in $\mathcal{X}_i$, for $i = 1 : s$, we say that $\mathcal{X}_i$ *is the matrix assignment for* $\mathbf{t}_i$ *induced by* $\mathcal{X}$.

The following simple lemma gives the block-structure of $M_{\mathbf{t}}(\mathcal{X})$ when $\mathbf{t}$ is a string consisting of nonnegative indices. We omit its proof since the result can be obtained by a direct computation.

LEMMA 4.5. *Let $\mathbf{t} = (a : b)$ be a string with indices from $\{0 : k-1\}$ and let $\mathcal{X} = (X_1, \ldots, X_{b-a+1})$ be an $n \times n$ matrix assignment for $\mathbf{t}$. Then, the $kn \times kn$ matrix $M_{\mathbf{t}}(\mathcal{X})$ is a block-diagonal matrix of the form*

$$M_{\mathbf{t}}(\mathcal{X}) \coloneqq I_{n(k-b-1)} \oplus \begin{bmatrix} X_{b-a+1} & I_n & & & \\ X_{b-a} & & I_n & & \\ \vdots & & & \ddots & \\ X_1 & & & & I_n \\ I_n & & & & \end{bmatrix} \oplus I_{n(a-1)} \tag{4.5}$$

if $a \neq 0$; and of the form

$$M_{\mathbf{t}}(\mathcal{X}) \coloneqq I_{n(k-b-1)} \oplus \begin{bmatrix} X_{b+1} & I_n & & & \\ X_b & & I_n & & \\ \vdots & & & \ddots & \\ X_2 & & & & I_n \\ X_1 & & & & \end{bmatrix} \tag{4.6}$$

if $a = 0$.

Notice that the matrix $M_{\mathbf{t}}(\mathcal{X})$ in (4.5)–(4.6) is *operation-free* (although it is defined as a product of matrices), that is, it is a block-matrix whose blocks are of the form 0_n, I_n, and the $n \times n$ matrices from the matrix assignment $\mathcal{X}$. Note also that the position where each of these blocks lies only depends on $\mathbf{t}$ [8, Definition 4.5]. To guarantee this operation-free property for more general index tuples $\mathbf{t}$, we need the following definition.

DEFINITION 4.6. [51, Definition 7] *Let $\mathbf{t} = (i_1, i_2, \ldots, i_r)$ be an index tuple of either nonnegative integers or negative integers. Then, $\mathbf{t}$ is said to satisfy the* Successor Infix Property (SIP) *if for every pair of indices $i_a, i_b \in \mathbf{t}$, with $1 \leq a < b \leq r$, satisfying $i_a = i_b$, there exists at least one index $i_c = i_a + 1$ with $a < c < b$.*

REMARK 4.7. *Any subtuple of consecutive indices of a tuple satisfying the SIP also satisfies the SIP. Moreover, the reversal of any tuple satisfying the SIP also satisfies the SIP.*

THEOREM 4.8. [8, Theorem 4.6] *Let $\mathbf{t}$ be a tuple with indices from either $\{0 : k-1\}$ or $\{-k : -1\}$. The tuple $\mathbf{t}$ is operation free in the sense of* [8, Definition 4.5] *if and only if $\mathbf{t}$ satisfies the SIP.*

SIP also guarantees that the block-transpose of a product of elementary matrices behaves as the regular transpose.

LEMMA 4.9. [8, Lemma 4.8] *Let k be a positive integer, let $\mathbf{t}$ be an index tuple satisfying the SIP with indices from $\{0 : k-1\}$ and let $\mathcal{X}$ be a matrix assignment for $\mathbf{t}$. Then, $M_{\mathbf{t}}(\mathcal{X})^{\mathcal{B}} = M_{\mathrm{rev}(\mathbf{t})}(\mathrm{rev}(\mathcal{X}))$.*

Due to the commutativity relations of elementary matrices explained in Remark 4.2, different index tuples $\mathbf{t}_1$ and $\mathbf{t}_2$ may give rise to the same product of elementary matrices, that is, given a matrix assignment $\mathcal{X}$ for both $\mathbf{t}_1$ and $\mathbf{t}_2$, we may have $M_{\mathbf{t}_1}(\mathcal{X}) = M_{\mathbf{t}_2}(\sigma(\mathcal{X}))$, where σ is a permutation that transforms t_1 into t_2 (see [8, Lemma 4.4]). Next, we introduce a canonical form for index tuples such that two index tuples with the same canonical form yield the same product of elementary matrices. But, first, notice that given an index tuple $\mathbf{t} = (t_1, t_2, \ldots, t_r)$ and a matrix assignment $\mathcal{X} = (X_1, \ldots, X_r)$ for $\mathbf{t}$, the elementary matrices $M_{t_i}(X_i)$ and $M_{t_{i+1}}(X_{i+1})$ commute whenever $||t_i| - |t_{i+1}|| > 1$ and $|t_i| \neq |t_{i+1}|$. For this reason, here and thereafter, we say that in this situation the *indices t_i and t_{i+1} in $\mathbf{t}$ commute*.

DEFINITION 4.10. [8, Definition 3.4] *Given two index tuples* $\mathbf{t}$ *and* $\mathbf{t}'$ *of nonnegative indices, we say that* $\mathbf{t}$ *is equivalent to* $\mathbf{t}'$ *(and write* $\mathbf{t} \sim \mathbf{t}'$*), if* $\mathbf{t} = \mathbf{t}'$ *or* $\mathbf{t}'$ *can be obtained from* $\mathbf{t}$ *by interchanging a finite number of times two distinct commuting indices in adjacent positions, that is, indices* t_i *and* t_{i+1} *such that* $|t_i - t_{i+1}| \neq 1$ *and* $t_i \neq t_{i+1}$.

REMARK 4.11. *The product of elementary matrices is invariant under the equivalence introduced in Definition 4.7, i.e., given an index tuple* $\mathbf{t}$ *and a matrix assignment* $\mathcal{X}$ *for* $\mathbf{t}$*, if* $\mathbf{t} \sim \mathbf{t}'$ *and* σ *is a permutation that transforms* $\mathbf{t}$ *into* $\mathbf{t}'$*, then* $M_{\mathbf{t}}(\mathcal{X}) = M_{\mathbf{t}'}(\sigma(\mathcal{X}))$*, where if* $\mathcal{X} = (X_1, \ldots, X_r)$*, then* $\sigma(X) = (X_{\sigma(1)}, \ldots, X_{\sigma(r)})$ *[8, Lemma 4,4].*

It turns out that if $\mathbf{t}$ satisfies the SIP property, the index tuple $\mathbf{t}$ is equivalent to a tuple with a special structure that we introduce in the following definition.

DEFINITION 4.12. [51, Theorem 1] *Let* $\mathbf{t}$ *be an index tuple with indices from* $\{0 : h\}$*,* $h \geq 0$*. Then* $\mathbf{t}$ *is said to be in* column standard form *if*

$$\mathbf{t} = (a_s : b_s, a_{s-1} : b_{s-1}, \ldots, a_2 : b_2, a_1 : b_1)\,,$$

with $h \geq b_s > b_{s-1} > \cdots > b_2 > b_1 \geq 0$ *and* $0 \leq a_j \leq b_j$*, for all* $j = 1 : s$*. We call each subtuple of consecutive indices* $(a_i : b_i)$ *a* string *of* $\mathbf{t}$.

In the next lemma, we show the connection between tuples satisfying the SIP and tuples in column standard form.

LEMMA 4.13. [51, Theorem 2] *Let* $\mathbf{t}$ *be an index tuple.*

(i) *If the indices of* $\mathbf{t}$ *are all nonnegative integers, then* $\mathbf{t}$ *satisfies the SIP if and only if* $\mathbf{t}$ *is equivalent to a tuple in column standard form.*

(ii) *If the indices of* $\mathbf{t}$ *are all negative integers and* a *is the minimum index in* $\mathbf{t}$*, then* $\mathbf{t}$ *satisfies the SIP if and only if* $-a + \mathbf{t}$ *is equivalent to a tuple in column standard form.*

Lemma 4.9, together with [5, Proposition 2.12], shows that two index tuples in column standard form are equivalent if and only if they coincide. Thus, we can introduce a canonical form for tuples satisfying the SIP under the equivalence relation introduced in Definition 4.7.

DEFINITION 4.14. [8, Definition 3.9] *The unique index tuple in column standard form equivalent to an index tuple* $\mathbf{t}$ *of nonnegative integers satisfying the SIP is called the* column standard form of $\mathbf{t}$ *and is denoted by* $\mathrm{csf}(\mathbf{t})$.

In the next definition, we introduce some special indices of tuples of nonnegative integers satisfying the SIP that will play a key role in the rest of the paper.

DEFINITION 4.15. *Given an index tuple* $\mathbf{t}$ *with* $\mathrm{csf}(\mathbf{t}) = (a_s : b_s, \ldots, a_1 : b_1)$*, we define* $\mathrm{heads}(\mathbf{t}) := \{b_i \mid 1 \leq i \leq s\}$*. Furthermore, we denote by* $\mathfrak{h}(\mathbf{t})$ *the cardinality of* $\mathrm{heads}(\mathbf{t})$.

Given an index tuple $\mathbf{t}$, note that $\mathfrak{h}(\mathbf{t})$ gives not only the cardinality of the set $\mathrm{heads}(\mathbf{t})$, but also the number of strings in $\mathrm{csf}(\mathbf{t})$.

EXAMPLE 4.16. *The tuples* $\mathbf{t}_1 = (3 : 4, 0 : 2)$ *and* $\mathbf{t}_2 = (1 : 4, 0 : 3, 0 : 2, 1, 0)$ *are tuples in column standard form with indices from* $\{0 : 4\}$*. Note that* $\mathrm{heads}(\mathbf{t}_1) = \{4, 2\}$ *and* $\mathfrak{h}(\mathbf{t}_1) = 2$*. Similarly,* $\mathrm{heads}(\mathbf{t}_2) = \{4, 3, 2, 1, 0\}$ *and* $\mathfrak{h}(\mathbf{t}_2) = 5$.

We introduce in Definition 4.12 an important distinction between some type of indices relative to a given index tuple.

DEFINITION 4.17. *Given an index tuple* $\mathbf{t}$ *and an index* x *such that* $(\mathbf{t}, x)$ *satisfies the SIP, we say that* x *is of* Type I *relative to* $\mathbf{t}$ *if* $\mathfrak{h}(\mathbf{t}, x) = \mathfrak{h}(\mathbf{t})$*, and of* Type II *otherwise. That is,* x *is of Type I relative to* $\mathbf{t}$ *if* $\mathrm{csf}(\mathbf{t}, x)$ *has the same number of heads (and, therefore of strings) as* $\mathrm{csf}(\mathbf{t})$*, and of* Type II *relative to* $\mathbf{t}$ *otherwise.*

EXAMPLE 4.18. *Let* $\mathbf{t} = (3 : 5, 2, 0 : 1)$*. Then* 3 *is an index of Type I relative to* $\mathbf{t}$ *since*

$$\mathrm{csf}(\mathbf{t}, 3) = (3 : 5, 2 : 3, 0 : 1)$$

has the same number of strings as $\mathbf{t}$ *(recall that* 3 *commutes with 0 and 1). However,* 4 *is an index of Type II relative to* $\mathbf{t}$ *since*

$$\mathrm{csf}(\mathbf{t}, 4) = (3 : 5, 4, 2, 0 : 1)$$

has more strings than $\mathbf{t}$.

The following technical lemmas and results are used in future sections. Here and thereafter we use the following notation. If $\mathbf{t}$ is an index tuple and a is an index of $\mathbf{t}$, we write $a \in \mathbf{t}$. Also, we denote by e_i the ith column of the identity matrix of arbitrary size, when the size is clear from the context.

In the first results we determine the set of heads of the concatenation $(\mathbf{t}_1, \mathbf{t}_2)$ of two index tuples that satisfies the SIP when the set of heads of either $\mathbf{t}_1$ or $\mathbf{t}_2$ is known.

LEMMA 4.19. *Let* k *be a positive integer and let* $\mathbf{t}$ *be an index tuple with indices from* $\{0 : k-1\}$. *Let* $(a : b)$ *be a string with indices from* $\{0 : k-2\}$. *Then, the tuple* $(\mathbf{t}, a : b)$ *satisfies the SIP property if and only if* $\mathbf{t}$ *satisfies the SIP and* $c \notin \mathrm{heads}(\mathbf{t})$, *for all* $c \in (a : b)$.

Proof. $\Longrightarrow$ Since $(\mathbf{t}, a : b)$ satisfies the SIP, so does $\mathbf{t}$. Thus, by Lemma 4.9, we may assume, without loss of generality, that $\mathbf{t}$ is in column standard form, that is, $\mathbf{t} = (a_s : b_s, \ldots, a_1 : b_1)$. The proof proceeds by contradiction. Assume that $c \in (a : b)$ and $c \in \mathrm{heads}(\mathbf{t})$. Then, there exists $1 \le i \le s$ such that $c = b_i$. Since $\mathbf{t}$ is a tuple in column standard form, the indices in the subtuple $(a_{i-1} : b_{i-1}, \ldots, a_1 : b_1)$ of $\mathbf{t}$ are in $\{0 : b_i - 1\}$. Therefore, there is no index between $b_i \in \mathbf{t}$ and $b_i = c \in (a : b)$ equal to $c+1$ which implies that $(\mathbf{t}, a : c)$ does not satisfy the SIP, a contradiction.

$\Longleftarrow$ To prove the implication, it is enough to show that, between $c \in (a : b)$ and the rightmost index equal to c in $\mathbf{t}$, if it exists, there is an index equal to $c+1$. Let c be such an index. Since $\mathbf{t}$ satisfies the SIP, without loss of generality, we may assume that $\mathbf{t} = (a_s : b_s, \ldots, a_1 : b_1)$, for some nonnegative integers a_i, b_i and $i = 1 : s$, that is, $\mathbf{t}$ is in column standard form. Let $(a_j : b_j)$ be the rightmost string of $\mathbf{t}$ such that $c \in (a_j : b_j)$. Since $c \neq b_i$, for all i, the tuple $(a_j : b_j)$ is of the form

$$(c, c+1, \ldots, b_j) \quad \text{or} \quad (a_j, \ldots, c, c+1, \ldots, b_j) \quad \text{or} \quad (a_j, \ldots, c, c+1),$$

which shows the desired result. □

PROPOSITION 4.20. *Let* $\mathbf{t} = (a_s : b_s, \ldots, a_2 : b_2, a_1 : b_1)$ *be a nonempty index tuple in column standard form with indices from* $\{0 : k-1\}$, *for* $k \ge 1$. *Let* x *be an index in* $\{0 : b_s - 1\}$ *such that* $(\mathbf{t}, x)$ *satisfies the SIP. Then* x *is of Type I relative to* $\mathbf{t}$ *if and only if* $x - 1 \in \mathrm{heads}(\mathbf{t})$. *In particular,* $x = 0$ *is always an index of Type II relative to* $\mathbf{t}$, *for every nonempty index tuple* $\mathbf{t}$ *in column standard form with nonnegative indices.*

Proof. Since $(\mathbf{t}, x)$ satisfies the SIP, we have $x \notin \mathrm{heads}(\mathbf{t})$ by Lemma 4.13. Therefore, since $x \in \{0 : b_s - 1\}$, there is either some $1 \le \ell < s$ with $b_{\ell+1} > x > b_\ell$ or $b_1 > x$.

Assume that $x - 1 \in \mathrm{heads}(\mathbf{t})$. Then, $x = b_j + 1$ for some $1 \le j \le s$. Thus, we have

$$\begin{aligned}(\mathbf{t}, x) &= (a_s : b_s, \ldots, a_j : b_j, \ldots, a_1 : b_1, b_j + 1) \\ &\sim (a_s : b_s, \ldots, a_j : b_j + 1, \ldots, a_1 : b_1) = \mathrm{csf}(\mathbf{t}, x),\end{aligned}$$

where the equivalence holds because $x > b_j > b_{j-1}$, and hence x commutes with every index of $\mathbf{t}$ ocurring to the right of b_j. Moreover, the second equality follows from the fact that $b_{j+1} \neq b_j + 1$, otherwise, the SIP does not hold. Since $\mathrm{csf}(\mathbf{t}, x)$ has the same number of strings (heads) as $\mathrm{csf}(\mathbf{t})$, it follows that x is of Type I relative to $\mathbf{t}$.

Now suppose that $x - 1 \notin \mathrm{heads}(\mathbf{t})$. If $b_1 > x$, then

$$\mathrm{csf}(\mathbf{t}, x) = (\mathbf{t}, x),$$

by definition of column standard form, which implies that $\mathrm{csf}(\mathbf{t}, x)$ has one more string than $\mathbf{t}$. Hence x is not of Type I relative to $\mathbf{t}$.

If $b_{\ell+1} > x > b_\ell$ for some $\ell > 0$ and $x - 1 \notin \mathrm{heads}(\mathbf{t})$, then $x > b_\ell + 1$. Therefore, x commutes with $(a_\ell : b_\ell, \ldots, a_1 : b_1)$ since all the indices of this tuple are smaller than or equal to b_ℓ. Therefore,

$$\begin{aligned}(\mathbf{t}, x) &= (a_s : b_s, \ldots, a_\ell : b_\ell, \ldots, a_1 : b_1, x) \\ &\sim (a_s : b_s, \ldots, a_{\ell+1} : b_{\ell+1}, x, a_\ell : b_\ell, \ldots, a_1 : b_1) = \mathrm{csf}(\mathbf{t}, x),\end{aligned}$$

Since the number of strings of $\mathrm{csf}(\mathbf{t}, x)$ is larger than the number of strings of $\mathbf{t}$, the index x is not of Type I relative to $\mathbf{t}$ and the result follows. □

REMARK 4.21. *Using the notation of Proposition 4.14, note that this proposition implies that* $\mathrm{heads}(\mathbf{t}, x) = (\mathrm{heads}(\mathbf{t}) \setminus \{b_j\}) \cup \{x\}$ *when x is an index of Type I relative to $\mathbf{t}$ and $x = b_j + 1$. Furthermore, the proof of Proposition 4.14 shows that if x is of Type II relative to $\mathbf{t}$, then* $\mathrm{heads}(\mathbf{t}, x) = \mathrm{heads}(\mathbf{t}) \cup \{x\}$.

LEMMA 4.22. *Let k be a positive integer and let $\mathbf{t}$ be an index tuple containing each index in $\{0 : k-1\}$ at least once. Let $(a : b)$ be a string with indices from $\{0 : k-2\}$ such that $(a : b, \mathbf{t})$ satisfies the SIP. Then* $\mathrm{heads}(a : b, \mathbf{t}) = \mathrm{heads}(\mathbf{t})$.

Proof. Since $(a : b, \mathbf{t})$ satisfies the SIP, so does $\mathbf{t}$. Therefore, the tuple $\mathbf{t}$ is equivalent to a unique tuple $\mathrm{csf}(\mathbf{t}) = (a_s : b_s, \ldots, a_1 : b_1)$ in column standard form by Lemma 4.9. Because $(a : b, \mathbf{t})$, and hence $(a : b, \mathrm{csf}(\mathbf{t}))$, satisfies the SIP, we know that the leftmost appearance of $b + 1$ in $\mathrm{csf}(\mathbf{t})$ is farther left than any appearance of b. We know that such appearances actually occur because each index in $\{0 : k-1\}$ appears in $\mathbf{t}$ at least once. It follows that the leftmost appearance of $b + 1$ in $\mathrm{csf}(\mathbf{t})$ must be of the form a_j for some $1 \le j \le s$, since otherwise b would occur to the left of this appearance of $b + 1$. Moreover, since $(a : b, \mathbf{t})$ satisfies the SIP, all indices in $(a_s : b_s, \ldots, a_{j+1} : b_{j+1})$ are larger than $b + 1$. Therefore, the indices $a, a+1, \ldots, b$ all commute with the indices in $(a_s : b_s, \ldots, a_{j+1} : b_{j+1})$ and we have

$$\begin{aligned}(a : b, \mathrm{csf}(\mathbf{t})) &= (a : b, a_s : b_s, \ldots, a_j : b_j, \ldots) \\ &\sim (a_s : b_s, \ldots, a : b, a_j : b_j, \ldots) \\ &= (a_s : b_s, \ldots, a : b_j, \ldots),\end{aligned}$$

which implies $\mathrm{heads}(a : b, \mathbf{t}) = \mathrm{heads}(\mathbf{t})$. □

We end this section with a result that provides some structural properties concerning the block-columns and block-rows of the product of elementary matrices $M_{\mathbf{t}}(\mathcal{X})$ defined in (4.3).

LEMMA 4.23. *Let k be a positive integer and let $\mathbf{t}$ be an index tuple in column standard form with indices from $\{0 : k-1\}$. Let $\mathcal{X}$ be an $n \times n$ matrix assignment for $\mathbf{t}$, and let us consider the $kn \times kn$ matrix $M_{\mathbf{t}}(\mathcal{X})$ as a $k \times k$ block-matrix with blocks of size $n \times n$. Then, all block-columns of $M_{\mathbf{t}}(\mathcal{X})$ are of the form $e_i \otimes I_n$ (for some $1 \le i \le k$) except for the $(k-h)$th block-column for each $h \in \mathrm{heads}(\mathbf{t})$. Moreover, all block-rows of the $kn \times kn$ matrix $M_{\mathbf{t}}(\mathcal{X})$ are of the form $e_i^T \otimes I_n$ (for some $1 \le i \le k$) except for the $(k-h)$th block-row for each $h \in \mathrm{heads}(\mathrm{rev}(\mathbf{t}))$.*

Proof. The proof proceeds by induction on the number s of strings of the tuple $\mathbf{t}$. Assume, first, that $s = 1$. In this case $\mathbf{t} = (a_1 : b_1)$, for some $a_1 \le b_1$. By Lemma 4.3, it is clear that all block-columns of $M_t(\mathcal{X})$ are of the form $e_i \otimes I_n$, except for the $(k - b_1)$th block-column. Therefore, the desired result for block-columns holds for $s = 1$.

Suppose now that the lemma holds for all s less than some $s_0 > 1$ and assume that $\mathbf{t}$ has s_0 strings. Let $\mathbf{t} = (a_{s_0} : b_{s_0}, \ldots, a_1 : b_1)$. For $1 \le j \le s_0$, let $\mathbf{t}_j := (a_j : b_j)$ and let $\mathcal{X}_j$ be

the matrix assignment for $\mathbf{t}_j$ induced by $\mathcal{X}$. Then

$$M_{\mathbf{t}}(\mathcal{X}) = M_{\mathbf{t}_{s_0}}(\mathcal{X}_{s_0}) \cdots M_{\mathbf{t}_1}(\mathcal{X}_1).$$

By the inductive hypothesis, all block-columns of $M_{\mathbf{t}_{s_0}}(\mathcal{X}_{s_0}) \cdots M_{\mathbf{t}_2}(\mathcal{X}_2)$, except for the $(k - b_j)$th block-column, for $2 \leq j \leq s_0$, are of the form $e_i \otimes I_n$. Thus,

$$M_{\mathbf{t}_{s_0}}(\mathcal{X}_{s_0}) \cdots M_{\mathbf{t}_2}(\mathcal{X}_2) = \begin{bmatrix} C & D \end{bmatrix},$$

where C is formed by the first $k - b_2$ block-columns of $M_{\mathbf{t}_{s_0}}(\mathcal{X}_{s_0}) \cdots M_{\mathbf{t}_2}(\mathcal{X}_2)$, and all block-columns of D are of the form $e_i \otimes I_n$. Recall that, since $\mathbf{t}$ is a tuple in column standard form, we have $b_{s_0} > b_{s_0-1} > \cdots > b_1$. Thus, from Lemma 4.3, multiplying $M_{\mathbf{t}_{s_0}}(\mathcal{X}_{s_0}) \cdots M_{\mathbf{t}_2}(\mathcal{X}_2)$ by $M_{\mathbf{t}_1}(\mathcal{X}_1)$ leaves unmodified the columns of C. Moreover, taking into account the block-structure of $M_{\mathbf{t}_1}(\mathcal{X}_1)$, all the block-columns of $M_{\mathbf{t}_{s_0}}(\mathcal{X}_{s_0}) \cdots M_{\mathbf{t}_1}(\mathcal{X}_1)$ from the $(k - b_2 + 1)$th to the last, with the exception of the $(k - b_1)$th column, are of the form $e_i \otimes I_n$ for some i, and the result for the block-columns follows.

To finish the proof, note that the block-rows of $M_{\mathbf{t}}(\mathcal{X})$ are the block-columns of $M_{\mathbf{t}}(\mathcal{X})^{\mathcal{B}}$, which, by Lemma 4.6, equals $M_{\mathrm{rev}(\mathbf{t})}(\mathrm{rev}(\mathcal{X}))$. Thus, the result for block-rows follows from the result for block-columns applied to $M_{\mathbf{t}}(\mathcal{X})^{\mathcal{B}}$. □

4.2. Fiedler-like pencils families. In this section, we recall the families of FP, proper GFP, FPR and GFPR (see also [2, 8, 13, 15, 23, 51]). Notice that in this work these families are defined in terms of the grades of the matrix polynomials they are associated with, in contrast to [2, 8, 13, 51], where they were originally defined in terms of the degrees.

We start by recalling the definition of FP. Although they have been defined for both square and rectangular matrix polynomials [15], here we focus on the square case since the other Fiedler-like families are only defined in the square case and we will use the Fiedler pencils as the base case in some inductive arguments. In this case, Fiedler pencils can be defined via elementary matrices.

DEFINITION 4.24. (Fiedler pencils) *Let* $P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i \in \mathbb{F}[\lambda]^{n \times n}$*, and let* $\mathbf{q}$ *be a permutation of* $\{0 : k - 1\}$*. Then, the* Fiedler pencil *associated with* $P(\lambda)$ *and* $\mathbf{q}$ *is*

$$F_{\mathbf{q}}(\lambda) := \lambda M_{-k}^{P} - M_{\mathbf{q}}^{P}.$$

EXAMPLE 4.25. *Let* $P(\lambda) = \sum_{i=0}^{6} A_i \lambda^i \in \mathbb{F}[\lambda]^{n \times n}$*, and let* $\mathbf{q} = (0, 2, 4, 1, 3, 5)$*. Then,*

$$F_{\mathbf{q}}(\lambda) = \begin{bmatrix} \lambda A_6 + A_5 & -I_n & 0 & 0 & 0 & 0 \\ A_4 & \lambda I_n & A_3 & -I_n & 0 & 0 \\ -I_n & 0 & \lambda I_n & 0 & 0 & 0 \\ 0 & 0 & A_2 & \lambda I_n & A_1 & -I_n \\ 0 & 0 & -I_n & 0 & \lambda I_n & 0 \\ 0 & 0 & 0 & 0 & A_0 & \lambda I_n \end{bmatrix} \tag{4.7}$$

is the Fiedler pencil associated with $P(\lambda)$ *and* $\mathbf{q}$*, which was considered in Section 3.4.*

Next, we introduce the proper GFP.

DEFINITION 4.26. (GFP and proper GFP) *Let* $P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i \in \mathbb{F}[\lambda]^{n \times n}$*, let* $\{C_0, C_1\}$ *be a partition of* $\{0 : k\}$*, where* $0 \in C_0$ *and* $k \in C_1$*, and let* $\mathbf{q}$ *and* $\mathbf{z}$ *be permutations of* C_0 *and* $-C_1$*, respectively. Then, the* proper generalized Fiedler pencil (GFP) *associated with* $P(\lambda)$ *and* $(\mathbf{q}, \mathbf{z})$ *is*

$$K_{\mathbf{q},\mathbf{z}}(\lambda) := \lambda M_{\mathbf{z}}^{P} - M_{\mathbf{q}}^{P}.$$

We show in Example 4.4 one proper generalized Fiedler pencil associated with a matrix polynomial of grade 6.

EXAMPLE 4.27. *Let* $P(\lambda) = \sum_{i=0}^{6} A_i\lambda^i \in \mathbb{F}[\lambda]^{n\times n}$*, let* $\mathbf{z} = (-1,-6,-5)$ *and* $\mathbf{q} = (3,4,2,0)$*. Then,*

$$K_{\mathbf{q},\mathbf{z}}(\lambda) = \begin{bmatrix} -I_n & \lambda A_6 & 0 & 0 & 0 & 0 \\ \lambda I_n & \lambda A_5 + A_4 & -I_n & 0 & 0 & 0 \\ 0 & A_3 & \lambda I_n & A_2 & -I_n & 0 \\ 0 & -I_n & 0 & \lambda I_n & 0 & 0 \\ 0 & 0 & 0 & -I_n & 0 & \lambda I_n \\ 0 & 0 & 0 & 0 & \lambda I_n & \lambda A_1 + A_0 \end{bmatrix}$$

is the proper generalized Fiedler pencil associated with $P(\lambda)$ *and* $(\mathbf{q},\mathbf{z})$*. This pencil is the proper generalized Fiedler pencil considered in Section 3.4.*

REMARK 4.28. *Notice that, for any given proper GFP* $K_{\mathbf{q},\mathbf{z}}(\lambda)$*, since* $(\mathbf{q},-\mathbf{z})$ *can be expressed in column standard form because it is a permutation of the tuple* $(0:k)$*, there is always an index* $h \in \{0:k-1\}$ *and a tuple* $\mathbf{m}$ *such that*

$$K_{\mathbf{q},\mathbf{z}}(\lambda) = \lambda M^P_{(\mathbf{m},-k:-h-1)} - M^P_{\mathbf{q}},$$

where $\widehat{\mathbf{q}} := (-\operatorname{rev}(\mathbf{m}),\mathbf{q})$ *is a permutation of* $\{0:h\}$*. We call* $(\widehat{\mathbf{q}},h)$ *the simple pair associated with* $K_{\mathbf{q},\mathbf{z}}(\lambda)$*.*

We finish the section by recalling the definition of FPR and GFPR. We note that, although we have assigned a special notation to denote FP and proper GFP, we will not do so with FPR and GFPR since their construction involve too many parameters.

DEFINITION 4.29. (FPR and GPFR) *Let* $P(\lambda) = \sum_{i=0}^{k} A_i\lambda^i \in \mathbb{F}[\lambda]^{n\times n}$*. Let* $h \in \{0:k-1\}$*, and let* $\mathbf{q}$ *and* $\mathbf{z}$ *be permutations of* $\{0:h\}$ *and* $\{-k:-h-1\}$*, respectively. Let* $\boldsymbol{\ell}_q$ *and* $\mathbf{r}_q$ *be tuples with indices from* $\{0:h-1\}$ *such that* $(\boldsymbol{\ell}_q,\mathbf{q},\mathbf{r}_q)$ *satisfies the SIP. Let* $\boldsymbol{\ell}_z$ *and* $\mathbf{r}_z$ *be tuples with indices from* $\{-k:-h-2\}$ *such that* $(\boldsymbol{\ell}_z,\mathbf{z},\mathbf{r}_z)$ *satisfies the SIP. Let* $\mathcal{X}$*,* $\mathcal{Y}$*,* $\mathcal{Z}$ *and* $\mathcal{W}$ *be* $n\times n$ *matrix assignments for* $\boldsymbol{\ell}_q$*,* $\mathbf{r}_q$*,* $\boldsymbol{\ell}_z$ *and* $\mathbf{r}_z$*, respectively. Then, the pencil*

$$L_P(\lambda) = M_{\boldsymbol{\ell}_q,\boldsymbol{\ell}_z}(\mathcal{X},\mathcal{Z})(\lambda M^P_{\mathbf{z}} - M^P_{\mathbf{q}})M_{\mathbf{r}_z,\mathbf{r}_q}(\mathcal{W},\mathcal{Y}) \tag{4.8}$$

is called a generalized Fiedler pencil with repetition *(GFPR) associated with* $P(\lambda)$*. When* $\mathcal{X}$*,* $\mathcal{Z}$*,* $\mathcal{W}$ *and* $\mathcal{Y}$ *are the trivial matrix assignments for* $\boldsymbol{\ell}_q$*,* $\boldsymbol{\ell}_z$*,* $\mathbf{r}_z$ *and* $\mathbf{r}_q$*, respectively, associated with* $P(\lambda)$*, then* $L_P(\lambda)$ *is called a* Fiedler pencil with repetition *(FPR).*

REMARK 4.30. *If* $\boldsymbol{\ell}_q$*,* $\boldsymbol{\ell}_z$*,* $\mathbf{r}_z$ *and* $\mathbf{r}_q$ *are all empty tuples, then* $L_P(\lambda) = K_{\mathbf{q},\mathbf{z}}(\lambda)$*, that is,* $L_P(\lambda)$ *is a special type of proper GFP. In particular, if* $\mathbf{z} = (-k)$*, then* $L_P(\lambda) = F_{\mathbf{q}}(\lambda)$*, that is,* $L_P(\lambda)$ *is a Fiedler pencil. Thus, the family of GFPR contains the Fiedler pencils, the FPR, and a subfamily of the proper GFP.*

We show in Example 4.5 one FPR associated with a matrix polynomial of grade 6.

EXAMPLE 4.31. *Let* $P(\lambda) = \sum_{i=0}^{6} A_i\lambda^i \in \mathbb{F}[\lambda]^{n\times n}$*, let* $\mathbf{z} = (-6:-1)$*,* $\mathbf{q} = (0)$ *and* $\mathbf{r}_z = (-6:-2,-6:-3,-6:-4,-6:-5,-6)$*. Notice that the index tuple* $(\mathbf{z},\mathbf{r}_z)$ *satisfies the SIP. Then, the following pencil*

$$(\lambda M^P_{\mathbf{z}} - M^P_{\mathbf{q}})M^P_{\mathbf{r}_z} = \begin{bmatrix} 0 & 0 & 0 & 0 & -A_6 & \lambda A_6 \\ 0 & 0 & 0 & -A_6 & \lambda A_6 - A_5 & \lambda A_5 \\ 0 & 0 & -A_6 & \lambda A_6 - A_5 & \lambda A_5 - A_4 & \lambda A_4 \\ 0 & -A_6 & \lambda A_6 - A_5 & \lambda A_5 - A_4 & \lambda A_4 - A_3 & \lambda A_3 \\ -A_6 & \lambda A_6 - A_5 & \lambda A_5 - A_4 & \lambda A_4 - A_3 & \lambda A_3 - A_2 & \lambda A_2 \\ \lambda A_6 & \lambda A_5 & \lambda A_4 & \lambda A_3 & \lambda A_2 & \lambda A_1 + A_0 \end{bmatrix}$$

is a Fiedler pencil with repetition associated with $P(\lambda)$. This pencil is the Fiedler pencil with repetition considered in Section 3.4. Moreover, it is also one of the pencils in the standard basis of the vector space $\mathbb{DL}(P)$.

We end this section presenting three structural lemmas for GFPR. But first, taking into consideration the commutativity relations (4.2), notice that the GFPR associated with a matrix polynomial $P(\lambda)$ given by (4.8) can be rewritten as follows

$$L(\lambda) = \lambda[M_{\boldsymbol{\ell}_q,\mathbf{r}_q}(\mathcal{X},\mathcal{Y})(M_{\boldsymbol{\ell}_z}(\mathcal{Z})M_{\mathbf{z}}^P M_{\mathbf{r}_z}(\mathcal{W}))] - [M_{\boldsymbol{\ell}_q}(\mathcal{X})M_{\mathbf{q}}^P M_{\mathbf{r}_q}(\mathcal{Y})M_{\boldsymbol{\ell}_z,\mathbf{r}_z}(\mathcal{Z},\mathcal{W})].$$

Lemma 4.20 gives some information about the block-structure of the factors that appear in the matrix coefficients of the pencil above. This lemma follows immediately just taking into account the range of indices in the tuples $\boldsymbol{\ell}_q$, $\boldsymbol{\ell}_z$, $\mathbf{r}_q$, $\mathbf{r}_z$, $\mathbf{z}$ and $\mathbf{q}$, so its proof is omitted.

LEMMA 4.32. *Let $L(\lambda) = M_{\boldsymbol{\ell}_q,\boldsymbol{\ell}_z}(\mathcal{X},\mathcal{Z})(\lambda M_{\mathbf{z}}^P - M_{\mathbf{q}}^P)M_{\mathbf{r}_z,\mathbf{r}_q}(\mathcal{W},\mathcal{Y})$ be a GFPR associated with a matrix polynomial $P(\lambda) = \sum_{i=0}^k A_i\lambda^i \in \mathbb{F}[\lambda]^{n\times n}$. Then, the following holds:*

$$\begin{aligned}
&M_{\boldsymbol{\ell}_q,\mathbf{r}_q}(\mathcal{X},\mathcal{Y}) = I_{n(k-h)} \oplus C_{22}, \quad \textit{for some } C_{22} \in \mathbb{F}^{nh\times nh},\\
&M_{\boldsymbol{\ell}_z}(\mathcal{Z})M_{\mathbf{z}}^P M_{\mathbf{r}_z}(\mathcal{W}) = C_{11} \oplus I_{nh}, \quad \textit{for some } C_{11} \in \mathbb{F}^{n(k-h)\times n(k-h)},\\
&M_{\boldsymbol{\ell}_q}(\mathcal{X})M_{\mathbf{q}}^P M_{\mathbf{r}_q}(\mathcal{Y}) = I_{n(k-h-1)} \oplus D_{22}, \quad \textit{for some } D_{22} \in \mathbb{F}^{n(h+1)\times n(h+1)}, \quad \textit{and},\\
&M_{\boldsymbol{\ell}_z,\mathbf{r}_z}(\mathcal{Z},\mathcal{W}) = D_{11} \oplus I_{n(h+1)}, \quad \textit{for some } D_{11} \in \mathbb{F}^{n(k-h-1)\times n(k-h-1)}.
\end{aligned}$$

In the proof of Theorem 8.1 (one of the main results in this paper), we use Lemma 4.21, where we present a structural result for the GFPR, which follows from Lemma 4.20.

LEMMA 4.33. [8, Theorem 5.3] *Let $P(\lambda)$ be an $n\times n$ matrix polynomial of grade k and*

$$L_P(\lambda) := \lambda L_1 - L_0 = M_{\boldsymbol{\ell}_q,\boldsymbol{\ell}_z}(\mathcal{X},\mathcal{Z})(\lambda M_{\mathbf{z}}^P - M_{\mathbf{q}}^P)M_{\mathbf{r}_z,\mathbf{r}_q}(\mathcal{W},\mathcal{Y})$$

be a GFPR associated with $P(\lambda)$. Then:

(a) *$L_1 = \operatorname{diag}(C_{11}, C_{22})$, where C_{11} is a $(k-h)\times(k-h)$ block-matrix which contains the blocks in the matrix assignments for the tuples $\mathbf{z}$, $\boldsymbol{\ell}_z$, and $\mathbf{r}_z$, and C_{22} is an $h\times h$ block-matrix which contains the blocks in the matrix assignments for the tuples $\boldsymbol{\ell}_q$ and $\mathbf{r}_q$.*

(b) *$L_0 = \operatorname{diag}(D_{11}, D_{22})$, where D_{11} is a $(k-h-1)\times(k-h-1)$ block-matrix which contains the blocks in the matrix assignments for the tuples $\boldsymbol{\ell}_z$ and $\mathbf{r}_z$, and D_{22} is a $(h+1)\times(h+1)$ block-matrix which contains the blocks in the matrix assignments for the tuples $\mathbf{q}$, $\boldsymbol{\ell}_q$, and $\mathbf{r}_q$.*

As an immediate consequence of Lemma 4.21, we obtain Lemma 4.22, which will be one of our main tools in future sections.

LEMMA 4.34. *Let $P(\lambda) = \sum_{i=0}^k A_i\lambda^i \in \mathbb{F}[\lambda]^{n\times n}$ be a matrix polynomial of grade k. Let $h \in \{0 : k-1\}$ and let $\mathbf{q}$ be a permutation of $\{0 : h\}$. Let $\mathbf{z}$ be a permutation of $\{-k : -h-1\}$ and let $L_P(\lambda)$ be a GFPR as in* (4.8)*. Then, the pencil $L_P(\lambda)$ can be partitioned as*

$$L_P(\lambda) = \left[\begin{array}{c|c|c} D_{\mathbf{z}}(\lambda) & y_{\mathbf{z}}(\lambda) & 0 \\ \hline x_{\mathbf{z}}(\lambda) & c(\lambda) & x_{\mathbf{q}}(\lambda) \\ \hline 0 & y_{\mathbf{q}}(\lambda) & D_{\mathbf{q}}(\lambda) \end{array}\right],$$

where $D_{\mathbf{q}}(\lambda) \in \mathbb{F}[\lambda]^{nh\times nh}$, $D_{\mathbf{z}}(\lambda) \in \mathbb{F}[\lambda]^{n(k-h-1)\times n(k-h-1)}$, $c(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$, $x_{\mathbf{z}}(\lambda) \in \mathbb{F}[\lambda]^{n\times n(k-h-1)}$, $x_{\mathbf{q}}(\lambda) \in \mathbb{F}[\lambda]^{n\times nh}$, $y_{\mathbf{z}}(\lambda) \in \mathbb{F}[\lambda]^{n(k-h-1)\times n}$ and $y_{\mathbf{q}}(\lambda) \in \mathbb{F}[\lambda]^{nh\times n}$, and

where the pencils

$$F(\lambda) := \begin{bmatrix} c(\lambda) & x_q(\lambda) \\ y_q(\lambda) & D_q(\lambda) \end{bmatrix}, \quad \textit{and} \quad G(\lambda) := \begin{bmatrix} D_z(\lambda) & y_z(\lambda) \\ x_z(\lambda) & c(\lambda) \end{bmatrix},$$

are GFPR associated with $Q(\lambda) := \lambda^{h+1}A_{h+1} + \lambda^h A_h + \cdots + \lambda A_1 + A_0$ *and* $Z(\lambda) := \lambda^{k-h}A_k + \lambda^{k-h-1}A_{k-1} + \cdots + \lambda A_{h+1} + A_h$*, respectively. More precisely, we have*

$$F(\lambda) = M_{\boldsymbol{\ell}_q}(\mathcal{X})(\lambda M^Q_{-h-1} - M^Q_{\mathbf{q}})M_{\mathbf{r}_q}(\mathcal{Y})$$

and

$$G(\lambda) = M_{\boldsymbol{\ell}_z}(\mathcal{Z})(\lambda M^Z_{h+\mathbf{z}} - M^Z_0)M_{\mathbf{r}_z}(\mathcal{W}).$$

5. Matrix pencils block-permutationally equivalent to extended block Kronecker pencils. As we have mentioned before, the main goal of the paper is to prove that almost all Fiedler-like pencils are block-permutationally equivalent to an extended block Kronecker pencil. To help us with this task, we introduce in this section some useful notation and some technical results concerning pencils that are block-permutationally equivalent to block Kronecker pencils or to extended block Kronecker pencils.

DEFINITION 5.1. *Let* $L(\lambda) \in \mathbb{F}[\lambda]^{kn\times kn}$ *be a* $k \times k$ *block-pencil with block-entries of size* $n \times n$*. Assume that there exist block-permutation matrices* $\Pi^n_{\boldsymbol{\ell}}$ *and* $\Pi^n_{\mathbf{r}}$ *such that* $C(\lambda) := (\Pi^n_{\boldsymbol{\ell}})^{\mathcal{B}} L(\lambda)\Pi^n_{\mathbf{r}}$ *is an extended* (p, n, q, n)*-block Kronecker pencil with* $k = p+q+1$*, partitioned as in* (3.4)*. We call* $M(\lambda)$ *the* body of $L(\lambda)$ *relative to* $(\boldsymbol{\ell}, \mathbf{r}, p, q)$*, and we call the* body block-rows *(resp.* body block-columns*) of* $L(\lambda)$ *relative to* $(\boldsymbol{\ell}, \mathbf{r}, p, q)$ *the block-rows (resp. block-columns) of* $L(\lambda)$ *that, after the permutations, occupy the first* $q+1$ *(resp.* $p+1$*) block-rows (resp. block-columns) of* $C(\lambda)$*. Additionally, we call the* wing block-rows *(resp.* wing block-columns*) of* $L(\lambda)$ *relative to* $(\boldsymbol{\ell}, \mathbf{r}, p, q)$ *the block-rows (resp. block-columns) of* $L(\lambda)$ *that are not body block-rows (resp. body block-columns) relative to* $(\boldsymbol{\ell}, \mathbf{r}, p, q)$*.*

In the following example we illustrate all the notions introduced in Definition 5.1.

EXAMPLE 5.2. *Let us consider again the proper GFP associated with the polynomial* $P(\lambda) = \sum_{i=0}^{6} A_i\lambda^i \in \mathbb{F}[\lambda]^{n\times n}$ *considered in Section 3.4, that is, the pencil*

$$K_{\mathbf{q},\mathbf{z}}(\lambda) = \begin{bmatrix} -I_n & \lambda A_6 & 0 & 0 & 0 & 0 \\ \lambda I_n & \lambda A_5 + A_4 & -I_n & 0 & 0 & 0 \\ 0 & A_3 & \lambda I_n & A_2 & -I_n & 0 \\ 0 & -I_n & 0 & \lambda I_n & 0 & 0 \\ 0 & 0 & 0 & -I_n & 0 & \lambda I_n \\ 0 & 0 & 0 & 0 & \lambda I_n & \lambda A_1 + A_0 \end{bmatrix}.$$

As we showed, we can transform $K_{\mathbf{q},\mathbf{z}}(\lambda)$ *into an (extended) block Kronecker pencil via block-row and block-column permutations, denoted by* $\Pi^n_{\boldsymbol{\ell}_2}$ *and* $\Pi^n_{\mathbf{r}_2}$*, to obtain*

$$(\Pi^n_{\boldsymbol{\ell}_2})^{\mathcal{B}} K_{\mathbf{q},\mathbf{z}}(\lambda)\Pi^n_{\mathbf{r}_2} = \left[\begin{array}{ccc|ccc} \lambda A_6 & 0 & 0 & -I_n & 0 & 0 \\ \lambda A_5 + A_4 & 0 & 0 & \lambda I_n & -I_n & 0 \\ A_3 & A_2 & 0 & 0 & \lambda I_n & -I_n \\ 0 & 0 & \lambda A_1 + A_0 & 0 & 0 & \lambda I_n \\ \hline -I_n & \lambda I_n & 0 & 0 & 0 & 0 \\ 0 & -I_n & \lambda I_n & 0 & 0 & 0 \end{array}\right],$$

which is a (2,n*,3,*n*)-block Kronecker pencil.*

Note that the first, second, third and sixth block-rows of $K_{\mathbf{q},\mathbf{z}}(\lambda)$ are its body block-rows relative to $(\boldsymbol{\ell}, \mathbf{r}, 2, 3)$, and the second, fourth and sixth block-columns of $K_{\mathbf{q},\mathbf{z}}(\lambda)$ are its body block-columns relative to $(\boldsymbol{\ell}, \mathbf{r}, 2, 3)$. Moreover, the fourth and fifth block-rows of $K_{\mathbf{q},\mathbf{z}}(\lambda)$ are its wing block-rows relative to $(\boldsymbol{\ell}, \mathbf{r}, 2, 3)$, and the first, third, and fifth block-columns of $K_{\mathbf{q},\mathbf{z}}(\lambda)$ are its wing block-columns relative to $(\boldsymbol{\ell}, \mathbf{r}, 2, 3)$.

REMARK 5.3. *Let $L(\lambda) \in \mathbb{F}[\lambda]^{kn\times kn}$ be a matrix pencil block-permutationally equivalent to an extended (p, n, q, n)-block Kronecker pencil with $p + q + 1 = k$, and let us denote by $\Pi^n_{\mathbf{r}}$ and $\Pi^n_{\boldsymbol{\ell}}$ the block-permutation matrices that transform the pencil $L(\lambda)$ in the extended (p, n, q, n)-block Kronecker pencil $(\Pi^n_{\boldsymbol{\ell}})^{\mathcal{B}} L(\lambda)\Pi^n_{\mathbf{r}}$. Let $\{r_1, \ldots, r_p\}$, $\{r_{p+1}, \ldots, r_k\}$, $\{c_1, \ldots, c_q\}$, $\{c_{q+1}, \ldots, c_k\}$ be, respectively, the sets of positions of the wing block-rows, body block-rows, wing block-columns and body block-columns of $L(\lambda)$, all relative to $(\boldsymbol{\ell}, \mathbf{r}, p, q)$, and let $L_1(\lambda) := (\Pi^n_{\boldsymbol{\ell}})^{\mathcal{B}} L(\lambda)$ and $L_2(\lambda) := L(\lambda)\Pi^n_{\mathbf{r}}$. Then, the following simple observations will be used freely.*

(i) *The pencils $L_1(\lambda)$ and $L_2(\lambda)$ are block-permutationally equivalent to the extended (p, n, q, n)-block Kronecker pencil $(\Pi^n_{\boldsymbol{\ell}})^{\mathcal{B}} L(\lambda)\Pi^n_{\mathbf{r}}$.*

(ii) *The sets $\{1 : q + 1\}$ and $\{q + 2 : k\}$ are, respectively, the sets of positions of the body block-rows and the wing block-rows of $L_1(\lambda)$, both relative to $(\mathbf{id}, \mathbf{r}, p, q)$.*

(iii) *The sets $\{c_1, \ldots, c_q\}$, $\{c_{q+1}, \ldots, c_k\}$ are, respectively, the sets of positions of the wing block-columns and the body block-columns of $L_1(\lambda)$, both relative to $(\mathbf{id}, \mathbf{r}, p, q)$.*

(iv) *The sets $\{1 : p+1\}$ and $\{p+2 : k\}$ are, respectively, the sets of positions of the body block-columns and the wing block-columns of $L_2(\lambda)$, both relative to $(\boldsymbol{\ell}, \mathbf{id}, p, q)$.*

(v) *The sets $\{r_1, \ldots, r_p\}$, $\{r_{p+1}, \ldots, r_k\}$ are, respectively, the sets of positions of the wing block-rows and the body block-rows of $L_2(\lambda)$, both relative to $(\boldsymbol{\ell}, \mathbf{id}, p, q)$.*

The following two lemmas will be useful for the proofs of the main results of this paper. Their simple proofs are omitted.

LEMMA 5.4. *Let $L_1(\lambda), L_2(\lambda) \in \mathbb{F}[\lambda]^{kn\times kn}$ be matrix pencils, and assume that there exist block-permutation matrices $\Pi^n_{\mathbf{r}}$, $\Pi^n_{\boldsymbol{\ell}_1}$ and $\Pi^n_{\boldsymbol{\ell}_2}$ such that $(\Pi^n_{\boldsymbol{\ell}_1})^{\mathcal{B}} L_1(\lambda)\Pi^n_{\mathbf{r}}$ and $(\Pi^n_{\boldsymbol{\ell}_2})^{\mathcal{B}} L_2(\lambda)\Pi^n_{\mathbf{r}}$ are both extended (p, n, q, n)-block Kronecker pencils. Then, the sets of positions of the body block-columns and the wing block-columns of $L_1(\lambda)$ relative to $(\boldsymbol{\ell}_1, \mathbf{r}, p, q)$ equal, respectively, the sets of positions of the body block-columns and the wing block-columns of $L_2(\lambda)$ relative to $(\boldsymbol{\ell}_2, \mathbf{r}, p, q)$.*

LEMMA 5.5. *Let $L_1(\lambda), L_2(\lambda) \in \mathbb{F}[\lambda]^{kn\times kn}$ be matrix pencils, and assume that there exist block-permutation matrices $\Pi^n_{\mathbf{r}_1}$, $\Pi^n_{\mathbf{r}_2}$ and $\Pi^n_{\boldsymbol{\ell}}$ such that $(\Pi^n_{\boldsymbol{\ell}})^{\mathcal{B}} L_1(\lambda)\Pi^n_{\mathbf{r}_1}$ and $(\Pi^n_{\boldsymbol{\ell}})^{\mathcal{B}} L_2(\lambda)\Pi^n_{\mathbf{r}_2}$ are both extended (p, n, q, n)-block Kronecker pencils. Then, the sets of positions of the body block-rows and the wing block-rows of $L_1(\lambda)$ relative to $(\boldsymbol{\ell}, \mathbf{r}_1, p, q)$ equal, respectively, the sets of positions of the body block-rows and the wing block-rows of $L_2(\lambda)$ relative to $(\boldsymbol{\ell}, \mathbf{r}_2, p, q)$.*

In the next two subsections we include two technical lemmas that show how the multiplication of pencils block-permutationally equivalent to an (extended) block Kronecker pencil by elementary matrices, in certain situations, produce new pencils permutationally equivalent to an extended block Kronecker pencil.

5.1. Product by elementary matrices associated with negative indices. The proof of the main result for GFP (Theorem 3.11 or Theorem 7.1) requires one key auxiliary result. This result is Lemma 5.4, which studies the effect of left-multiplications by elementary matrices associated with negative indices in some relevant situations.

LEMMA 5.6. *Let $P(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ be a matrix polynomial of grade k, let $L(\lambda) \in \mathbb{F}[\lambda]^{kn\times kn}$ be a pencil block-permutationally equivalent to a (p, n, q, n)-block Kronecker*

pencil with $p+q+1=k$, i.e., there exist block-permutation matrices $\Pi_{\boldsymbol{\ell}}^n, \Pi_{\mathbf{r}}^n$ such that

$$(\Pi_{\boldsymbol{\ell}}^n)^{\mathcal{B}} L(\lambda) \Pi_{\mathbf{r}}^n = \left[\begin{array}{c|c} M(\lambda) & L_q(\lambda)^T \otimes I_n \\ \hline L_p(\lambda) \otimes I_n & 0 \end{array}\right].$$

Assume that $M(\lambda)$ satisfies the AS condition for $P(\lambda)$. Let $\{r_1, \dots, r_p\}$, $\{r_{p+1}, \dots, r_k\}$, $\{c_1, \dots, c_q\}$, $\{c_{q+1}, \dots, c_k\}$ be, respectively, the sets of positions of the wing block-rows, the body block-rows, the wing block-columns and the body block-columns of $L(\lambda)$, all relative to $(\boldsymbol{\ell}, \mathbf{r}, p, q)$. Let $x \in \{1 : k-1\}$ be such that $k-x+1 \in \{r_1, \dots, r_p\}$ and $k-x \in \{r_{p+1}, \dots, r_k\}$. Define $\widetilde{L}(\lambda) := M_{-x}(X) L(\lambda)$, where $X \in \mathbb{F}^{n \times n}$. Then, the following statements hold.

(i) *There exists a block-permutation matrix $\Pi_{\widetilde{\boldsymbol{\ell}}}^n$ such that*

$$(\Pi_{\widetilde{\boldsymbol{\ell}}}^n)^{\mathcal{B}} \widetilde{L}(\lambda) \Pi_{\mathbf{r}}^n = \left[\begin{array}{c|c} \widetilde{M}(\lambda) & L_q(\lambda)^T \otimes I_n \\ \hline L_p(\lambda) \otimes I_n & 0 \end{array}\right] \tag{5.1}$$

is an extended (p, n, q, n)-block Kronecker pencil whose body $\widetilde{M}(\lambda)$ satisfies the AS condition for $P(\lambda)$.

(ii.a) *For $j = 1 : k$, if $j \neq k-x, k-x+1$, then the jth block-row of $\widetilde{L}(\lambda)$ is equal to the jth block-row of $L(\lambda)$.*

(ii.b) *The $(k-x)$th block-row of $\widetilde{L}(\lambda)$ is equal to the $(k-x+1)$th block-row of $L(\lambda)$.*

(ii.c) *The set $(\{r_1, \dots, r_p\} \setminus \{k-x+1\}) \cup \{k-x\}$ is the set of positions of the wing block-rows of $\widetilde{L}(\lambda)$ relative to $(\widetilde{\boldsymbol{\ell}}, \mathbf{r}, p, q)$.*

(ii.d) *The set $(\{r_{p+1}, \dots, r_k\} \setminus \{k-x\}) \cup \{k-x+1\}$ is the set of positions of the body block-rows of $\widetilde{L}(\lambda)$ relative to $(\widetilde{\boldsymbol{\ell}}, \mathbf{r}, p, q)$.*

(iii.a) *The set $\{c_1, \dots, c_q\}$ is the set of positions of the wing block-columns of $\widetilde{L}(\lambda)$ relative to $(\widetilde{\boldsymbol{\ell}}, \mathbf{r}, p, q)$.*

(iii.b) *The set $\{c_{q+1}, \dots, c_k\}$ is the set of positions of the body block-columns of $\widetilde{L}(\lambda)$ relative to $(\widetilde{\boldsymbol{\ell}}, \mathbf{r}, p, q)$.*

Proof. Notice that the product on the left by the matrix $M_{-x}(X)$ only affects the $(k-x)$th and the $(k-x+1)$th block-rows of $L(\lambda)\Pi_{\mathbf{r}}^n$. Moreover, since the $(k-x+1)$th block-row of $L(\lambda)$ is a wing block-row relative to $(\boldsymbol{\ell}, \mathbf{r}, p, q)$, the submatrix of $L(\lambda)\Pi_{\mathbf{r}}^n$ consisting of its $(k-x)$th and $(k-x+1)$th block-rows is of the form

$$\begin{bmatrix} R_1(\lambda) & R_2(\lambda) & R_3(\lambda) & R_4(\lambda) & R_5(\lambda) \\ 0 & -I_n & \lambda I_n & 0 & 0 \end{bmatrix},$$

for some matrix pencils $R_1(\lambda)$, $R_2(\lambda)$, $R_3(\lambda)$, $R_4(\lambda)$ and $R_5(\lambda)$, where some of the block-columns containing a zero block other than the last one may not be present. This, in turn, implies that the submatrix of $M_{-x}(X)L(\lambda)\Pi_{\mathbf{r}}^n$ formed by its $(k-x)$th and $(k-x+1)$th block-rows is of the form

$$\begin{bmatrix} 0 & -I_n & \lambda I_n & 0 & 0 \\ R_1(\lambda) & R_2(\lambda) - X & R_3(\lambda) - \lambda X & R_4(\lambda) & R_5(\lambda) \end{bmatrix}.$$

Therefore, setting

$$\mathbf{t} := (1 : k-x-1, k-x+1, k-x, k-x+2 : k)$$

and introducing the block-permutation matrix $\Pi^n_{\widetilde{\boldsymbol{\ell}}} := \Pi^n_{\mathbf{t}}\Pi^n_{\boldsymbol{\ell}}$, it is clear that (5.1) holds for some pencil $\widetilde{M}(\lambda)$. Furthermore, note that

$$\widetilde{M}(\lambda) = M(\lambda) + (e_t \otimes I_n)\begin{bmatrix} 0 & -I_n & \lambda I_n & 0 & 0 \end{bmatrix},$$

for some $t \in \{0 : q+1\}$. Thus, the fact that the body of $\widetilde{L}(\lambda)$ satisfies the AS condition for $P(\lambda)$ follows from Lemma 3.10. Therefore, part (i) is true. Parts (ii.a), (ii.b), (ii.c) and (ii.d) follows from applying Lemma 5.3 to $L(\lambda)$ and $(\Pi^n_{\mathbf{t}})^{\mathcal{B}}L(\lambda)$, together with the simple fact that left-multiplication by $M_{-x}(X)$ only affects the $(k-x)$th and $(k-x+1)$th block-rows of $L(\lambda)$, and replaces the $(k-x)$th block-row by the $(k-x+1)$th block-row. Finally, parts (iii.a) and (iii.b) are direct consequences of Lemma 5.2. □

5.2. Product by elementary matrices associated with nonnegative indices. The proof of the main result for GFPR (Theorem 3.12 or Theorem 8.1) requires one key auxiliary result. This result is Lemma 5.5, which studies the effect of right-multiplications by elementary matrices associated with nonnegative indices in some relevant situations.

LEMMA 5.7. *Let $P(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ be a matrix polynomial of grade k, let $L(\lambda) \in \mathbb{F}[\lambda]^{kn\times kn}$ be a pencil block-permutationally equivalent to an extended (p,n,q,n)-block Kronecker pencil with $p+q+1=k$, i.e., there exist block-permutation matrices $\Pi^n_{\boldsymbol{\ell}}, \Pi^n_{\mathbf{r}}$ such that*

$$(\Pi^n_{\boldsymbol{\ell}})^{\mathcal{B}}L(\lambda)\Pi^n_{\mathbf{r}} = \left[\begin{array}{c|c} M(\lambda) & K_2(\lambda)^T \\ \hline K_1(\lambda) & 0 \end{array}\right],$$

where $K_1(\lambda)$ and $K_2(\lambda)$ are wing pencils, and assume that $M(\lambda)$ satisfies the AS condition for $P(\lambda)$. Let $\{r_1,\dots,r_p\}$, $\{r_{p+1},\dots,r_k\}$, $\{c_1,\dots,c_q\}$ and $\{c_{q+1},\dots,c_k\}$ be, respectively, the sets of positions of the wing block-rows, the body block-rows, the wing block-columns and the body block-columns of $L(\lambda)$, all relative to $(\boldsymbol{\ell},\mathbf{r},p,q)$. Assume, additionally, that x is an index such that $k-x \in \{c_1,\dots,c_q\}$. Define $\widetilde{L}(\lambda) := L(\lambda)M_x(X)$, where $X \in \mathbb{F}^{n\times n}$. Then, the following statements hold.

(i) *There exists a block-permutation matrix $\Pi^n_{\widetilde{\mathbf{r}}}$ such that*

$$(\Pi^n_{\boldsymbol{\ell}})^{\mathcal{B}}\widetilde{L}(\lambda)\Pi^n_{\widetilde{\mathbf{r}}} = \left[\begin{array}{c|c} \widetilde{M}(\lambda) & \widetilde{K}_2(\lambda)^T \\ \hline K_1(\lambda) & 0 \end{array}\right] \tag{5.2}$$

is an extended (p,n,q,n)-block Kronecker pencil whose body $\widetilde{M}(\lambda)$ satisfies the AS condition for $P(\lambda)$.

(ii.a) *If $x=0$, the kth block-column of $\widetilde{L}(\lambda)$ is a wing block-column of $\widetilde{L}(\lambda)$ relative to $(\boldsymbol{\ell},\widetilde{\mathbf{r}},p,q)$.*

(ii.b) *If $x\neq 0$, then the $(k-x+1)$th block-column of $\widetilde{L}(\lambda)$ is a wing block-column of $\widetilde{L}(\lambda)$ relative to $(\boldsymbol{\ell},\widetilde{\mathbf{r}},p,q)$, and it is equal to the $(k-x)$th block-column of $L(\lambda)$.*

(ii.c) *If $x\neq 0$, then the $(k-x)$th block-column of $\widetilde{L}(\lambda)$ is a wing block-column of $\widetilde{L}(\lambda)$ relative to $(\boldsymbol{\ell},\widetilde{\mathbf{r}},p,q)$ if and only if the $(k-x+1)$th block-column of $L(\lambda)$ is a wing block-column of $L(\lambda)$ relative to $(\boldsymbol{\ell},\mathbf{r},p,q)$.*

(iii.a) *For $j=1:q$, if $c_j \neq k-x, k-x+1$, then the c_j-th block-column of $\widetilde{L}(\lambda)$ is a wing block-column of $\widetilde{L}(\lambda)$ relative to $(\boldsymbol{\ell},\widetilde{\mathbf{r}},p,q)$, and it is equal to the c_j-th block-column of $L(\lambda)$.*

(iii.b) *For $j=q+1:k$, if $c_j \neq k-x+1$, then the c_j-th block-column of $\widetilde{L}(\lambda)$ is a body block-column of $\widetilde{L}(\lambda)$ relative to $(\boldsymbol{\ell},\widetilde{\mathbf{r}},p,q)$.*

(iv.a) *The set* $\{r_1, \dots, r_p\}$ *is the set of positions of the wing block-rows of* $\widetilde{L}(\lambda)$ *relative to* $(\boldsymbol{\ell}, \widetilde{\mathbf{r}}, p, q)$.

(iv.b) *The set* $\{r_{p+1}, \dots, r_k\}$ *is the set of positions of the body block-rows of* $\widetilde{L}(\lambda)$ *relative to* $(\boldsymbol{\ell}, \widetilde{\mathbf{r}}, p, q)$.

Proof. First, let us assume that $x = 0$. By the hypotheses of the theorem, the kth block-column of $L(\lambda)$ is a wing block-column relative to $(\boldsymbol{\ell}, \mathbf{r}, p, q)$. This implies that the wing block-rows of $L(\lambda)$ and the block-columns of $L(\lambda)$ other than its kth block-column are not changed after the right-multiplication of $L(\lambda)$ by $M_0(X)$. Thus, setting $\Pi^n_{\widetilde{\mathbf{r}}} = \Pi^n_{\mathbf{r}}$, we obtain that (5.2) holds with $\widetilde{M}(\lambda) = M(\lambda)$, for some matrix pencil $\widetilde{K}_2(\lambda) \in \mathbb{F}[\lambda]^{nq \times n(q+1)}$. Moreover, we claim that $\widetilde{K}_2(\lambda)$ is a wing pencil. This claim follows from the fact that $\widetilde{K}_2(\lambda)$ is obtained by multiplying $K_2(\lambda)$ by a block diagonal matrix whose diagonal blocks are I_n except for the last one, which is equal to X, together with Proposition 2.6. Thus, part (i) is true when $x = 0$. Part (ii.a) also follows immediately. Parts (iii.a) and (iii.b) follow from Lemma 5.2 and the fact that the block-columns of $L(\lambda)$ and $\widetilde{L}(\lambda)$, other than their kth block-columns, are equal. Finally, parts (iv.a) and (iv.b) follow from Lemma 5.3.

Let us assume, now, that $x \neq 0$. By the hypotheses of the theorem, the $(k-x)$th block-column of $L(\lambda)$ is a wing block-column relative to $(\boldsymbol{\ell}, \mathbf{r}, p, q)$. Notice that the right-multiplication of $L(\lambda)$ by the matrix $M_x(X)$ only affects the $(k-x)$th and the $(k-x+1)$th block-columns of $L(\lambda)$. Then, to prove all the results, we have to distinguish two cases.

Case I: Assume that the $(k-x+1)$th block-column of $L(\lambda)$ is a body block-column of $L(\lambda)$ relative to $(\boldsymbol{\ell}, \mathbf{r}, p, q)$. Thus, the $(k-x)$th and the $(k-x+1)$th block-columns of $(\Pi^n_{\boldsymbol{\ell}})^{\mathcal{B}} L(\lambda)$ are, respectively, of the form $\begin{bmatrix} R(\lambda)^T & 0 \end{bmatrix}^T$ and $\begin{bmatrix} B(\lambda)^T & C(\lambda)^T \end{bmatrix}^T$, with $B(\lambda), R(\lambda) \in \mathbb{F}[\lambda]^{n(q+1)\times n}$ and $C(\lambda) \in \mathbb{F}[\lambda]^{np \times n}$. Then, set

$$\mathbf{t} := (1 : k-x-1, k-x+1, k-x, k-x+2 : k),$$

and let $\widehat{L}(\lambda) := \widetilde{L}(\lambda)\Pi^n_{\mathbf{t}}$. Notice that the submatrices of $(\Pi^n_{\boldsymbol{\ell}})^{\mathcal{B}} L(\lambda)$, $(\Pi^n_{\boldsymbol{\ell}})^{\mathcal{B}} \widetilde{L}(\lambda)$ and $(\Pi^n_{\boldsymbol{\ell}})^{\mathcal{B}} \widehat{L}(\lambda)$ formed by their $(k-x)$th and $(k-x+1)$th block-columns are, respectively, given by

$$\begin{bmatrix} R(\lambda) & B(\lambda) \\ 0 & C(\lambda) \end{bmatrix}, \quad \begin{bmatrix} R(\lambda) & B(\lambda) \\ 0 & C(\lambda) \end{bmatrix} \begin{bmatrix} X & I_n \\ I_n & 0 \end{bmatrix} = \begin{bmatrix} R(\lambda)X + B(\lambda) & R(\lambda) \\ C(\lambda) & 0 \end{bmatrix}, \quad \text{and}$$
$$\begin{bmatrix} R(\lambda) & B(\lambda) \\ 0 & C(\lambda) \end{bmatrix} \begin{bmatrix} X & I_n \\ I_n & 0 \end{bmatrix} \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} = \begin{bmatrix} R(\lambda) & R(\lambda)X + B(\lambda) \\ 0 & C(\lambda) \end{bmatrix}.$$

This implies that all the wing block-columns and the wing block-rows of $L(\lambda)$ relative to $(\boldsymbol{\ell}, \mathbf{r}, p, q)$ remain unchanged after the right-multiplication by $M_x(X)\Pi^n_{\mathbf{t}}$. Therefore, setting $\Pi^n_{\widetilde{\mathbf{r}}} := \Pi^n_{\mathbf{t}}\Pi^n_{\mathbf{r}}$, we obtain that (5.2) holds with $\widetilde{K}_2(\lambda) = K_2(\lambda)$ and $\widetilde{M}(\lambda) = M(\lambda) + R(\lambda)X(e_t^T \otimes I_n)$, for some $t \in \{1 : p+1\}$, where e_t denotes the tth column of the $(p+1) \times (p+1)$ identity matrix. Note that $(\Lambda_q(\lambda) \otimes I_n)R(\lambda)X = 0$ since $R(\lambda)$ is a block-column of the wing pencil $K_2(\lambda)^T$. Thus, from Lemma 3.10, we obtain that the pencil $\widetilde{M}(\lambda)$ satisfies the AS condition for $P(\lambda)$, and part (i) follows. Parts (ii.b), (iii.a) and (iii.b) follow from Lemma 5.2 applied to $L(\lambda)$ and $\widehat{L}(\lambda)$, together with the simple fact that the only difference between $\widehat{L}(\lambda)$ and $\widetilde{L}(\lambda)$ is that their $(k-x)$th and $(k-x+1)$th block-columns are permuted. Parts (iv.a) and (iv.b) follow from Lemma 5.3 applied to $L(\lambda)$ and $\widetilde{L}(\lambda)$.

Case II: Assume that the $(k-x+1)$th block-column of $L(\lambda)$ is a wing block-column relative to $(\boldsymbol{\ell}, \mathbf{r}, p, q)$. Thus, the $(k-x)$th and the $(k-x+1)$th block-columns of $(\Pi^n_{\boldsymbol{\ell}})^{\mathcal{B}} L(\lambda)$ are, respectively, of the form $\begin{bmatrix} R_1(\lambda)^T & 0 \end{bmatrix}^T$ and $\begin{bmatrix} R_2(\lambda)^T & 0 \end{bmatrix}^T$, with $R_1(\lambda), R_2(\lambda) \in$

$\mathbb{F}[\lambda]^{n(q+1)\times n}$. Then, notice that the submatrix formed by the $(k-x)$th and the $(k-x+1)$th block-columns of the pencil $(\Pi_{\boldsymbol{\ell}}^n)^{\mathcal{B}} L(\lambda) M_x(X)$ is given by

$$\begin{bmatrix} R_1(\lambda) & R_2(\lambda) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X & I_n \\ I_n & 0 \end{bmatrix} = \begin{bmatrix} R_1(\lambda)X + R_2(\lambda) & R_1(\lambda) \\ 0 & 0 \end{bmatrix}.$$

Therefore, setting $\Pi_{\widetilde{\mathbf{r}}}^n = \Pi_{\mathbf{r}}^n$, we obtain that (5.2) holds with $\widetilde{M}(\lambda) = M(\lambda)$ and $\widetilde{K}_2(\lambda)^T = K_2(\lambda)^T + R_1(\lambda)X(e_t^T \otimes I_n)$, for some $t \in \{1:q\}$, where e_t denotes the tth column of the $q \times q$ identity matrix. Since $\widetilde{K}_2(\lambda)$ can be written as $BK_2(\lambda)$, for some nonsingular matrix B, by Corollary 2.7, $\widetilde{K}_2(\lambda)$ is a wing pencil, and part (i) follows. Note that the body of $\widetilde{L}(\lambda)$ relative to $(\boldsymbol{\ell}, \mathbf{r}, p, q)$ and the body of $L(\lambda)$ relative to $(\boldsymbol{\ell}, \mathbf{r}, p, q)$ are the same, and therefore, the body of $\widetilde{L}(\lambda)$ satisfies the AS condition for $P(\lambda)$. Parts (ii.b), (iii.a) and (iii.b) follow from Lemma 5.2 applied to $L(\lambda)$ and $\widetilde{L}(\lambda)$, together with the simple fact that the $(k-x)$th and the $(k-x+1)$th block-columns of $L(\lambda)$ and $\widetilde{L}(\lambda)$ are permuted. Parts (iv.a) and (iv.b) follow immediately from Lemma 5.3.

Finally, notice that in the proofs of **Case I** and **Case II**, we have shown that the $(k-x)$th block-column of $\widetilde{L}(\lambda)$ is a wing block-column (resp. a body block-column) of $\widetilde{L}(\lambda)$ relative to $(\boldsymbol{\ell}, \widetilde{\mathbf{r}}, p, q)$ whenever the $(k-x+1)$th block-column of $L(\lambda)$ is a wing block-column (resp. a body block-column) of $L(\lambda)$ relative to $(\boldsymbol{\ell}, \mathbf{r}, p, q)$. Thus, part (ii.c) holds. □

6. Fiedler pencils as block Kronecker pencils. In [18, Theorem 4.5] it was proven that all Fiedler pencils associated with a matrix polynomial $P(\lambda)$ are block Kronecker pencils with bodies satisfying the AS condition for $P(\lambda)$, after permuting some of its block-rows and block-columns. The main theorem of this section (Theorem 6.3) gives a more detailed description of this block Kronecker form for Fiedler pencils that will be useful to show that almost all Fiedler-like pencils associated with $P(\lambda)$ are extended block Kronecker pencils with bodies satisfying the AS condition for $P(\lambda)$, modulo permutations. However, we warn the reader that Theorem 6.3 is only valid when $n = m$, in contrast to [18, Theorem 4.5] which is valid also when $n \neq m$.

We start by relating the notation used in [13] to work with and construct Fiedler pencils with the index tuple notation introduced in Section 4. To this end, we recall the notion of consecutions and inversions of a permutation $\mathbf{q}$ of the set $\{0:k-1\}$.

DEFINITION 6.1. *Let k be a positive integer and let $\mathbf{q}$ be a permutation of the set $\{0:k-1\}$. For $i \in \{0:k-2\}$, we say that $\mathbf{q}$ has a* consecution *at i if i occurs to the left of $i+1$ in $\mathbf{q}$, and that $\mathbf{q}$ has an* inversion *at i if i occurs to the right of $i+1$.*

Next, we relate the consecutions and inverstions of a permutation $\mathbf{q}$ of $\{0:k-1\}$ with the sets $\mathrm{heads}(\mathbf{q})$ and $\mathrm{heads}(\mathrm{rev}(\mathbf{q}))$ (recall that $\mathfrak{h}(\mathbf{t})$ denotes the cardinality of the set $\mathrm{heads}(\mathbf{t})$). In the proof of this lemma and in the rest of the section, we use the following notation. We denote by $\mathfrak{i}(\mathbf{q})$ and $\mathfrak{c}(\mathbf{q})$ the total number of inversions and consecutions of $\mathbf{q}$, respectively. We also denote by $\mathfrak{C}_q$ and $\mathfrak{I}_q$ the set of indices at which the permutation $\mathbf{q}$ has a consecution and an inversion, respectively.

LEMMA 6.2. *Let $\mathbf{q}$ be a permutation of $\{0:k-1\}$. Then,*

(i) $\mathfrak{i}(\mathbf{q}) = \mathfrak{h}(\mathbf{q}) - 1$ *and* $\mathfrak{c}(\mathbf{q}) = \mathfrak{h}(\mathrm{rev}(\mathbf{q})) - 1$*;*

(ii) $k - \mathfrak{C}_q = \{k-h : h \notin \mathrm{heads}(\mathbf{q})\}$ *and* $k - \mathfrak{I}_q = \{k-h : h \notin \mathrm{heads}(\mathrm{rev}(\mathbf{q}))\}$;

(iii) $\mathfrak{I}_q \cup \mathfrak{C}_q = \{0:k-2\}$ *and* $\mathfrak{I}_q \cap \mathfrak{C}_q = \emptyset$.

Proof. Note that a permutation $\mathbf{q}$ of the set $\{0:k-1\}$ has an inversion at an index i if and only if $i \neq k-1$ and $i \in \mathrm{heads}(\mathbf{q})$; and $\mathbf{q}$ has a consecution at i if and only if $i \neq k-1$ and $i \notin \mathrm{heads}(\mathbf{q})$, or equivalently, if $i \neq k-1$ and $i \in \mathrm{heads}(\mathrm{rev}(\mathbf{q}))$. Thus, taking into account that $k-1 \in \mathrm{heads}(\mathbf{q}) \cap \mathrm{heads}(\mathrm{rev}(\mathbf{q}))$, we get parts (i) and (ii). Moreover, part

(iii) follows since $\mathbf{q}$ cannot have a consecution and an inversion at the same index and $\mathbf{q}$ has neither a consecution nor an inversion at $k-1$. □

Theorem 6.3, the main result of this section, recalls that, modulo block-permutations, FP are block Kronecker pencils, and identify the body block-rows, the body block-columns, the wing block-rows and the wing block-columns of any FP relative to some block-permutations that transform it into a block Kronecker pencil.

THEOREM 6.3. *Let $P(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ be a matrix polynomial of degree k as in* (2.1)*. Let $\mathbf{q}$ be a permutation of $\{0:k-1\}$ and let*

$$F_{\mathbf{q}}(\lambda) = \lambda M^P_{-k} - M^P_{\mathbf{q}}$$

be the Fiedler pencil associated with $P(\lambda)$ and $\mathbf{q}$. Then, there exist block-permutation matrices $\Pi^n_{\mathbf{r}}$ and $\Pi^n_{\boldsymbol{\ell}}$ such that

$$C(\lambda) := (\Pi^n_{\boldsymbol{\ell}})^{\mathcal{B}} F_{\mathbf{q}}(\lambda)\Pi^n_{\mathbf{r}} = \left[\begin{array}{c|c} M(\lambda) & L_{\mathfrak{h}(\mathrm{rev}(\mathbf{q}))-1}(\lambda)^T \otimes I_n \\ \hline L_{\mathfrak{h}(\mathbf{q})-1}(\lambda)\otimes I_n & 0 \end{array}\right], \qquad (6.1)$$

is a $(\mathfrak{h}(\mathbf{q})-1, n, \mathfrak{h}(\mathrm{rev}(\mathbf{q}))-1, n)$-block Kronecker pencil with body $M(\lambda)$ satisfying the AS condition for $P(\lambda)$. Moreover, the following statements hold.

(a) *The wing block-columns of $(\Pi^n_{\boldsymbol{\ell}})^{\mathcal{B}} F_{\mathbf{q}}(\lambda)$ relative to $(\mathbf{id}, \mathbf{r}, \mathfrak{h}(\mathbf{q})-1, \mathfrak{h}(\mathrm{rev}(\mathbf{q}))-1)$ are of the form $-e_i\otimes I_n + \lambda e_{i+1}\otimes I_n$, for $1\le i\le \mathfrak{h}(\mathrm{rev}(\mathbf{q}))-1$, and are located in positions $k-j$, where $j\in\{0:k-2\}$ and $j\notin \mathrm{heads}(\mathbf{q})$, or, equivalently, $j\in\{0:k-2\}$ and $(\mathbf{q}, j)$ satisfies the SIP.*

(b) *The wing block-rows of $F_{\mathbf{q}}(\lambda)\Pi^n_{\mathbf{r}}$ relative to $(\boldsymbol{\ell}, \mathbf{id}, \mathfrak{h}(\mathbf{q})-1, \mathfrak{h}(\mathrm{rev}(\mathbf{q}))-1)$ are of the form $-e_i^T\otimes I_n + \lambda e_{i+1}^T\otimes I_n$, for $1\le i<\mathfrak{h}(\mathbf{q})-1$, and are located in positions $k-j$, where $j\in\{0:k-2\}$ and $j\notin \mathrm{heads}(\mathrm{rev}(\mathbf{q}))$, or, equivalently, $j\in\{0:k-2\}$ and $(j,\mathbf{q})$ satisfies the SIP.*

(c) *The first block-row and the first block-column of $F_{\mathbf{q}}(\lambda)$ are, respectively, the first body block-row and the first body block-column of $F_{\mathbf{q}}(\lambda)$ relative to $(\boldsymbol{\ell}, \mathbf{r}, \mathfrak{h}(\mathbf{q})-1, \mathfrak{h}(\mathrm{rev}(\mathbf{q}))-1)$. Moreover, the block-entry of $M(\lambda)$ in position $(1,1)$ equals $\lambda A_k + A_{k-1}$.*

Proof. Let $p := \mathfrak{h}(\mathbf{q})-1$ and $q := \mathfrak{h}(\mathrm{rev}(\mathbf{q}))-1$. By [18, Theorem 4.5], and taking into account part (i) in Lemma 6.2, there exist block-permutation matrices $\Pi^n_{\boldsymbol{\ell}}$ and $\Pi^n_{\mathbf{r}}$ such that (6.1) holds with $M(\lambda)$ with the so-called staircase pattern for $\lambda A_k + A_{k-1}, A_{k-2}, \ldots, A_0$ (see Definition 4.3 and Theorem 4.5 in [18]). Thus, the FP $F_{\mathbf{q}}(\lambda)$ is block-permutationally equivalent to a (p,n,q,n)-block Kronecker pencil. Furthermore, it is not difficult to check that the staircase pattern of $M(\lambda)$ implies that it satisfies the AS condition for $P(\lambda)$.

We now prove parts (a) and (b). It is clear, by (6.1), that the wing block-columns of $(\Pi^n_{\boldsymbol{\ell}})^{\mathcal{B}} F_{\mathbf{q}}(\lambda)$ relative to $(\mathbf{id}, \mathbf{r}, p, q)$ are of the form $-e_i\otimes I_n + \lambda e_{i+1}\otimes I_n$ for some $i\le q$, and that the wing block-rows of $F_{\mathbf{q}}(\lambda)\Pi^n_{\mathbf{r}}$ relative to $(\boldsymbol{\ell}, \mathbf{id}, p, q)$ are of the form $-e_i^T\otimes I_n + \lambda e_{i+1}^T\otimes I_n$, for some $i\le p$. This implies the first claim in parts (a) and (b).

To prove the second claim in parts (a) and (b), notice that the wing block-columns of $F_{\mathbf{q}}(\lambda)$ relative to $(\boldsymbol{\ell}, \mathbf{r}, p, q)$ and the wing block-columns of $(\Pi^n_{\boldsymbol{\ell}})^{\mathcal{B}} F_{\mathbf{q}}(\lambda)$ relative to $(\mathbf{id}, \mathbf{r}, p, q)$ are in exactly the same positions. Similarly, the wing block-rows of $F_{\mathbf{q}}(\lambda)$ relative to $(\boldsymbol{\ell}, \mathbf{r}, p, q)$ and the wing block-rows of $F_{\mathbf{q}}(\lambda)\Pi^n_{\mathbf{r}}$ relative to $(\boldsymbol{\ell}, \mathbf{id}, p, q)$ are located in exactly the same positions (see also Remark 5.1). Thus, to prove the second claim in parts (a) and (b), we just need to identify the wing block-rows and the wing block-columns of $F_{\mathbf{q}}(\lambda)$ relative to $(\boldsymbol{\ell}, \mathbf{r}, p, q)$. By Lemma 4.16, the block-columns (resp. block-rows) of $M^P_{\mathbf{q}}$ of the form $e_i\otimes I_n$ (resp. $e_i^T\otimes I_n$), for some $1\le i\le k$, are precisely those not in positions $k-\mathrm{heads}(\mathbf{q})$ (resp. not in positions $k-\mathrm{heads}(\mathrm{rev}(\mathbf{q}))$ or, equivalently, by part (ii) in Lemma 6.2, those in positions $k-\mathfrak{C}_q$ (resp. $k-\mathfrak{I}_q$). Taking into account part (c) in Lemma 6.2, the set of positions of

the block-columns of the form $e_i \otimes I_n$ and the set of positions of the block-rows of the form $e_i^T \otimes I_n$ are disjoint and their union is $\{2:k\}$. Since $M_{-k}^P = A_k \oplus I_{n(k-1)}$, the blocks of M_{-k}^P in positions (i,i), with $i \in \{2:k\}$, are of the form λI_n. Thus, the block-columns (resp. the block-rows) of $F_{\mathbf{q}}(\lambda)$ of the form $-e_i \otimes I_n + \lambda e_j \otimes I_n$ (resp. $-e_i^T \otimes I_n + \lambda e_j^T \otimes I_n$) for some $i,j \in \{1:k\}$ are precisely those in $k - \mathfrak{C}_q$ (resp. $k - \mathfrak{I}_q$). Hence, the second claim in parts (a) and (b) follows. The equivalent condition for the position of the wing block-rows and the wing block-columns in terms of tuples satisfying the SIP property follows from Lemma 4.13 and the definition of reversal tuple.

To prove part (c), notice that, since $k-1$ is the largest index in $\mathbf{q}$, we have $k - 1 \in \operatorname{heads}(\mathbf{q}) \cap \operatorname{heads}(\operatorname{rev}(\mathbf{q}))$. Therefore, by parts (b) and (c), the first block-row and the first block-column of $F_{\mathbf{q}}(\lambda)$ are not, respectively, a wing block-row of $F_{\mathbf{q}}(\lambda)\Pi_{\mathbf{r}}^n$ relative to $(\boldsymbol{\ell}, \mathbf{id}, p, q)$, or a wing block-column of $(\Pi_{\boldsymbol{\ell}}^n)^{\mathcal{B}} F_{\mathbf{q}}(\lambda)$ relative to $(\mathbf{id}, \mathbf{r}, p, q)$. Since the positions of the wing block-rows (resp. the wing block-columns) of $F_{\mathbf{q}}(\lambda)\Pi_{\mathbf{r}}^n$ (resp. $(\Pi_{\boldsymbol{\ell}}^n)^{\mathcal{B}} F_{\mathbf{q}}(\lambda)$) are the same as the positions of the wing block-rows (resp. wing block-columns) of $F_{\mathbf{q}}(\lambda)$, the first claim in part (c) follows. The second claim follows from the fact that $M(\lambda)$ follows the staircase pattern for $\lambda A_k + A_{k-1}, A_{k-2}, \dots, A_0$. ☐

REMARK 6.4. *We note that part (c) in Theorem 6.3 implies that the block-permutations* $\Pi_{\mathbf{r}}^n$ *and* $\Pi_{\boldsymbol{\ell}}^n$ *in* (6.1) *are, respectively, of the form* $I_n \oplus \Pi_{\widetilde{\mathbf{r}}}^n$ *and* $I_n \oplus \Pi_{\widetilde{\boldsymbol{\ell}}}^n$*, for some permutations* $\widetilde{\mathbf{r}}$ *and* $\widetilde{\boldsymbol{\ell}}$ *of the set* $\{1:k-1\}$.

7. The proper GFP as block-Kronecker pencils. In this section, we prove that the proper GFP are, up to permutations, block Kronecker pencils. The precise statement of this result is Theorem 7.1, however, we postpone its proof to subsection 7.1.

Recall from Remark 4.6 the concept of a simple pair associated with a proper GFP.

THEOREM 7.1. *Let* $P(\lambda) = \sum_{i=0}^k A_i \lambda^i \in \mathbb{F}[\lambda]^{n\times n}$ *be a matrix polynomial of grade* k*, and let*

$$K_{\mathbf{q},\mathbf{z}}(\lambda) = \lambda M_{\mathbf{z}}^P - M_{\mathbf{q}}^P$$

be the proper GFP associated with $P(\lambda)$ *and* $(\mathbf{q}, \mathbf{z})$*. Let* $(\widehat{\mathbf{q}}, h)$ *be the simple pair associated with* $K_{\mathbf{q},\mathbf{z}}(\lambda)$*. Then, there exist block-permutation matrices* $\Pi_{\boldsymbol{\ell}}^n$ *and* $\Pi_{\mathbf{r}}^n$ *such that*

$$(\Pi_{\boldsymbol{\ell}}^n)^{\mathcal{B}} K_{\mathbf{q},\mathbf{z}}(\lambda)\Pi_{\mathbf{r}}^n = \left[\begin{array}{c|c} M(\lambda) & L_{\mathfrak{h}(\operatorname{rev}(\widehat{\mathbf{q}}))+k-h-2}(\lambda)^T \otimes I_n \\ \hline L_{\mathfrak{h}(\widehat{\mathbf{q}})-1}(\lambda) \otimes I_n & 0 \end{array}\right] \tag{7.1}$$

is a $(\mathfrak{h}(\widehat{\mathbf{q}})-1, n, \mathfrak{h}(\operatorname{rev}(\widehat{\mathbf{q}})) + k - h - 2, n)$*-block Kronecker pencil, where the body* $M(\lambda)$ *satisfies the AS condition for* $P(\lambda)$*. Moreover, setting* $p := \mathfrak{h}(\widehat{\mathbf{q}}) - 1$ *and* $q := \mathfrak{h}(\operatorname{rev}(\widehat{\mathbf{q}})) + k - h - 2$*, the following statements hold.*

(a) *The wing block-columns of* $(\Pi_{\boldsymbol{\ell}}^n)^{\mathcal{B}} K_{\mathbf{q},\mathbf{z}}(\lambda)$ *relative to* $(\mathbf{id}, \mathbf{r}, p, q)$ *are of the form* $-e_i \otimes I_n + \lambda e_{i+1} \otimes I_n$*, for some* $1 \le i \le q$*, and are located in positions* $j \in \{1:k\}$ *such that* $(\mathbf{q}, k-j)$ *and* $(\mathbf{z}+k, j-1)$ *satisfy the SIP.*

(b) *The wing block-rows of* $K_{\mathbf{q},\mathbf{z}}(\lambda)\Pi_{\mathbf{r}}^n$ *relative to* $(\boldsymbol{\ell}, \mathbf{id}, p, q)$ *are of the form* $-e_i^T \otimes I_n + \lambda e_{i+1}^T \otimes I_n$*, for some* $1 \le i \le p$*, and are located in positions* $j \in \{1:k\}$ *such that* $(k-j, \mathbf{q})$ *and* $(j-1, \mathbf{z}+k)$ *satisfy the SIP.*

REMARK 7.2. *We note that Theorem 7.1, together with Theorem 3.7, implies that the (extended) block Kronecker pencil* $(\Pi_{\boldsymbol{\ell}}^n)^{\mathcal{B}} K_{\mathbf{q},\mathbf{z}}(\lambda)\Pi_{\mathbf{r}}^n$ *in* (7.1) *is a strong linearization of the matrix polynomial* $P(\lambda) = \sum_{i=0}^k A_i \lambda^i$.

We start by showing that Theorem 7.1 holds for a subfamily of proper GFP.

THEOREM 7.3. *Let* $P(\lambda) = \sum_{i=0}^k A_i \lambda^i \in \mathbb{F}[\lambda]^{n\times n}$ *be a matrix polynomial of grade* k*. Let* $h \in \{1:k-1\}$*, let* $\mathbf{q}$ *be a permutation of the set* $\{0:h\}$*, and let* $K_{\mathbf{q},\mathbf{z}}(\lambda)$ *be the*

following proper generalized Fiedler pencil

$$K_{\mathbf{q},\mathbf{z}}(\lambda) = \lambda M^P_{-k:-h-1} - M^P_{\mathbf{q}},$$

associated with $P(\lambda)$. Then, Theorem 7.1 holds for $K_{\mathbf{q},\mathbf{z}}(\lambda)$.

Proof. Let $\mathbf{z} = (-k : -h-1)$. First note that the simple pair associated with $K_{\mathbf{q},\mathbf{z}}(\lambda)$ is $(\mathbf{q}, h)$. By applying Lemma 4.22 to the pencil $K_{\mathbf{q},\mathbf{z}}(\lambda)$, which is a also a GFPR, we obtain that $K_{\mathbf{q},\mathbf{z}}(\lambda)$ can be partitioned as follows

$$K_{\mathbf{q},\mathbf{z}}(\lambda) = \left[\begin{array}{c|c|c} D_{\mathbf{z}}(\lambda) & y_{\mathbf{z}}(\lambda) & 0 \\ \hline x_{\mathbf{z}}(\lambda) & c(\lambda) & x_{\mathbf{q}}(\lambda) \\ \hline 0 & y_{\mathbf{q}}(\lambda) & D_{\mathbf{q}}(\lambda) \end{array}\right],$$

where $D_{\mathbf{z}}(\lambda) \in \mathbb{F}[\lambda]^{(k-h-1)n\times(k-h-1)n}$, $x_{\mathbf{z}}(\lambda) \in \mathbb{F}[\lambda]^{n\times(k-h-1)n}$, $y_{\mathbf{z}}(\lambda) \in \mathbb{F}[\lambda]^{(k-h-1)n\times n}$, $D_{\mathbf{q}}(\lambda) \in \mathbb{F}[\lambda]^{nh\times nh}$, $x_{\mathbf{q}}(\lambda) \in \mathbb{F}[\lambda]^{n\times nh}$, $y_{\mathbf{q}}(\lambda) \in \mathbb{F}[\lambda]^{nh\times n}$ and $c(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$. Moreover, Lemma 4.22 also tells us that the pencil

$$F(\lambda) := \begin{bmatrix} c(\lambda) & x_{\mathbf{q}}(\lambda) \\ y_{\mathbf{q}}(\lambda) & D_{\mathbf{q}}(\lambda) \end{bmatrix}$$

is a Fiedler pencil associated with the matrix polynomial $Q(\lambda) := \lambda^{h+1}A_{h+1} + \lambda^h A_h + \cdots + \lambda A_1 + A_0$ since $F(\lambda) = \lambda M^Q_{-h-1} - M^Q_{\mathbf{q}}$. Similarly, the pencil

$$G(\lambda) := \begin{bmatrix} D_{\mathbf{z}}(\lambda) & y_{\mathbf{z}}(\lambda) \\ x_{\mathbf{z}}(\lambda) & c(\lambda) \end{bmatrix}$$

is a proper GFP associated with the matrix polynomial $Z(\lambda) := \lambda^{k-h}A_k + \lambda^{k-h-1}A_{k-1} + \cdots + \lambda A_{h+1} + A_h$ since $G(\lambda) = \lambda M^Q_{h-k:-1} - M^Q_0$. Furthermore, a direct matrix multiplication similar to the computations necessary to prove Lemma 4.3 shows that

$$\begin{bmatrix} D_{\mathbf{z}}(\lambda) \\ x_{\mathbf{z}}(\lambda) \end{bmatrix} = L_{k-h-1}(\lambda)^T \otimes I_n, \quad \text{and} \quad \begin{bmatrix} y_{\mathbf{z}}(\lambda) \\ c(\lambda) \end{bmatrix} = \begin{bmatrix} \lambda A_k \\ \lambda A_{k-1} \\ \vdots \\ \lambda A_{h+2} \\ \lambda A_{h+1} + A_h \end{bmatrix}.$$

Next, applying Theorem 6.3 to the pencil $F(\lambda)$ and to according Remark 6.1, we deduce that $c(\lambda) = \lambda A_{h+1} + A_h$ and obtain that there exist block-permutation matrices $\Pi_{\mathbf{r}_1} = I_n \oplus \Pi_{\widetilde{\mathbf{r}}}$ and $\Pi_{\boldsymbol{\ell}_1} = I_n \oplus \Pi_{\widetilde{\boldsymbol{\ell}}}$ such that

$$\begin{aligned} \Pi^{\mathcal{B}}_{\boldsymbol{\ell}_1} F(\lambda) \Pi_{\mathbf{r}_1} = \begin{bmatrix} I_n & 0 \\ 0 & \Pi^{\mathcal{B}}_{\widetilde{\boldsymbol{\ell}}} \end{bmatrix} \begin{bmatrix} \lambda A_{h+1} + A_h & x_{\mathbf{q}}(\lambda) \\ y_{\mathbf{q}}(\lambda) & D_{\mathbf{q}}(\lambda) \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & \Pi_{\widetilde{\mathbf{r}}} \end{bmatrix} =: \\ \left[\begin{array}{c|c} M(\lambda) & L_{q_1}(\lambda)^T \otimes I_n \\ \hline L_p(\lambda) \otimes I_n & 0 \end{array}\right] =: \\ \left[\begin{array}{cc|c} \lambda A_{h+1} + A_h & m_1(\lambda) & \ell_{q_1}(\lambda)^T \otimes I_n \\ m_2(\lambda) & \widetilde{M}(\lambda) & \widetilde{L}_{q_1}(\lambda)^T \otimes I_n \\ \hline \ell_p(\lambda) \otimes I_n & \widetilde{L}_p(\lambda) \otimes I_n & 0 \end{array}\right] \end{aligned}$$

is a (p, n, q_1, n)-block Kronecker pencil for $Q(\lambda)$, with $p = \mathfrak{h}(\mathbf{q})-1$ and $q_1 := \mathfrak{h}(\mathrm{rev}(\mathbf{q}))-1$. Then, notice that

$$\begin{bmatrix} I_{(k-h-1)n} & & \\ & I_n & \\ & & \Pi^{\mathcal{B}}_{\widetilde{\boldsymbol{\ell}}} \end{bmatrix} \begin{bmatrix} D_{\mathbf{z}}(\lambda) & y_{\mathbf{z}}(\lambda) & 0 \\ x_{\mathbf{z}}(\lambda) & \lambda A_{h+1} + A_h & x_{\mathbf{q}}(\lambda) \\ 0 & y_{\mathbf{q}}(\lambda) & D_{\mathbf{q}}(\lambda) \end{bmatrix} \begin{bmatrix} I_{(k-h-1)n} & & \\ & I_n & \\ & & \Pi_{\widetilde{\mathbf{r}}} \end{bmatrix} = \begin{bmatrix} D_{\mathbf{z}}(\lambda) & y_{\mathbf{z}}(\lambda) & 0 & 0 \\ x_{\mathbf{z}}(\lambda) & \lambda A_{h+1} + A_h & m_1(\lambda) & \ell_{q_1}(\lambda)^T \otimes I_n \\ 0 & m_2(\lambda) & \widetilde{M}(\lambda) & \widetilde{L}_{q_1}(\lambda)^T \otimes I_n \\ 0 & \ell_p(\lambda) \otimes I_n & \widetilde{L}_p(\lambda) \otimes I_n & 0 \end{bmatrix}, \tag{7.2}$$

which is block-permutationally equivalent to the matrix pencil

$$\left[\begin{array}{cc|cc} y_{\mathbf{z}}(\lambda) & 0 & D_{\mathbf{z}}(\lambda) & 0 \\ \lambda A_{h+1} + A_h & m_1(\lambda) & x_{\mathbf{z}}(\lambda) & \ell_{q_1}(\lambda)^T \otimes I_n \\ m_2(\lambda) & \widetilde{M}(\lambda) & 0 & \widetilde{L}_{q_1}(\lambda)^T \otimes I_n \\ \hline \ell_p(\lambda) \otimes I_n & \widetilde{L}_p(\lambda) \otimes I_n & 0 & 0 \end{array}\right]. \tag{7.3}$$

Thus, there exist matrix permutations $\Pi^n_{\boldsymbol{\ell}}$ and $\Pi^n_{\mathbf{r}}$ such that $(\Pi^n_{\boldsymbol{\ell}})^{\mathcal{B}} K_{\mathbf{q},\mathbf{z}}(\lambda)\Pi^n_{\mathbf{r}}$ is a $(p, n, q_1 + k - h - 1)$-block-Kronecker pencil with body

$$H(\lambda) = \begin{bmatrix} y_{\mathbf{z}}(\lambda) & 0 \\ \lambda A_{h+1} + A_h & m_1(\lambda) \\ m_2(\lambda) & \widetilde{M}(\lambda) \end{bmatrix}.$$

To prove that the pencil $H(\lambda)$ satisfies the AS condition for $P(\lambda)$, we decompose $H(\lambda)$ as

$$\underbrace{\begin{bmatrix} y_{\mathbf{z}}(\lambda) & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{H_1(\lambda)} + \underbrace{\begin{bmatrix} 0 & 0 \\ \lambda A_{h+1} + A_h & m_1(\lambda) \\ m_2(\lambda) & \widetilde{M}(\lambda) \end{bmatrix}}_{H_2(\lambda)},$$

and notice that

$$\mathrm{AS}(H, s) = \mathrm{AS}(H_1, s) + \mathrm{AS}(H_2, s) = A_s, \quad \text{for } s = 0 : k,$$

which follows from the structure of the block-vector $y_{\mathbf{z}}(\lambda)$, the fact that the body of $\Pi^{\mathcal{B}}_{\boldsymbol{\ell}_1} F(\lambda)\Pi_{\mathbf{r}_1}$ satisfies the AS condition for $Q(\lambda)$, and the linearity of the antidiagonal sum.

To prove part (a) of Theorem 7.1, notice first that Theorem 6.3 implies that the wing block-columns of the FP $F(\lambda)$ relative to $(\boldsymbol{\ell}_1, \mathbf{r}_1, p, q_1)$ are in positions $h + 1 - j$, where $j \in \{0 : h - 1\}$ and $(\mathbf{q}, j)$ satisfies the SIP. Then, notice that, for $q := q_1 + k - h - 2$, the wing block-columns of $(\Pi^n_{\boldsymbol{\ell}})^{\mathcal{B}} K_{\mathbf{q},\mathbf{z}}(\lambda)$ relative to $(\mathbf{id}, \mathbf{r}, p, q)$ are in positions $\{1 : k - h - 1\} \cup \{(k - h - 1) + h + 1 - i\colon i \in \{0 : h - 1\}$ and $(\mathbf{q}, i)$ satisfies the $SIP\}$, or equivalently, in positions

$$\{1 : k - h - 1\} \cup \{j : \ j \in \{k - h - 1 : k\} \text{ and } (\mathbf{q}, k - j) \text{ satisfies the SIP}\}.$$

Note also that, in this case, $\mathbf{z} + k = 0 : k - h - 1$ and $(\mathbf{z} + k, s)$ satisfies the SIP if and only if $s \in \{0 : k - h - 2\}$. Moreover, we observe that all the wing block-columns are of the form $-e_i \otimes I_n + \lambda e_{i+1} \otimes I_n$ for some $1 \le i \le q$, which implies part (a).

Finally, to prove part (b), recall from Theorem 6.3 that the wing block-rows of $F(\lambda)$ relative to $(\boldsymbol{\ell}_1, \mathbf{r}_1, p, q_1)$ are located in positions $h+1-j$, where $j \in \{0:h-1\}$ and $(j,\mathbf{q})$ satisfies the SIP. Then, notice that, no index s is such that $(s, \mathbf{z}+k)$ satisfies the SIP and the wing block-rows of $K_{\mathbf{q},\mathbf{z}}(\lambda)\Pi_{\mathbf{r}}^n$ relative to $(\boldsymbol{\ell}, \mathbf{id}, p, q)$ are located in positions

$$\{(k-h-1)+h+1-j:\ j \in \{0:h-1\} \text{ and } (j,\mathbf{q}) \text{ satisfies the SIP}\}$$

and have the desired form, which implies part (b). □

7.1. Proof of Theorem 7.1. Armed with Lemma 5.4 and Theorem 7.2, we are in a position to prove Theorem 7.1.

Proof. (of Theorem 7.1) Recall from Remark 4.6 that the proper GFP $K_{\mathbf{q},\mathbf{z}}(\lambda)$ can be written in the form

$$K_{\mathbf{q},\mathbf{z}}(\lambda) = M_{\mathbf{m}}^P(\lambda M_{-k:-h-1}^P - M_{\widehat{\mathbf{q}}}^P),$$

where $\widehat{\mathbf{q}} = (-\operatorname{rev}(\mathbf{m}), \mathbf{q})$ is a permutation of $\{0:h\}$, and $\mathbf{m}$ is a tuple with indices from $\{-1:-h\}$. Additionally, let us introduce the notation $\mathbf{m} = (-i_s, -i_{s-1}, \ldots, -i_1)$ for the indices of the tuple $\mathbf{m}$, and let us write

$$K_{\mathbf{q},\mathbf{z}}(\lambda) = M_{(-i_s,-i_{s-1},\ldots,-i_1)}^P(\lambda M_{-k:-h-1}^P - M_{(i_1,\ldots,i_{s-1},i_s,\mathbf{q})}^P). \tag{7.4}$$

The proof proceeds by induction on the number s of factors in $M_{(-i_s,-i_{s-1},\ldots,-i_1)}^P$. When $s=0$, we have the pencil $\lambda M_{-k:-h-1}^P - M_{\widehat{\mathbf{q}}}^P$. In this case, the result follows from Theorem 7.2. Assume, now, that the result is true for the proper GFP

$$\begin{aligned} L(\lambda) :=& M_{(-i_{s-1},\ldots,-i_1)}^P(\lambda M_{-k:-h-1}^P - M_{(i_1,\ldots,i_{s-1},i_s,\mathbf{q})}^P) = \\ & \lambda M_{(-i_{s-1},\ldots,-i_1,-k:-h-1)}^P - M_{(i_s,\mathbf{q})}^P, \end{aligned}$$

and let us show that the result is true for the pencil $K_{\mathbf{q},\mathbf{z}}(\lambda) = M_{-i_s}^P L(\lambda)$. Since $(\widehat{\mathbf{q}}, h)$ is also a simple pair associated with $L(\lambda)$, the inductive hypothesis implies that there exist block-permutation matrices $\Pi_{\widetilde{\boldsymbol{\ell}}}^n$ and $\Pi_{\mathbf{r}}^n$ such that

$$(\Pi_{\widetilde{\boldsymbol{\ell}}}^n)^{\mathcal{B}} L(\lambda)\Pi_{\mathbf{r}}^n = \left[\begin{array}{c|c} M'(\lambda) & L_{\mathfrak{h}(\operatorname{rev}(\widehat{\mathbf{q}}))+k-h-2}(\lambda)^T \otimes I_n \\ \hline L_{\mathfrak{h}(\widehat{\mathbf{q}})-1}(\lambda) \otimes I_n & 0 \end{array}\right], \tag{7.5}$$

is an extended (p,n,q,n)-block Kronecker pencil, whose body $M'(\lambda)$ satisfies the AS condition for $P(\lambda)$, where $p = \mathfrak{h}(\widehat{q})-1$ and $q = \mathfrak{h}(\operatorname{rev}(\widehat{\mathbf{q}}))+k-h-2$. To prove that the result holds as well for $K_{\mathbf{q},\mathbf{z}}(\lambda)$, we will apply Lemma 5.4, so we start by showing that the $(k-i_s)$th and $(k-i_s+1)$th block-rows of $L(\lambda)$ are, respectively, a body block-row and a wing block-row of $L(\lambda)$ relative to $(\widetilde{\boldsymbol{\ell}}, \mathbf{r}, p, q)$. First, notice that $(i_s, (i_s, \mathbf{q}))$ does not satisfy the SIP. Thus, part (b) of the inductive hypothesis implies that the $(k-i_s)$th block-row of $L(\lambda)$ is a body block-row relative to $(\widetilde{\boldsymbol{\ell}}, \mathbf{r}, p, q)$. Next, notice that each index in $\{0:k\}$ appears either in the tuple $(i_{s-1}, \ldots, i_1, k:h+1)$ or in the tuple $(i_s, \mathbf{q})$, and it appears in those tuples at most one time. This, in turn, implies that both $(i_s-1, (i_s, \mathbf{q}))$ and $(k-i_s, (k-i_{s-1}, \ldots, k-i_1, 0:k-h-1))$ satisfy the SIP. By part (b) of the inductive hypothesis, it follows, then, that the $(k-i_s+1)$th block-row of $L(\lambda)$ is a wing block-row relative to $(\widetilde{\boldsymbol{\ell}}, \mathbf{r}, p, q)$. Therefore, by part (i) of Lemma 5.4, there exist block-permutation matrices $\Pi_{\boldsymbol{\ell}}^n$ and $\Pi_{\mathbf{r}}^n$ such that (7.1) holds with a body $M(\lambda)$ satisfying the AS condition for $P(\lambda)$.

Now, we show that parts (a) and (b) hold for $K_{\mathbf{q},\mathbf{z}}(\lambda)$. To this end, we introduce the notation $\mathbf{t} := (k - i_{s-1}, \ldots, k - i_1, 0 : k - h - 1)$. Notice that $\mathbf{z} + k = (\mathbf{m}, -k : -h - 1) + k = (k - i_s, \mathbf{t})$. We start by proving part (a).

Let us denote by $\{c_1, \ldots, c_q\}$ the set of positions of the wing block-columns of $L(\lambda)$ relative to $(\widetilde{\boldsymbol{\ell}}, \mathbf{r}, p, q)$. From parts (iiia) and (iiib) of Lemma 5.4, we obtain that $\{c_1, \ldots, c_q\}$ is also the set of positions of the wing block-columns of $K_{\mathbf{q},\mathbf{z}}(\lambda)$ relative to $(\boldsymbol{\ell}, \mathbf{r}, p, q)$. Since $L(\lambda)$ and $K_{\mathbf{q},\mathbf{z}}(\lambda)$ have the same number of wing block-columns and are located in the same positions, part (a) for $K_{\mathbf{q},\mathbf{z}}(\lambda)$ follows from part (a) for $L(\lambda)$ (which holds by the inductive hypothesis) if we prove that the tuples $(\mathbf{q}, k - j)$ and $((k - i_s, \mathbf{t}), j - 1)$ satisfy the SIP if, respectively, the tuples $((i_s, \mathbf{q}), k - j)$ and $(\mathbf{t}, j - 1)$ do. So, let us assume that $((i_s, \mathbf{q}), k - j)$ and $(\mathbf{t}, j - 1)$ satisfy the SIP for some $j \in \{1 : k\}$. The assumptions clearly imply that the tuple $(\mathbf{q}, k - j)$ satisfy the SIP, since this is a subtuple of consecutive indices of $((i_s, \mathbf{q}), k - j)$. Then, notice that, since $(\mathbf{t}, j - 1)$ satisfies the SIP, since each different index of $\mathbf{t}$ appears only one time in $\mathbf{t}$, and $k - i_s \notin \mathbf{t}$, to prove that the tuple $((k - i_s, \mathbf{t}), j - 1)$ satisfies the SIP, it is only necessary to check the case $j = k - i_s + 1$, that is, we have to prove that $(k - i_s, \mathbf{t}, k - i_s)$ satisfies the SIP when $((i_s, \mathbf{q}), i_s - 1)$ and $(\mathbf{t}, k - i_s)$ satisfy the SIP. The proof proceeds by contradiction. Assume that $((i_s, \mathbf{q}), i_s - 1)$ and $(\mathbf{t}, k - i_s)$ satisfy the SIP and that $(k - i_s, \mathbf{t}, k - i_s)$ does not satisfy the SIP. The latter assumption implies that $k - i_s + 1 \notin \mathbf{t}$, which, in turn, implies that $i_s - 1 \in \mathbf{q}$. Since each different index in $(i_s, \mathbf{q})$ appears only once, the statement $i_s - 1 \in \mathbf{q}$ implies that $((i_s, \mathbf{q}), i_s - 1)$ does not satisfy the SIP, which contradicts our assumptions. Thus, (a) follows.

Finally, we prove that part (b) is true. Let us denote by $\{r_1, \ldots, r_p\}$ the set of positions of the wing block-rows of $L(\lambda)$ relative to $(\widetilde{\boldsymbol{\ell}}, \mathbf{r}, p, q)$, and recall that $k - i_s \notin \{r_1, \ldots, r_p\}$ and $k - i_s + 1 \in \{r_1, \ldots, r_p\}$. From parts (ii.c) and (ii.d) of Lemma 5.4, we obtain that the set of positions of the wing block-rows of $K_{\mathbf{q},\mathbf{z}}(\lambda)$ relative to $(\boldsymbol{\ell}, \mathbf{r}, p, q)$ is $(\{r_1, \ldots, r_p\} \setminus \{k - i_s + 1\}) \cup \{k - i_s\}$. Since $L(\lambda)$ and $K_{\mathbf{q},\mathbf{z}}(\lambda)$ have the same number of wing block-rows, part (b) for $K_{\mathbf{q},\mathbf{z}}(\lambda)$ follows from part (b) for $L(\lambda)$ (by the inductive hypothesis) if the following three statements hold: (i) since $k - i_s$ is the position of a wing block-row of $K_{\mathbf{q},\mathbf{z}}(\lambda)$ relative to $(\boldsymbol{\ell}, \mathbf{r}, p, q)$, the tuples $(i_s, \mathbf{q})$ and $(k - i_s - 1, (k - i_s, \mathbf{t}))$ satisfy the SIP; (ii) since $k - i_s + 1$ is not a wing block-row of $K_{\mathbf{q},\mathbf{z}}(\lambda)$ relative to $(\boldsymbol{\ell}, \mathbf{r}, p, q)$, one of the following tuples $(i_s - 1, \mathbf{q})$ and $(k - i_s, (k - i_s, \mathbf{t}))$, or both, does not satisfy the SIP; (iii) when $j \neq k - i_s, k - i_s + 1$, if the tuples $(k - j, (i_s, \mathbf{q}))$ and $(j - 1, \mathbf{t})$ satisfy the SIP, then $(k - j, \mathbf{q})$ and $(j - 1, (k - i_s, \mathbf{t}))$ also satisfy the SIP. Since (i) and (ii) are immediate to prove, we focus on proving (iii). Assume that $(k - j, (i_s, \mathbf{q}))$ and $(j - 1, \mathbf{t})$ satisfy the SIP, and let $j \neq k - i_s, k - i_s + 1$. First, if $j \neq k - i_s - 1$, then the tuple $(i_s, k - j, \mathbf{q}) \sim (k - j, i_s, \mathbf{q})$ satisfies the SIP, which, in turn, implies that its subtuple $(k - j, \mathbf{q})$ satisfies the SIP as well. Second, if $j = k - i_s - 1$, the assumption that $(i_s + 1, (i_s, \mathbf{q}))$ satisfies the SIP, together with the fact that each different index of $(i_s, \mathbf{q})$ appears only one time, implies that $(i_s + 1, \mathbf{q})$ satisfies the SIP as well. Finally, since $k - i_s \notin \mathbf{t}$ and $j - 1 \neq k - i_s, k - i_s - 1$, we immediately obtain that $(j - 1, (k - i_s, \mathbf{t}))$ satisfies the SIP. □

8. The GFPR as extended block Kronecker pencils. We prove in this section that all GFPR associated with a matrix polynomial $P(\lambda)$ are, up to permutations of block-rows and block-columns, extended block Kroneckers pencils with bodies satisfying the AS condition for $P(\lambda)$. This result is stated in Theorem 8.1, which is one of the main results of this paper. However, we postpone its proof to Section 8.1.

THEOREM 8.1. *Let $P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i \in \mathbb{F}[\lambda]^{n \times n}$ be a matrix polynomial of grade k, and let*

$$L_P(\lambda) = M_{\boldsymbol{\ell}_q, \boldsymbol{\ell}_z}(\mathcal{X}, \mathcal{Z})(\lambda M_{\mathbf{z}}^P - M_{\mathbf{q}}^P) M_{\mathbf{r}_z, \mathbf{r}_q}(\mathcal{W}, \mathcal{Y}) \tag{8.1}$$

be a GFPR associated with $P(\lambda)$. Then, there exist block-permutation matrices $\Pi_{\boldsymbol{\ell}}^n, \Pi_{\mathbf{r}}^n$ such that

$$(\Pi_{\boldsymbol{\ell}}^n)^{\mathcal{B}} L_P(\lambda) \Pi_{\mathbf{r}}^n = \left[\begin{array}{c|c} M(\lambda) & K_2(\lambda)^T \\ \hline K_1(\lambda) & 0 \end{array}\right]$$

is an extended $(\mathfrak{h}(\mathbf{q}) + \mathfrak{h}(k+\mathbf{z}) - 2, n, \mathfrak{h}(\mathrm{rev}(\mathbf{q})) + \mathfrak{h}(\mathrm{rev}(k+\mathbf{z})) - 2, n)$-block Kronecker pencil, whose body $M(\lambda)$ satisfies the AS condition for $P(\lambda)$. Moreover, if $K_1(\lambda)$ and $K_2(\lambda)$ are minimal bases, then $(\Pi_{\boldsymbol{\ell}}^n)^{\mathcal{B}} L_P(\lambda) \Pi_{\mathbf{r}}^n$ is a strong linearization of $P(\lambda)$.

REMARK 8.2. *We note that the fact that $(\Pi_{\boldsymbol{\ell}}^n)^{\mathcal{B}} L_P(\lambda) \Pi_{\mathbf{r}}^n$ is a strong linearization of $P(\lambda)$ if $K_1(\lambda)$ and $K_2(\lambda)$ are minimal bases follows from Theorem 3.7. Thus, in this section, we focus on proving the first claim in Theorem 8.1.*

The main idea behind the proof of Theorem 8.1 is that right multiplications by elementary matrices preserve, in some relevant cases, the property of being block-permutationally equivalent to an extended block Kronecker pencil whose body satisfies the AS condition for some $P(\lambda)$, as we showed in Lemma 5.5. With this result, next we present and prove Theorems 8.2 and 8.3, which will be key to prove Theorem 8.1. Theorems 8.2 and 8.3 can be seen as particular instances of the general result in Theorem 8.1 together with some structural information concerning block-rows and block-columns of the particular GFPR they focus on.

THEOREM 8.3. *Let $P(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ be a matrix polynomial of grade $k \geq 2$. Let $\mathbf{q}$ be a permutation of $\{0 : k-1\}$ and let*

$$L_P(\lambda) = M_{\boldsymbol{\ell}_q}(\mathcal{X})(\lambda M_{-k}^P - M_{\mathbf{q}}^P) M_{\mathbf{r}_q}(\mathcal{Y}),$$

be a GFPR associated with $P(\lambda)$. Then, there exist block-permutation matrices $\Pi_{\boldsymbol{\ell}}^n$ and $\Pi_{\mathbf{r}}^n$ such that

$$(\Pi_{\boldsymbol{\ell}}^n)^{\mathcal{B}} L_P(\lambda) \Pi_{\mathbf{r}}^n = \left[\begin{array}{c|c} M(\lambda) & K_2(\lambda)^T \\ \hline K_1(\lambda) & 0 \end{array}\right], \tag{8.2}$$

is an extended $(\mathfrak{h}(\mathbf{q}) - 1, n, \mathfrak{h}(\mathrm{rev}(\mathbf{q})) - 1, n)$-block Kronecker pencil, whose body $M(\lambda)$ satisfies the AS condition for $P(\lambda)$. Moreover, the following statements hold.

(a) *The first block-row and the first block-column of $L_P(\lambda)$ are, respectively, the first body block-row and the first body block-column of $L_P(\lambda)$ relative to $(\boldsymbol{\ell}, \mathbf{r}, \mathfrak{h}(\mathbf{q})-1, \mathfrak{h}(\mathrm{rev}(\mathbf{q}))-1)$. Moreover, the block-entry of $M(\lambda)$ in position $(1,1)$ equals $\lambda A_k + A_{k-1}$.*

(b) *The wing block-columns of $(\Pi_{\boldsymbol{\ell}}^n)^{\mathcal{B}} L_P(\lambda)$ relative to $(\mathbf{id}, \mathbf{r}, \mathfrak{h}(\mathbf{q}) - 1, \mathfrak{h}(\mathrm{rev}(\mathbf{q})) - 1)$ of the form $-e_i \otimes I_n + \lambda e_{i+1} \otimes I_n$, for $1 \leq i \leq \mathfrak{h}(\mathrm{rev}(\mathbf{q})) - 1$, are precisely those located in positions $k-j$, where $j \in \{0 : k-2\}$ and $j \notin \mathrm{heads}(\boldsymbol{\ell}_q, \mathbf{q}, \mathbf{r}_q)$, or, equivalently, $j \in \{0 : k-2\}$ and $(\boldsymbol{\ell}_q, \mathbf{q}, \mathbf{r}_q, j)$ satisfies the SIP.*

(c) *The wing block-rows of $L_P(\lambda)\Pi_{\mathbf{r}}^n$ relative to $(\boldsymbol{\ell}, \mathbf{id}, \mathfrak{h}(\mathbf{q}) - 1, \mathfrak{h}(\mathrm{rev}(\mathbf{q})) - 1)$ of the form $-e_i^T \otimes I_n + \lambda e_{i+1}^T \otimes I_n$, for $1 \leq i \leq \mathfrak{h}(\mathbf{q}) - 1$, are precisely those located in positions $k-j$, where $j \in \{0 : k-2\}$ and $j \notin \mathrm{heads}(\mathrm{rev}(\mathbf{r}_q), \mathrm{rev}(\mathbf{q}), \mathrm{rev}(\boldsymbol{\ell}_q))$, or, equivalently, $j \in \{0 : k-2\}$ and $(\mathrm{rev}(\mathbf{r}_q), \mathrm{rev}(\mathbf{q}), \mathrm{rev}(\boldsymbol{\ell}_q), j)$ satisfies the SIP.*

Proof. Let $p := \mathfrak{h}(\mathbf{q}) - 1$ and $q := \mathfrak{h}(\mathrm{rev}(\mathbf{q})) - 1$. We prove the result by induction on the number of indices in $\boldsymbol{\ell}_q$ and in $\mathbf{r}_q$. If both $\boldsymbol{\ell}_q$ and $\mathbf{r}_q$ are empty, then $L_P(\lambda)$ is a Fiedler pencil and the result follows by Theorem 6.3.

Assume that $\boldsymbol{\ell}_q$ is such that the result holds for $\boldsymbol{\ell}_q$ and for tuples $\mathbf{r}_q'$ with at most t indices, with $t \geq 0$. The case of Fiedler pencils above demonstrates that some such $\boldsymbol{\ell}_q$ and t exist, namely, $\boldsymbol{\ell}_q = \varnothing$ and $t = 0$. Now, suppose that $\mathbf{r}_q = (\mathbf{r}_q', x)$ has $t+1$ indices, where $x \in \{0 : k-2\}$. Let $\mathcal{Y} = (\mathcal{Y}', Y_0)$ be an $n \times n$ matrix assignment for $\mathbf{r}_q$, where $\mathcal{Y}'$ and Y_0 are, respectively, the matrix assignments for $\mathbf{r}_q'$ and x induced by $\mathcal{Y}$, and let $L'(\lambda) :=$

$M_{\boldsymbol{\ell}_q}(\mathcal{X})(\lambda M^P_{-k} - M^P_{\mathbf{q}})M_{\mathbf{r}'_q}(\mathcal{Y}')$. Since $\mathbf{r}'_q$ has t indices, by the inductive hypothesis, there exist block-permutation matrices $\Pi^n_{\mathbf{r}'}$ and $\Pi^n_{\boldsymbol{\ell}}$ such that

$$(\Pi^n_{\boldsymbol{\ell}})^{\mathcal{B}} L'(\lambda)\Pi^n_{\mathbf{r}'} = \left[\begin{array}{c|c} M'(\lambda) & K_2'(\lambda)^T \\ \hline K_1(\lambda) & 0 \end{array}\right] \tag{8.3}$$

is an extended (p, n, q, n)-block Kronecker pencil whose body $M'(\lambda)$ satisfies the AS condition for $P(\lambda)$. Since the index tuple $(\boldsymbol{\ell}_q, \mathbf{q}, \mathbf{r}'_q, x)$ satisfies the SIP, by (b) applied to $L'(\lambda)$, the $(k-x)$th block-column of $L'(\lambda)$ is one of its wing block-columns relative to $(\boldsymbol{\ell}, \mathbf{r}', p, q)$. Thus, by Lemma 5.5, there exists a block-permutation matrix $\Pi^n_{\mathbf{r}}$ such that (8.2) holds, with $M(\lambda)$ satisfying the AS condition for $P(\lambda)$.

Now, we prove part (a). By (iv.b) of Lemma 5.5, together with part (a) for $L'(\lambda)$ (by the inductive hypothesis), we deduce that the first block-row of $L_P(\lambda)$ is a body block-row relative to $(\boldsymbol{\ell}, \mathbf{r}, p, q)$. Since $x \leq k-2$, by (a) for $L'(\lambda)$ and by part (iii.b) of Lemma 5.5, the first block-column of $L_P(\lambda)$ is a body block-column relative to $(\boldsymbol{\ell}, \mathbf{r}, p, q)$. Moreover, since the right-multiplication by the matrix $M_x(Y_0)$ does not affect the first block entry of $L'(\lambda)$ because $x \leq k-2$, the block-entry of $M(\lambda)$ in position $(1,1)$ equals $\lambda A_k + A_{k-1}$. Therefore, part (a) is true for $L_P(\lambda)$.

Next, we prove part (b). We have to distinguish several cases.

Assume, first, that $x = 0$. Then,

$$(\Pi^n_{\ell})^{\mathcal{B}} L_P(\lambda) = (\Pi^n_{\ell})^{\mathcal{B}} L'(\lambda) M_0(Y_0) = \left[\begin{array}{c|c|c} N_1(\lambda) & u(\lambda) \end{array}\right] (I_{n(k-1)} \oplus Y_0) \tag{8.4}$$

$$= \left[\begin{array}{c|c} N_1(\lambda) & u(\lambda)Y_0 \end{array}\right], \tag{8.5}$$

where $N_1(\lambda)$ consists of the first $k-1$ block-columns of $(\Pi^n_{\ell})^{\mathcal{B}} L'(\lambda)$ and, hence, $u(\lambda)$ is the kth block-column of $(\Pi^n_{\ell})^{\mathcal{B}} L'(\lambda)$. Part (iii.a) in Lemma 5.5 implies that the wing block-columns of $L_P(\lambda)$ relative to $(\boldsymbol{\ell}, \mathbf{r}, p, q)$ and the wing block-columns of $L'(\lambda)$ relative to $(\boldsymbol{\ell}, \mathbf{r}', p, q)$, other than the kth block-column, are equal and are located at the same positions. Thus, the wing block-columns of $(\Pi^n_{\boldsymbol{\ell}})^{\mathcal{B}} L_P(\lambda)$ relative to $(\mathbf{id}, \mathbf{r}, p, q)$ and the wing block-columns of $(\Pi^n_{\boldsymbol{\ell}})^{\mathcal{B}} L'(\lambda)$ relative to $(\mathbf{id}, \mathbf{r}', p, q)$, other than the kth block-column, are equal and are located at the same positions as well.

Since 0 is an index of Type II relative to $(\boldsymbol{\ell}_q, \mathbf{q}, \mathbf{r}'_q)$ by Proposition 4.14, and in view of Remark 4.5,

$$\operatorname{heads}(\boldsymbol{\ell}_q, \mathbf{q}, \mathbf{r}'_q, x) = \operatorname{heads}(\boldsymbol{\ell}_q, \mathbf{q}, \mathbf{r}'_q) \cup \{x\}.$$

This, together with Lemma 4.13 and part (iii.a) of Lemma 5.5, implies that, for part (b) to hold for $L_P(\lambda)$, we only need to show that the kth block-column of $(\Pi^n_{\ell})^{\mathcal{B}} L_P(\lambda)$ is not of the form $-e_i \otimes I_n + \lambda e_{i+1} \otimes I_n$. Notice that the induction hypothesis and (b) imply that the kth block-column of $(\Pi^n_{\boldsymbol{\ell}})^{\mathcal{B}} L'(\lambda)$ is of the form $-e_i \otimes I_n + \lambda e_{i+1} \otimes I_n$, for some $1 \leq i \leq q$, which, in turns, implies that the kth block-column of $(\Pi^n_{\boldsymbol{\ell}})^{\mathcal{B}} L_P(\lambda)$ is not generically of the form $-e_i \otimes I_n + \lambda e_{i+1} \otimes I_n$.

Assume, now, that $x \neq 0$. We have

$$(\Pi^n_{\ell})^{\mathcal{B}} L_P(\lambda) = (\Pi^n_{\ell})^{\mathcal{B}} L'(\lambda) M_x(Y_0) \tag{8.6}$$

$$= \left[\begin{array}{c|c|c} N_1(\lambda) & U(\lambda) & N_2(\lambda) \end{array}\right] (I_{n(k-x-1)} \oplus \left[\begin{array}{cc} Y_0 & I_n \\ I_n & 0 \end{array}\right] \oplus I_{n(x-1)}) \tag{8.7}$$

$$= \left[\begin{array}{c|c|c} N_1(\lambda) & U(\lambda) \cdot \left[\begin{array}{cc} Y_0 & I_n \\ I_n & 0 \end{array}\right] & N_2(\lambda) \end{array}\right], \tag{8.8}$$

where $N_1(\lambda)$ and $N_2(\lambda)$ consist, respectively, of the block-columns $1 : k-x-1$ and $k-x+2 : k$ of $(\Pi_\ell^n)^{\mathcal{B}}L'(\lambda)$. Let us denote by $u_1(\lambda)$ and $u_2(\lambda)$ the first and second block-columns of $U(\lambda)$, which are, respectively, the $(k-x)$th and the $(k-x+1)$th block-columns of $(\Pi_\ell^n)^{\mathcal{B}}L'(\lambda)$. Since $(\boldsymbol{\ell}_q, \mathbf{q}, \mathbf{r}'_q, x)$ satisfies the SIP, by the inductive hypothesis, the $(k-x)$th block-column of $(\Pi_\ell^n)^{\mathcal{B}}L'(\lambda)$, that is, $u_1(\lambda)$, is a wing block-column of $(\Pi_\ell^n)^{\mathcal{B}}L'(\lambda)$ relative to $(\mathbf{id}, \mathbf{r}', p, q)$ of the form $-e_i \otimes I_n + \lambda e_{i+1} \otimes I_n$, with $i \leq q$. Observe also that the second block-column $u'_2(\lambda)$ of $U(\lambda)\left[\begin{smallmatrix} Y_0 & I_n \\ I_n & 0 \end{smallmatrix}\right]$, which is the $(k-x+1)$th block-column of $(\Pi_\ell^n)^{\mathcal{B}}L_P(\lambda)$, equals $u_1(\lambda)$ and, therefore, it is of the form $-e_i \otimes I_n + \lambda e_{i+1} \otimes I_n$, with $i \leq q$. Moreover, the first block-column $u'_1(\lambda)$ of $U(\lambda)\left[\begin{smallmatrix} Y_0 & I_n \\ I_n & 0 \end{smallmatrix}\right]$, which is the $(k-x)$th block-column of $(\Pi_\ell^n)^{\mathcal{B}}L_P(\lambda)$, equals $u_2(\lambda) + u_1(\lambda)Y_0$. Thus, the $(k-x)$th block-column of $(\Pi_\ell^n)^{\mathcal{B}}L_P(\lambda)$ is, generically, not of the form $-e_i \otimes I_n + \lambda e_{i+1} \otimes I_n$.

We consider two cases, namely, the index x is a Type I or a Type II index relative to the tuple $(\boldsymbol{\ell}_q, \mathbf{q}, \mathbf{r}'_q)$.

Case I: Assume that x is a Type I index relative to $(\boldsymbol{\ell}_q, \mathbf{q}, \mathbf{r}'_q)$. In view of Remark 4.5,

$$\operatorname{heads}(\boldsymbol{\ell}_q, \mathbf{q}, \mathbf{r}'_q, x) = (\operatorname{heads}(\boldsymbol{\ell}_q, \mathbf{q}, \mathbf{r}'_q) \cup \{x\}) \setminus \{x-1\}.$$

This, together with Lemma 4.13 and part (iii.a) of Lemma 5.5 implies that, for (b) to hold for $L_P(\lambda)$ we only need to show that the $(k-x)$th block-column of $(\Pi_\ell^n)^{\mathcal{B}}L_P(\lambda)$ is not of the form $-e_i \otimes I_n + \lambda e_{i+1} \otimes I_n$ (which we already proved above) and the $(k-x+1)$th block-column is a wing column of the form $-e_i \otimes I_n + \lambda e_{i+1} \otimes I_n$, which follows from (ii.b) in Lemma 5.5 and the comments above.

Case II: Assume that x is a Type II index relative to $(\boldsymbol{\ell}_q, \mathbf{q}, \mathbf{r}'_q)$. In view of Remark 4.5,

$$\operatorname{heads}(\boldsymbol{\ell}_q, \mathbf{q}, \mathbf{r}'_q, x) = \operatorname{heads}(\boldsymbol{\ell}_q, \mathbf{q}, \mathbf{r}'_q) \cup \{x\}.$$

This, together with Lemma 4.13 and part (iii.a) of Lemma 5.5, implies that, for part (b) to hold for $L_P(\lambda)$, we only need to show that the $(k-x)$th block-column of $(\Pi_\ell^n)^{\mathcal{B}}L_P(\lambda)$ is not of the form $-e_i \otimes I_n + \lambda e_{i+1} \otimes I_n$, which we already proved above.

Finally, we prove part (c). Notice that part (iv.a) in Lemma 5.5 implies that the wing block-rows of $L_P(\lambda)$ relative to $(\boldsymbol{\ell}, \mathbf{r}, p, q)$ and the wing block-rows of $L'(\lambda)$ relative to $(\boldsymbol{\ell}, \mathbf{r}', p, q)$ are located at the same positions. This, in turn, implies that the wing block-rows of $L'(\lambda)\Pi_{\mathbf{r}'}^n$ relative to $(\boldsymbol{\ell}, \mathbf{id}, p, q)$ and the wing block-rows of $L_P(\lambda)\Pi_{\mathbf{r}}^n$ relative to $(\boldsymbol{\ell}, \mathbf{id}, p, q)$ are located also at the same positions. Moreover, in view of (8.2) and (8.3), the wing block-rows of $L'(\lambda)\Pi_{\mathbf{r}'}^n$ and $L_P(\lambda)\Pi_{\mathbf{r}}^n$ are equal. Additionally, Lemma 4.15 implies $\operatorname{heads}(x, \operatorname{rev}(\mathbf{r}'_q), \operatorname{rev}(\mathbf{q}), \operatorname{rev}(\boldsymbol{\ell}_q)) = \operatorname{heads}(\operatorname{rev}(\mathbf{r}'_q), \operatorname{rev}(\mathbf{q}), \operatorname{rev}(\boldsymbol{\ell}_q))$. Thus, $j \in \operatorname{heads}(x, \operatorname{rev}(\mathbf{r}'_q), \operatorname{rev}(\mathbf{q}), \operatorname{rev}(\boldsymbol{\ell}_q))$ if and only if $j \in \operatorname{heads}(\operatorname{rev}(\mathbf{r}'_q), \operatorname{rev}(\mathbf{q}), \operatorname{rev}(\boldsymbol{\ell}_q))$. This, together with Lemma 4.13, implies part (c).

It now follows that, when this lemma holds for a GFPR of the form $M_{\boldsymbol{\ell}_q}(\mathcal{X})(\lambda M_{-k}^P - M_{\mathbf{q}}^P)$, then it also holds for $M_{\boldsymbol{\ell}_q}(\mathcal{X})(\lambda M_{-k}^P - M_{\mathbf{q}}^P)M_{\mathbf{r}_q}(\mathcal{Y})$ for arbitrary $\mathbf{r}_q$ such that $(\boldsymbol{\ell}_q, \mathbf{q}, \mathbf{r}_q)$ satisfies the SIP and arbitrary matrix assignment $\mathcal{Y}$. Since this lemma holds for Fiedler pencils $\lambda M_{-k}^P - M_{\mathbf{q}}^P$, we have proven that it holds for $(\lambda M_{-k}^P - M_{\mathbf{q}}^P)M_{\mathbf{r}_q}(\mathcal{Y})$ for arbitrary $\mathbf{r}_q$ such that $(\mathbf{q}, \mathbf{r}_q)$ satisfies the SIP and arbitrary $\mathcal{Y}$. We have to show that the lemma holds for $M_{\boldsymbol{\ell}_q}(\mathcal{X})(\lambda M_{-k}^P - M_{\mathbf{q}}^P)M_{\mathbf{r}_q}(\mathcal{Y})$, when $\boldsymbol{\ell}_q \neq \emptyset$. To this end, let $Q(\lambda)$ be a GFPR associated with $P(\lambda)$ for which this lemma holds. We now show that this lemma also holds for $Q(\lambda)^{\mathcal{B}}$.

Let $\Pi_{\boldsymbol{\ell}}^n$ and $\Pi_{\mathbf{r}}^n$ be block-permutation matrices such that $(\Pi_{\boldsymbol{\ell}}^n)^{\mathcal{B}}Q(\lambda)\Pi_{\mathbf{r}}^n$ is an extended (p, n, q, n)-block Kronecker pencil whose body satisfies the AS condition for $P(\lambda)$ partitioned as in (3.4). Then,

$$((\Pi_{\boldsymbol{\ell}}^n)^{\mathcal{B}}Q(\lambda)\Pi_{\mathbf{r}}^n)^{\mathcal{B}} = (\Pi_{\mathbf{r}}^n)^{\mathcal{B}}Q(\lambda)^{\mathcal{B}}\Pi_{\boldsymbol{\ell}}^n = \left[\begin{array}{c|c} M(\lambda)^{\mathcal{B}} & K_1(\lambda)^{\mathcal{B}} \\ \hline (K_2(\lambda)^T)^{\mathcal{B}} & 0 \end{array}\right].$$

It is not difficult to show that the pencil above is an extended (q, n, p, n)-block Kronecker pencil. Moreover, since block-transposition maps each antidiagonal of a block-matrix into itself, the pencil $M(\lambda)^{\mathcal{B}}$ satisfies the AS condition for $P(\lambda)$, as $M(\lambda)$ does. Property (a) holds for $Q(\lambda)^{\mathcal{B}}$ because its block-rows and block-columns are, respectively, the block-columns and the block-rows of $Q(\lambda)$, for which property (a) holds by assumption. Furthermore, properties (b) and (c) for $Q(\lambda)$ imply, respectively, that properties (c) and (b) hold for $Q(\lambda)^{\mathcal{B}}$. Now, since this lemma holds for $Q(\lambda) = (\lambda M_{-k}^P - M_{\mathbf{q}}^P)M_{\mathbf{r}_q}(\mathcal{Y})$ as noted above, it also holds for $Q(\lambda)^{\mathcal{B}} = M_{\mathrm{rev}(\mathbf{r}_q)}(\mathrm{rev}(\mathcal{Y}))(\lambda M_{-k}^P - M_{\mathrm{rev}(\mathbf{q})}^P)$ (recall Lemma 4.6). Consider some arbitrary $\boldsymbol{\ell}_q$ such that $(\boldsymbol{\ell}_q, \mathbf{q}, \mathbf{r}_q)$ satisfies the SIP and an arbitrary matrix assignment $\mathcal{X}$ for $\boldsymbol{\ell}_q$. By the comments above, this lemma also holds for $M_{\mathrm{rev}(\mathbf{r}_q)}(\mathrm{rev}(\mathcal{Y}))(\lambda M_{-k}^P - M_{\mathrm{rev}(\mathbf{q})}^P)M_{\mathrm{rev}(\boldsymbol{\ell}_q)}(\mathrm{rev}(\mathcal{X}))$, and hence for $M_{\boldsymbol{\ell}_q}(\mathcal{X})(\lambda M_{-k}^P - M_{\mathbf{q}}^P)M_{\mathbf{r}_q}(\mathcal{Y})$. This establishes the lemma for arbitrary $\boldsymbol{\ell}_q$ and $\mathbf{r}_q$. □

REMARK 8.4. *We note that part (c) in Theorem 8.2 implies that the block-permutations* $\Pi_{\mathbf{r}}^n$ *and* $\Pi_{\boldsymbol{\ell}}^n$ *in* (8.2) *are, respectively, of the form* $I_n \oplus \Pi_{\widetilde{\mathbf{r}}}^n$ *and* $I_n \oplus \Pi_{\widetilde{\boldsymbol{\ell}}}^n$*, for some permutations* $\widetilde{\mathbf{r}}$ *and* $\widetilde{\boldsymbol{\ell}}$ *of the set* $\{1 : k-1\}$.

Another auxiliary result to prove Theorem 8.1 is Theorem 8.3. In the proof of this result, we will make use of the following properties of elementary matrices:

$$R_{k,n}M_i(B)R_{k,n} = M_{-k+i}(B), \quad \text{for } i = 0 : k-1 \text{ and arbitrary } B, \tag{8.9}$$

$$R_{k,n}M_{-i}(B)R_{k,n} = M_{k-i}(B), \quad \text{for } i = 1 : k \text{ and arbitrary } B, \tag{8.10}$$

where $R_{s,n}$ is the block sip matrix defined in (2.5). Moreover, to keep the notation simple, we will omit the second index of $R_{s,n}$ and just write R_s, since it is going to remain constant and equal to n throughout the whole proof.

THEOREM 8.5. *Let* $P(\lambda) = \sum_{i=0}^k A_i\lambda^i \in \mathbb{F}[\lambda]^{n\times n}$ *be a matrix polynomial of grade* $k \geq 2$*. Let* $\mathbf{z}$ *be a permutation of* $\{-k : -1\}$ *and let*

$$L_P(\lambda) = M_{\boldsymbol{\ell}_z}(\mathcal{Z})(\lambda M_{\mathbf{z}}^P - M_0^P)M_{\mathbf{r}_z}(\mathcal{W}),$$

be a GFPR associated with $P(\lambda)$*. Then, there exist block-permutation matrices* $\Pi_{\boldsymbol{\ell}}^n$ *and* $\Pi_{\mathbf{r}}^n$ *such that*

$$C(\lambda) := (\Pi_{\boldsymbol{\ell}}^n)^{\mathcal{B}} L_P(\lambda)\Pi_{\mathbf{r}}^n = \left[\begin{array}{c|c} 0 & L_1(\lambda) \\ \hline L_2(\lambda)^T & N(\lambda) \end{array}\right] \begin{array}{l} \} \, (\mathfrak{h}(k+\mathbf{z})-1)n \\ \} \, \mathfrak{h}(\mathrm{rev}(k+\mathbf{z}))n \end{array}, \tag{8.11}$$

$$\underbrace{\qquad}_{(\mathfrak{h}(\mathrm{rev}(k+\mathbf{z}))-1)n} \quad \underbrace{\qquad}_{\mathfrak{h}(k+\mathbf{z})n}$$

where the pencil $N(\lambda)$ *satisfies the AS condition for* $P(\lambda)$*, and where* $L_1(\lambda)$ *and* $L_2(\lambda)$ *are wing pencils. Moreover, the last block-row of* $C(\lambda)$ *is the last block-row of* $(\Pi_{\boldsymbol{\ell}}^n)^{\mathcal{B}}L_P(\lambda)$*, and the last block-column of* $C(\lambda)$ *is the last block-column of* $L_P(\lambda)\Pi_{\mathbf{r}}^n$*, respectively. Additionally, the block-entry of* $N(\lambda)$ *in position* $(\mathfrak{h}(\mathrm{rev}(k+\mathbf{z})), \mathfrak{h}(k+\mathbf{z}))$ *equals* $\lambda A_1 + A_0$.

Proof. Let $\widehat{P}(\lambda) := \mathrm{rev}(-P(\lambda)) = \sum_{i=0}^k -A_{k-i}\lambda^i$ and let $L_P(\lambda) =: \lambda L_1 - L_0$. Let us consider the pencil

$$\begin{aligned}\widehat{L}(\lambda) :=& R_k\,\mathrm{rev}(-L_P(\lambda))R_k = \lambda R_k L_0 R_k - R_k L_1 R_k = \\ & \lambda R_k M_{\boldsymbol{\ell}_z,\mathbf{r}_z}(\mathcal{Z},\mathcal{W})M_0^P R_k - R_k M_{\boldsymbol{\ell}_z}(\mathcal{Z})M_{\mathbf{z}}^P M_{\mathbf{r}_z}(\mathcal{W})R_k.\end{aligned}$$

Since $\mathbf{z}$ is a permutation of $\{-k : -1\}$, there exists a permutation $\mathbf{q}$ of $\{0 : k-1\}$ such that $\mathbf{z} = -k + \mathbf{q}$. Taking into account (8.9)–(8.10) and the fact that R_k is nonsingular with $R_k^{-1} = R_k$, it is not difficult to see that

$$\widehat{L}(\lambda) = M_{k+\boldsymbol{\ell}_z}(\mathcal{Z})(\lambda M_{-k}^{\widehat{P}} - M_{\mathbf{q}}^{\widehat{P}})M_{k+\mathbf{r}_z}(\mathcal{W}),$$

which is a GFPR associated with $\widehat{P}(\lambda)$. Notice that if $(\boldsymbol{\ell}_z, \mathbf{z}, \mathbf{r}_z)$ satisfies the SIP, so does $k + (\boldsymbol{\ell}_z, \mathbf{z}, \mathbf{r}_z)$. By Theorem 8.2, there exist block-permutation matrices $\Pi^n_{\boldsymbol{\ell}'}$ and $\Pi^n_{\mathbf{r}'}$ such that

$$(\Pi^n_{\boldsymbol{\ell}'})^{\mathcal{B}} \widehat{L}(\lambda) \Pi^n_{\mathbf{r}'} = \left[\begin{array}{c|c} M(\lambda) & K_2(\lambda)^T \\ \hline K_1(\lambda) & 0 \end{array}\right]. \tag{8.12}$$

is an extended $(\mathfrak{h}(\mathbf{q}) - 1, n, \mathfrak{h}(\mathrm{rev}(\mathbf{q})) - 1, n)$-block Kronecker pencil, whose body $M(\lambda)$ satisfies the AS condition for $\widehat{P}(\lambda)$. Then, notice that

$$\begin{aligned} -\mathrm{rev}(R_k(\Pi^n_{\boldsymbol{\ell}'})^{\mathcal{B}} \widehat{L}(\lambda) \Pi^n_{\mathbf{r}'} R_k) &= -\mathrm{rev}[R_k(\Pi^n_{\boldsymbol{\ell}'})^{\mathcal{B}} R_k \,\mathrm{rev}(-L(\lambda)) R_k \Pi^n_{\mathbf{r}'} R_k] \\ &= \mathrm{rev}[(R_k(\Pi^n_{\boldsymbol{\ell}'})^{\mathcal{B}} R_k)\,\mathrm{rev}(L(\lambda))(R_k \Pi^n_{\mathbf{r}'} R_k)] \\ &= (R_k(\Pi^n_{\boldsymbol{\ell}'})^{\mathcal{B}} R_k) L(\lambda) (R_k \Pi^n_{\mathbf{r}'} R_k) \\ &= (\Pi^n_{\boldsymbol{\ell}})^{\mathcal{B}} L_P(\lambda) \Pi^n_{\mathbf{r}}, \end{aligned}$$

where $\Pi^n_{\boldsymbol{\ell}} := R_k \Pi^n_{\boldsymbol{\ell}'} R_k$ and $\Pi^n_{\mathbf{r}} := R_k \Pi^n_{\mathbf{r}'} R_k$ are block-permutation matrices. On the other hand, setting $p := \mathfrak{h}(\mathbf{q}) - 1$ and $q := \mathfrak{h}(\mathrm{rev}(\mathbf{q})) - 1$, from (8.12), we get

$$\begin{aligned} -\mathrm{rev}(R_k(\Pi^n_{\boldsymbol{\ell}'})^{\mathcal{B}} \widehat{L}(\lambda) \Pi^n_{\mathbf{r}'} R_k) &= -\mathrm{rev}\left(R_k \left[\begin{array}{c|c} M(\lambda) & K_2(\lambda)^T \\ \hline K_1(\lambda) & 0 \end{array}\right] R_k\right) \\ &= \left[\begin{array}{cc} 0 & -\mathrm{rev}(R_p K_1(\lambda) R_{p+1}) \\ -\mathrm{rev}(R_q K_2(\lambda) R_{q+1})^T & -\mathrm{rev}(R_{q+1} M(\lambda) R_{p+1}) \end{array}\right]. \end{aligned}$$

Note that, if $K(\lambda)$ is a wing pencil, then $-\mathrm{rev}(RK(\lambda)R)$ is also a wing pencil. Thus, letting $N(\lambda) := -\mathrm{rev}(R_{q+1} M(\lambda) R_{p+1})$, $L_1(\lambda) := -\mathrm{rev}(R_p K_1(\lambda) R_{p+1})$ and $L_2(\lambda) := -\mathrm{rev}(R_q K_2(\lambda) R_{q+1})$, we obtain that (8.11) holds.

Writing $M(\lambda) = \lambda H_1 + H_0$, next, we show that $N(\lambda) = -\lambda R_{q+1} H_0 R_{p+1} - R_{q+1} H_1 R_{p+1}$ satisfies the AS condition for $P(\lambda)$. As shown above, the pencil $M(\lambda)$ is a $(q+1) \times (p+1)$ block-pencil with blocks of size $n \times n$, since $(\Pi^n_{\boldsymbol{\ell}'})^{\mathcal{B}} \widehat{L}(\lambda) \Pi^n_{\mathbf{r}'}$ is an extended (p, n, q, n)-block Kronecker pencil. Moreover, $p + q + 1 = k$ and $M(\lambda)$ satisfies the AS condition for $\widehat{P}(\lambda)$. We observe that $[R_{q+1} M(\lambda) R_{p+1}]_{ij} = [M(\lambda)]_{q+2-i, p+2-j}$. Thus, using the notation in Definition 3.6, we have

$$\begin{aligned} AS(N, s) &= \sum_{i+j=k+2-s} [-R_{q+1} H_0 R_{p+1}]_{ij} + \sum_{i+j=k+1-s} [-R_{q+1} H_1 R_{p+1}]_{ij} \\ &= -\left\{ \sum_{i+j=k+2-s} [H_0]_{q+2-i, p+2-j} + \sum_{i+j=k+1-s} [H_1]_{q+2-i, p+2-j} \right\} \\ &= -\left\{ \sum_{i+j=k+1-(k-s)} [H_0]_{i,j} + \sum_{i+j=k+2-(k-s)} [H_1]_{i,j} \right\} = -(-A_{k-(k-s)}) = A_s, \end{aligned}$$

where the third equality follows using the fact that $p + q + 1 = k$ and the fourth equality follows from the fact that $M(\lambda)$ satisfies the AS condition for $\widehat{P}(\lambda)$.

Finally, notice that, by part (a) in Theorem 8.2, the first block-row and the first block-column of the pencil $\widehat{L}(\lambda)$ are, respectively, the first block-row and the first block-column of the pencil $(\Pi^n_{\boldsymbol{\ell}'})^{\mathcal{B}} \widehat{L}(\lambda) \Pi^n_{\mathbf{r}'}$, and the block-entry in position $(1,1)$ of $M(\lambda)$ is $-\lambda A_0 - A_1$. Since $N(\lambda) = -\mathrm{rev}(R_{q+1} M(\lambda) R_{p+1})$, the block-entry in position $(q+1, p+1)$ of $N(\lambda)$

is $\lambda A_1 + A_0$. Moreover, since $L_P(\lambda) = -\operatorname{rev}(R_k \widehat{L}(\lambda) R_k)$ and $C(\lambda) = (\Pi_{\boldsymbol{\ell}}^n)^{\mathcal{B}} L_P(\lambda) \Pi_{\mathbf{r}}^n = -\operatorname{rev}[R_k (\Pi_{\boldsymbol{\ell}'}^n)^{\mathcal{B}} \widehat{L}(\lambda) \Pi_{\mathbf{r}'}^n R_k]$, the claim about the last block-row and the last block-column of $L_P(\lambda)$ and $C(\lambda)$ follows. □

We will refer to pencils as in (8.11) as a *reversed* extended block Kronecker pencils.

REMARK 8.6. *We note that Theorem 8.2 implies that* $\Pi_{\mathbf{r}}^n$ *and* $\Pi_{\boldsymbol{\ell}}^n$ *in* (8.11) *are, respectively, of the form* $\Pi_{\widetilde{\mathbf{r}}}^n \oplus I_n$ *and* $\Pi_{\widetilde{\boldsymbol{\ell}}}^n \oplus I_n$*, for some permutations* $\widetilde{\mathbf{r}}$ *and* $\widetilde{\boldsymbol{\ell}}$ *of the set* $\{1 : k-1\}$.

8.1. Proof of Theorem 8.1. Armed with Lemma 4.22 together with Theorems 8.2 and 8.3, we are in a position to prove Theorem 8.1.

Proof. (of Theorem 8.1) Let $P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i \in \mathbb{F}[\lambda]^{n \times n}$ be a matrix polynomial of degree k. Let $h \in \{0 : k-1\}$ and let $\mathbf{q}$ be a permutation of $\{0 : h\}$. Let $\mathbf{z}$ be a permutation of $\{-k : -h-1\}$ and let $L_P(\lambda)$ be a GFPR as in (4.8). By Lemma 4.22, the pencil $L_P(\lambda)$ can be written in the following form:

$$L_P(\lambda) = \left[\begin{array}{c|c|c} D_{\mathbf{z}}(\lambda) & y_{\mathbf{z}}(\lambda) & 0 \\ \hline x_{\mathbf{z}}(\lambda) & c(\lambda) & x_{\mathbf{q}}(\lambda) \\ \hline 0 & y_{\mathbf{q}}(\lambda) & D_{\mathbf{q}}(\lambda) \end{array}\right],$$

where $D_{\mathbf{q}}(\lambda) \in \mathbb{F}[\lambda]^{nh \times nh}$, $D_{\mathbf{z}}(\lambda) \in \mathbb{F}[\lambda]^{n(k-h-1) \times n(k-h-1)}$, $c(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$, $x_{\mathbf{z}}(\lambda) \in \mathbb{F}[\lambda]^{n \times n(k-h-1)}$, $x_{\mathbf{q}}(\lambda) \in \mathbb{F}[\lambda]^{n \times nh}$, $y_{\mathbf{z}}(\lambda) \in \mathbb{F}[\lambda]^{n(k-h-1) \times n}$ and $y_{\mathbf{q}}(\lambda) \in \mathbb{F}[\lambda]^{nh \times n}$. In addition, let us denote

$$F(\lambda) := \left[\begin{array}{cc} c(\lambda) & x_{\mathbf{q}}(\lambda) \\ y_{\mathbf{q}}(\lambda) & D_{\mathbf{q}}(\lambda) \end{array}\right], \quad \text{and} \quad G(\lambda) := \left[\begin{array}{cc} D_{\mathbf{z}}(\lambda) & y_{\mathbf{z}}(\lambda) \\ x_{\mathbf{z}}(\lambda) & c(\lambda) \end{array}\right].$$

Then, by Theorem 4.22, we get that

$$F(\lambda) = M_{\boldsymbol{\ell}_q}(\mathcal{X})(\lambda M_{-h-1}^{Q} - M_{\mathbf{q}}^{Q}) M_{\mathbf{r}_q}(\mathcal{Y})$$

is a GFPR associated with $Q(\lambda) := \lambda^{h+1} A_{h+1} + \lambda^h A_h + \cdots + \lambda A_1 + A_0$. Thus, from part (a) in Theorem 8.2, we obtain that $c(\lambda) = \lambda A_{h+1} + A_h$. Also by Theorem 8.2 and according to Remark 8.2, there exist block-permutation matrices $\Pi_{\mathbf{r}_1} = I_n \oplus \Pi_{\widetilde{\mathbf{r}}_1}$ and $\Pi_{\boldsymbol{\ell}_1} = I_n \oplus \Pi_{\widetilde{\boldsymbol{\ell}}_1}$ such that

$$\Pi_{\boldsymbol{\ell}_1}^{\mathcal{B}} F(\lambda) \Pi_{\mathbf{r}_1} = \begin{bmatrix} I_n & 0 \\ 0 & \Pi_{\widetilde{\boldsymbol{\ell}}_1}^{\mathcal{B}} \end{bmatrix} \begin{bmatrix} \lambda A_{h+1} + A_h & x_{\mathbf{q}}(\lambda) \\ y_{\mathbf{q}}(\lambda) & D_{\mathbf{q}}(\lambda) \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & \Pi_{\widetilde{\mathbf{r}}_1} \end{bmatrix} =$$

$$\begin{bmatrix} \lambda A_{h+1} + A_h & x_{\mathbf{q}}(\lambda) \Pi_{\widetilde{\mathbf{r}}_1} \\ \Pi_{\widetilde{\boldsymbol{\ell}}_1}^{\mathcal{B}} y_{\mathbf{q}}(\lambda) & \Pi_{\widetilde{\boldsymbol{\ell}}_1}^{\mathcal{B}} D_{\mathbf{q}}(\lambda) \Pi_{\widetilde{\mathbf{r}}_1} \end{bmatrix} = \left[\begin{array}{c|c} M(\lambda) & K_2(\lambda)^T \\ \hline K_1(\lambda) & 0 \end{array}\right] =:$$

$$\left[\begin{array}{cc|c} \lambda A_{h+1} + A_h & m_1(\lambda) & k_2(\lambda)^T \\ m_2(\lambda) & \widetilde{M}(\lambda) & \widetilde{K}_2(\lambda)^T \\ \hline k_1(\lambda) & \widetilde{K}_1(\lambda) & 0 \end{array}\right]$$

is an extended $(\mathfrak{h}(\mathbf{q}) - 1, n, \mathfrak{h}(\operatorname{rev}(\mathbf{q})) - 1, n)$-block Kronecker pencil for $Q(\lambda)$.

Furthermore, by Theorem 4.22, the pencil

$$G(\lambda) = M_{\boldsymbol{\ell}_z}(\mathcal{Z})(\lambda M_{h+\mathbf{z}}^{Z} - M_0^{Z}) M_{\mathbf{r}_z}(\mathcal{W})$$

is a GFPR associated with $Z(\lambda) := \lambda^{k-h} A_k + \lambda^{k-h-1} A_{k-1} + \cdots + \lambda A_{h+1} + A_h$. By Theorem 8.3 and according to Remark 8.3, there exist block-permutation matrices $\Pi_{\mathbf{r}_2} = \Pi_{\widetilde{\mathbf{r}}_2} \oplus I_n$

and $\Pi_{\boldsymbol{\ell}_2} = \Pi_{\widetilde{\boldsymbol{\ell}}_2} \oplus I_n$ such that

$$\Pi^{\mathcal{B}}_{\boldsymbol{\ell}_2} G(\lambda) \Pi_{\mathbf{r}_2} = \begin{bmatrix} \Pi^{\mathcal{B}}_{\widetilde{\boldsymbol{\ell}}_2} & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} D_{\mathbf{z}}(\lambda) & y_{\mathbf{z}}(\lambda) \\ x_{\mathbf{z}}(\lambda) & \lambda A_{h+1} + A_h \end{bmatrix} \begin{bmatrix} \Pi_{\widetilde{\mathbf{r}}_2} & 0 \\ 0 & I_n \end{bmatrix} =$$
$$\begin{bmatrix} \Pi^{\mathcal{B}}_{\widetilde{\boldsymbol{\ell}}_2} D_{\mathbf{z}}(\lambda) \Pi_{\widetilde{\mathbf{r}}_2} & \Pi^{\mathcal{B}}_{\widetilde{\boldsymbol{\ell}}_2} y_{\mathbf{z}}(\lambda) \\ x_{\mathbf{z}}(\lambda) \Pi_{\widetilde{\mathbf{r}}_2} & \lambda A_{h+1} + A_h \end{bmatrix} = \left[\begin{array}{c|c} 0 & L_1(\lambda) \\ \hline L_2(\lambda)^T & N(\lambda) \end{array}\right] =:$$
$$\left[\begin{array}{c|cc} 0 & \widetilde{T}_1(\lambda) & t_1(\lambda) \\ \hline \widetilde{T}_2(\lambda)^T & \widetilde{N}(\lambda) & n_1(\lambda) \\ t_2(\lambda)^T & n_2(\lambda) & \lambda A_{h+1} + A_h \end{array}\right]$$

is a reversed block Kronecker pencil for $Z(\lambda)$. Then, notice that

$$\begin{bmatrix} \Pi^{\mathcal{B}}_{\widetilde{\boldsymbol{\ell}}_2} & & \\ & I_n & \\ & & \Pi^{\mathcal{B}}_{\widetilde{\boldsymbol{\ell}}_1} \end{bmatrix} \begin{bmatrix} D_{\mathbf{z}}(\lambda) & y_{\mathbf{z}}(\lambda) & 0 \\ x_{\mathbf{z}}(\lambda) & \lambda A_{h+1} + A_h & x_{\mathbf{q}}(\lambda) \\ 0 & y_{\mathbf{q}}(\lambda) & D_{\mathbf{q}}(\lambda) \end{bmatrix} \begin{bmatrix} \Pi_{\widetilde{\mathbf{r}}_2} & & \\ & I_n & \\ & & \Pi_{\widetilde{\mathbf{r}}_1} \end{bmatrix} =$$
$$\begin{bmatrix} 0 & \widetilde{T}_1(\lambda) & t_1(\lambda) & 0 & 0 \\ \widetilde{T}_2(\lambda)^T & \widetilde{N}(\lambda) & n_1(\lambda) & 0 & 0 \\ t_2(\lambda)^T & n_2(\lambda) & \lambda A_{h+1} + A_h & m_1(\lambda) & k_2(\lambda)^T \\ 0 & 0 & m_2(\lambda) & \widetilde{M}(\lambda) & \widetilde{K}_2(\lambda)^T \\ 0 & 0 & k_1(\lambda) & \widetilde{K}_1(\lambda) & 0 \end{bmatrix},$$

which is block-permutationally equivalent to the pencil

$$\left[\begin{array}{ccc|cc} \widetilde{N}(\lambda) & n_1(\lambda) & 0 & \widetilde{T}_2(\lambda)^T & 0 \\ n_2(\lambda) & \lambda A_{h+1} + A_h & m_1(\lambda) & t_2(\lambda)^T & k_2(\lambda)^T \\ 0 & m_2(\lambda) & \widetilde{M}(\lambda) & 0 & \widetilde{K}_2(\lambda)^T \\ \hline \widetilde{T}_1(\lambda) & t_1(\lambda) & 0 & 0 & 0 \\ 0 & k_1(\lambda) & \widetilde{K}_1(\lambda) & 0 & 0 \end{array}\right] =: \left[\begin{array}{c|c} H(\lambda) & S_2(\lambda)^T \\ \hline S_1(\lambda)^T & 0 \end{array}\right],$$

where $S_1(\lambda)$ and $S_2(\lambda)$ are wing pencils by Theorem 2.8. Thus, to finish the proof, we only need to check that the pencil $H(\lambda)$ satisfies the AS condition for $P(\lambda)$. Indeed, decomposing the pencil $H(\lambda)$ as follows

$$\underbrace{\begin{bmatrix} \widetilde{N}(\lambda) & n_1(\lambda) & 0 \\ n_2(\lambda) & \lambda A_{h+1} + A_h & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{=:H_1(\lambda)} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda A_{h+1} + A_h & m_1(\lambda) \\ 0 & m_2(\lambda) & \widetilde{M}(\lambda) \end{bmatrix}}_{=:H_2(\lambda)} - \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda A_{h+1} + A_h & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{=:H_3(\lambda)},$$

and using the linearity of the block antidiagonal sum, we obtain $\mathrm{AS}(H, s) = \mathrm{AS}(H_1, s) + \mathrm{AS}(H_2, s) - \mathrm{AS}(H_3, s)$, for $s = 0 : k$. Finally, using that $M(\lambda)$ and $N(\lambda)$ satisfy the AS condition for $Q(\lambda)$ and $Z(\lambda)$, respectively, it follows easily from the equation above that the pencil $H(\lambda)$ satisfies the AS condition for $P(\lambda)$. □

9. Conclusions. In the last decade, many new families of linearizations for a matrix polynomial have been constructed. The families of square Fiedler-like pencils, that include the Fiedler pencils, the generalized Fiedler pencils, the Fiedler pencils with repetition and the generalized Fiedler pencils with repetition, consist of pencils with good properties but whose

definition involves products of the so-called elementary matrices. This fact is a disadvantage when proving theorems about these pencils since the proofs can become very involved. Also, each of these families, although related, are studied separately in the literature. Recently, a new family of pencils associated with a matrix polynomial $P(\lambda)$, called the block minimal bases pencils, was introduced. The pencils in this family are defined in terms of their block-structure and it is straightforward to determine when they are strong linearizations of $P(\lambda)$. A subfamily of this family, called the block-Kronecker pencils was also identified and it was proven that the Fiedler pencils are in this family, up to permutations of rows and columns. In this paper, we prove with considerable effort that, with the exception of the non-proper generalized Fiedler pencils, all Fiedler-like pencils are, up to permutations, inside a new class of pencils called extended block-Kronecker pencils that can be easily described in terms of their block entries. In addition, these extended block-Kronecker pencils are under some generic nonsingularity conditions strong block minimal bases pencils, and, so, strong linearizations. As a consequence, essentially all Fiedler-like pencils and their main properties can now be described in a very simple way, avoiding, if desired, at all the complicated use of index tuples and matrix products. We expect this result will contribute to get further developments in the area.

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