ALGORITHMS, DEHN FUNCTIONS, AND AUTOMATIC GROUPS

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Abstract. Ever since it was shown in the mid 1980s that being word hyperbolic was equivalent to the existence of a linear time solution to the word problem, computational and algorithmic issues have been intimately intertwined with geometric and topological ones in the study of infinite groups. The scope of this chapter is indicated by the three topics in its title: algorithms, Dehn functions, and automatic groups.

Contents

This chapter focuses on the 1-dimensional aspects of geometric group theory to such an extent that an alternative title could have been "The role of graphs in geometric group theory". Virtually every concepts focused on in this chapter can be defined and thought about in terms of graphs with implicit group actions. In particular, Cayley graphs play a starring role.

Date: May 9, 2007
NOT EVEN A DRAFT YET.

As you will quickly notice, first two parts are currently incoherent scaps of notes to myself, while part 3 is close to a rough draft. Since the conference has already started, I thought I should go ahead and post this anyway so people have SOMETHING to look at, but I'll keep on working on fleshing things out as the week continues.

Part 1. Decidability, Algorithms, and Computational Complexity

There are three basic steps in the evolution of our understanding of a problem from a computational viewpoint and they coincide with the way in which the questions have have been organized.

1. Can we do X? (decidability),
2. How do we do X? (algorithms), and
3. How fast can we do X? (computational complexity).

All questions involving all three stages have been around for nearly a century and these types of issues continue to thoroughly permeate combinatorial and computational group theory. In order to keep the scope of the questions in this part within reason, I have tried to limit them to questions that are relevant to *geometric* group theory and geometric group theorists.

1. Decision Problems

The obvious place to begin is with Max Dehn's three famous problems from his 1910(12?) article add cite: the word problem (WP), conjugacy problem (CP) and the isomorphism problem (IP). To these we will include the generalized word problem (GWP). [briefly describe in one sentence]

Random notes.

The group $\mathbb{F}_2 \times \mathbb{F}_2$ has an unsolvable GWP, so subgroups of biautomatic, semi-hyperbolic, or CAT(0) groups can exhibit undecidable behavior.

Even stronger, Rips has a construction that can be used to import bad behavior into negatively curved gorups.

unsolvable problems in small cancellation and word-hyperbolic groups [3].

The CP is NOT a geometric invariant and is not true when passing to or from a subgroup of finite index. Look up the two different index 2 counterexamples.

For the early history of these types of problems there is the 1971 book [30] by Chuck Miller and his 1992 survey article [31]. More broadly there is Bruce Chandler's history of Combinatorial Group Theory [16].

Specific Classes of Groups. The order of difficulty is typically WP/CP/GWP and the IP is of a different sort. Different standard classes of groups have are further along towards resolving these four classical decision problems. Here are three examples.

Question 1 (IP for Coxeter groups)**.** *The word and conjugacy problems for Coxeter groups are known to be solvable and much recent progress has been made on the isomorphism problem. Complete the solution of the isomorphism problem for Coxeter groups.*

Question 2 (CP for one-relator groups)**.** *The word problem for one-relator groups was solved by Magnus in 193x and the conjugacy problem is known for certain subclasses of one-relator groups. Solve the conjugacy problem for one-relator groups. groups.*

Question 3 (WP for Artin groups)**.** *Solve the word problem for an arbitrary Artin group.*

Polycyclic Groups. Polycyclic groups are one of the largest classes of groups for which all four fundamental problems can be solved. The class of polycyclic groups contains the class of nilpotent groups and it is contained in the class of solvable groups. All finite solvable groups are polycyclic so the class of finite polycyclic groups coincides with the class of finite solvable groups. In particular, the nice computational properties possessed by all polycyclic groups enhances their claim to be the appropriate (computational) generalization of finite solvable groups to the world of infinite group theory.

The basic theory of polycyclic groups and, in fact, a good survey of large portions of computational group theory are contained in the book "Computations with finitely presented groups" by Charles Sims [37].

Hyperbolic Groups. +WP and +CP. IP for torsion-free due to Sela.

3-Nanifold Groups. Jason's article. Only now, as a consequence of geometrization (Perleman) is the WP for compact 3-mfds known to be decidable. Is CP known? Ask Jason

Include the stuff that Sal Schleimer has been doing somewhere

Mapping Class Groups. automatic so +WP. +CP due to add cite. It is relatively straightforward to embed $\mathbb{F}_2 \times \mathbb{F}_2$ into many mapping class groups so those groups do not have a solvable GWP, (and IP is fairly trivial)

Free Group Automorphisms. +WP Nielsen, Whitehead, ask Karen for the appropriate references.

Coxeter Groups. Coxeter Groups are both CAT(0) groups (Moussong []) and automatic groups (Brink-Howlett [13]), both of which imply that the word and conjugacy problems are solvable. On the other hand, it is relatively straightforward to embed $\mathbb{F}_2 \times \mathbb{F}_2$ into a Coxeter group so the generalized membership problem is not always solvable. The isomorphism problem was virtually ignored for many years before attracting a flurry of activity since 2000.

add lots of citations: Charney-Davis, BMMN, Bahls, Mihalik, Muehlheer, etc. Talk about diagram twisting, and the even isomorphism theorem

Problem 4. *Complete the classification of Coxeter groups up to isometry.*

Artin Groups. (essentially nothing is known along these lines. ?WP ?CP ?IP and -GWP since $\mathbb{F}_2 \times \mathbb{F}_2$ is itself an Artin groups

One-relator Groups.

Question 5. *The word problem for one-relator groups was shown to be decidable by Magnus. Show that the conjugacy problem for one-relator groups is decidable.*

Question 6. *Is the generalized word problem solvable for one-relator groups?*

Question 7. *Is the isomorphism problem for one-relator groups decidable?*

Quadratic Dehn Function Groups. There are many classes of groups that are known to have a quadratic Dehn function as well as a decidable conjugacy problem. The conjugacy problem in general is open for these groups. The isomorphism problem is also open for these groups (as it is for the subclasses CAT(0), biautomatic, automatic, semi-hyperbolic, and free-by-cyclic). It is known to be undecidable for the class of groups with Dehn function at most $n^2 logn$. There are also such groups with undecidable conjugacy problems.

2. Algorithms

(how much to include?)

Question 8. *Is there a polynomial time algorithm to solve the conjugacy problem in mapping class groups?*

3-manifolds. algorithms for 3-manifolds. Stuff from Jason Mannings algorithms article (again).

Hyperbolic Groups. Finding δ for hyperbolic groups. (what else?)

CAT(0) Groups. PE and CAT(0) algorithms
Stuff with Murray, why its hard, reference the Durham survey article.

Question 9 (NPC for PE complexes)**.** *Find an effective algorithm that determines whether or not a finite piecewise Euclidean simplicial complex is non-positively curved.*

Automatic and Biautomatic Groups.

Question 10. *Given an word in the generators of an automatic group, is there an algorithm to determine whether it is a torsion element?*

Miscellaneous. Commutator length and stable commutator length. Cite Calegari-Fujiwara.

Question 11. *If G is perfect with +WP, can you compute the commutator length of an element by a uniform algorithm. What about genus 1 in particular? What about its stable commutator length? (i.e. its translation length with respect to the commutator generating set). CP?*

3. Computational Complexity

Most of the stuff that I can think of for this section would be much more computational group theory then geometric. Are there truly geometric group theory questions that focus on computational complexity?
Misha Kapovich.

Question 12. *Compute presentations for latttice in Lie groups (explicitly!!!)*

Part 2. Dehn Functions and the Isoperimetric Spectrum

Documents: Martin's survey article [7]
Tim Riley's long article on filling functions [39]

4. Dehn functions

Start with Gersten's result

Theorem 1. *For a finite presentation, the following are equivalent:*

- *The word problem is decidable.*
- $\text{AREA} : \mathbb{N} \to \mathbb{N}$ *is a recursive function, and*
- $\text{AREA} : \mathbb{N} \to \mathbb{N}$ *is bounded above by a recursive function.*

(pull some of the stuff from Tim Riley [39])
(look up the Magnus problem list)
The group $B = \langle a, b \mid a^{a^b} = a^2 \rangle$ was introduced by Gilbert Baumslag in add cite and it is sometimes called *Baumslag's group*. It was also studied by Gersten and Gersten's student add cites.

Question 13 (Gersten)**.** *Is the Dehn function of every one-relator group bounded above by the Dehn function of Baumslag's group* $B = \langle a, b \mid a^{a^b} = a^2 \rangle$*?*

Question 14. *What is the Dehn function for* $\text{AUT}(\mathbb{F}_n)$ *and* $\text{OUT}(\mathbb{F}_n)$*.*

It is known that the Dehn function is at least exponential and at most expontial (there are restrictions on these statements. Check with Karen)

5. Isoperimetric spectrum

Basic results

Theorem 2. *A group is hyperbolic iff it has a linear Dehn function, which is true iff it has a subquadratic Dehn function.*

Proofs that subquadratic is linear Bowditch [4], Olshanski, Gromov.
Group with quadratic Dehn function: CAT(0), semihyperbolic, automatic, biautomatic groups, free by $\mathbb{Z}$ groups have a quadratic Dehn function.
In [5] Noel Brady and Martin Bridson considered the following class of groups.

$$G_{p,q} = \langle a, b, s, t \mid ab = ba, s^{-1}a^q s = a^p b, t^{-1}a^q t = a^p b^{-1} \rangle$$

Theorem 3. *The Dehn function for* $G_{p,q}$ *is* $n^{2\log_2(2p/q)}$*. As a consequence, there is only one gap in the isoperimetric spectrum.*

These are examples of groups that they called *tubular groups*, iterated cyclic HNN extensions of $\mathbb{Z} \times \mathbb{Z}$.
The Birget-Olshanskii-Rips-Sapir results,
(contact Noel)

6. Other Filling Functions

Tim Riley stuff. How many of the variations should be included?

7. Higher order Dehn functions

Brady led results. Snowflake groups. Others?

Jason Behrstock mentioned something today about the asymptotic cones implying that the higher-order Dehn functions of mapping class groups are polynomial. Ask him about this.

Part 3. Automatic and Biautomatic Groups

In the late 1980s and early 1990s the theory of automatic groups was rapidly developed by a number of different researchers and the basic theory was recorded in a book and a survey article that were both published in 1992. These continue to be two of the main sources of basic information about automatic groups. The book is "Word processing in groups" by David Epstein, Jim Cannon, Derek Holt, Silvio Levy, Michael Paterson and Bill Thurston [21]. It carefully defines automatic, biautomatic, asynchronously automatic, combable, bicombable and asynchronously bicombable groups, and many foundational results about these classes of groups appear there in print for the first time.[1] The survey article is "Automatic Groups: A Guided Tour" by Benson Farb [22]. It is a short and leisurely introduction to the theory that surveys the basic ideas, major results, key examples and non-examples without delving too far into the details of the proofs. (A third source of basic information is the proceedings of the 1989 MSRI conference on algorithms in combinatorial group theory [2], also published in 1992.) The main drawback of these sources is that they were published 15 years ago, and thus their descriptions of open problems need to be brought up to date.

Since 1990 there have been significant papers on

- biautomaticity and small cancellation groups (Gersten and Short [26, 25]),
- rational subgroups of biautomatic groups (Gersten and Short [27]),
- automaticity of Coxeter groups (Brink and Howlett [13]).
- biautomaticity of finite-type Artin groups (Charney [17]),
- automaticity of mapping class groups (Mosher [32]),
- central quotients of biautomatic groups (Mosher [33]),
- biautomaticity of fundamental groups of nonpositively curved cube complexes (Niblo and Reeves [35]),
- introduction and biautomaticity of Garside groups (Dehornoy [18]),
- biautomaticity of many Coxeter groups (Niblo and Reeves [36])

(via actions on cube complexes), and the 2003 paper by Martin Bridson that finally distinguished the classes of automatic, combable and bicombable groups [8].

Q: Any other highpoints that I overlooked? include Neumann-Shapiro, and Rebecchi somewhere

8. Eight Families of Groups

Let Γ be a connected graph with a cocompact group action. There are eight closely-related families of groups that are defined by the existence of such a graph Γ together with a collection of distinguished paths in Γ with additional properties. Such properties might include that the collection be recognized by a finite state automaton, that every path in the collection must be a (k, ϵ)-quasi-geodesic, or that the paths in the collection that start and/or end close together must k-fellow travel. The schematic relationships among the eight families are shown in Figure 1 and their definitions can easily be found in the references, particularly [21], except for semi-hyperbolic groups which were only introduced by Alonso and Bridson in 1995 [1]. For semi-hyperbolic groups, [11] is a good reference.

In Figure 1 an arrow $A \to B$ indicate that every group of type A is a group of type B. Thus Gromov hyperbolic groups are the most and asynchronously combable groups are the less restrictive of the eight classes depicted. The question marks and inequality signs indicate whether or not these inclusions are known to

[1] The introduction also contains a detailed description of the early history of these concepts.

$$\begin{tikzcd}
\text{hyperbolic} \arrow[r, "\neq"] \arrow[d, "\neq"'] & \text{biautomatic} \arrow[r, "?"] \arrow[d, "\neq"'] & \text{automatic} \arrow[r, "\neq"] \arrow[d, "\neq"'] & \text{asyn. automatic} \arrow[d, "\neq"'] \\
\text{semi-hyperbolic} \arrow[r, "?"] & \text{bicombable} \arrow[r, "\neq"] & \text{combable} \arrow[r, "?"] & \text{asyn. combable}
\end{tikzcd}$$

FIGURE 1. Eight related families of groups.

be strict. Some of the counterexamples are easy. The group $\mathbb{Z}^2$ is biautomatic and semi-hyperbolic but not hyperbolic and it is shown in [21] that the non-solvable Baumslag-Solitar groups are asynchronouslly automatic but not automatic. For a long time it was an open question whether combable groups were automatic or bicombable, but in 2003 Martin Bridson gave examples of combable groups that are not bicombable and examples of combable groups that are not automatic [8]. An asynchronously combable but not asynchronously automatic example can be found in [6] and is further explored in [9] and [10]. In addition, there is a free-by-cyclic group that a bicombable but not biautomatic group constructed by Brady and Bridson (unpublished) and this example is not even automatic (Bridson and Reeves, also unpublished). At this point there are at least three major questions along these lines that are still wide open.

Some of the other questionmarks probably have easy counterexamples that I am overlooking.

Question 15 (Automatic vs. Biautomatic)**.** *Is every automatic group biautomatic?*

Most experts agree that the answer is probably no, and several potential counterexamples have been suggested.

add citations

Question 16 (CAT(0) vs. Biautomatic)**.** *Is every* CAT(0) *group biautomatic?*

The class of CAT(0) groups is contained in the class of semi-hyperbolic groups and its relationship to hyperbolicity is a well-known open problem. Some partial positive results are discussed below in §10. The converse of Question 16 is known to be false. The third question along these lines is the biautomatic analog of Thurston's hyperbolization conjecture.

add cites

Question 17 (Hyperbolic vs. Biautomatic)**.** *Is every biautomatic group with no* $\mathbb{Z} \times \mathbb{Z}$ *subgroup word hyperbolic?*

9. Basic Structural Properties

There were two early problem collections focused on automatic groups: one collected by Steve Gersten and published in [23] and the other in Benson Farb's survey article [22]. The two lists have significant overlap, and neither seriously addresses the open questions about the then recently invented theory of biautomatic groups.

Finish pulling together the next several jotted notes

The principal examples of automatic groups are finite groups, Euclidean groups, negatively curved groups (including word-hyperbolic groups), various small cancellation groups, Coxeter groups, geometrically finite groups and braid groups. Many of these

Groups that have been shown to not support an automatic structre include infinite torsion groups, nilpotent groups that are not virtually abelian, $\mathrm{SL}_n(Z)$ for $n \geq 3$, and non-solvable Baumslag-Solitar groups.

Some of the most important properties of automatic groups are that they are finitely presented and have solvable word problem. They are closed under direct and free products (with finite amalgamated subgroup) and under finite extension.

Biautomatic groups are known to have a solvable conjugacy problem, polycyclic subgroups of biautomatic groups are abelian by finite, the Baumslag-Solitar group $BS(1,2)$ cannot be isomorphic to a subgroup of a biautomatic group, and the center of a biautomatic group is always finitely generated.

Biautomatic groups include all word hyperbolic groups, all fundamental groups of finite volume hyperbolic and Euclidean orbifolds, the braid groups and central extnensions of word hyperbolic groups.

Several important structural results about subgroups and quotients of biautomatic groups were shown by Gersten and Short in [27] and Lee Mosher in [33]. Together these two papers establish all of the following results.

Theorem 4. *If G is a biautomatic group and H is the centralizer in G of a finite set of elements, then H is biautomatic. If C is a subgroup of the center of G then G/C is biautomatic. If L is a direct factor of G then L is biautomatic. If K is a polycyclic subgroup of G then K is virtually abelian. And finally, the group G does not contain any subgroups isomorphic to $BS(k,l)$ with $|k| \neq |l|$.*

Combined with the foundational results shown in [21], we thus know that free products, direct products, free factors and direct factors of biautomatic groups are biautomatic.

Structural Questions. The care that needs to be taken to distinguish between automatic and biautomatic is illustrated by their known closure properties. For the most part, every closure property that is known to hold for automatic groups, is also known to hold for biautomatic groups with one glaring exception: every virtually automatic group (i.e. a group that contains a finite index automatic subgroup) is itself automatic. This is an open question for biautomatic groups.

Question 18 (Finite Extensions)**.** *Is a virtually biautomatic group biautomatic?*

With that sole exception, the rest of the questions posed here are either open for both automatic and biautomatic groups or they are known for biaumatic groups and open for automatic groups.

Any progress on these? Are any others known for biautomatic?

Problem 19 (Conjugacy Problem)**.** *Biautomatic groups are known to have a solvable conjugacy problem. Determine whether or not every automatic group has a solvable conjugacy problem.*

Question 20 (Tits alternative)**.** *Do either automatic or biautomatic groups satisfy the Tits alternative? That is, does every subgroup that is not virtually abelian contain a copy of a non-abelian free group?*

Are these both still open?

Question 21 (Hopfian)**.** *Are automatic groups Hopfian? Are biautomatic groups?*

Question 22 (Center)**.** *Is the center of an automatic group finitely generated? Is there a bound on the order of the torsion elements in the center?*

The center of a biautomatic group is known to be finitely generated.

add cite

Question 23 (Retracts)**.** *Is a retract of an automatic group automatic?*

Known for biautomatic?

Question 24 (Cyclic Amalgamations)**.** *Let $G = A *_C B$ where A and B are automatic and C is infinite cyclic. Is G automatic?* Known for biautomatic?

The next four questions ask about properties of subgroups of automatic and biautomatic groups.

Question 25 (Finite-index subgroups)**.** *Is every automatic group residually finite? Does an automatic group necessarily have even one nontrivial subgroup of finite index? Similar questions can be asked about biautomatic groups.*

Question 26 (Cyclic subgroups)**.** *Let G be a automatic group and let $x \in G$ be an element of infinite order. Is the map of the set of integers into the Cayley graph of G given by $n \mapsto x^n$ a quasi-isometry?*

Question 27 (Polycyclic subgroups)**.** *Every polycyclic subgroup of a biautomatic group and every virtuallly nilpotent subgroup of an automatic group is virtually abelian. Must every polycyclic subgroup of an automatic group be virtually abelian?*

Question 28 (Baumslag-Solitar subgroups)**.** *Biautomatic groups do not contain any subgroups isomorphic to $BS(k,l)$ with $|k| \neq |l|$. Is the same true for automatic groups? More specifically, can an automatic group contain a $BS(1,2)$ subgroup?*

10. Connections to Negative and Nonpositive Curvature

There are many results that connect presence of negative or nonpositive curvature to the existence of an automatic or biautomatic struture.

Riemannian Manifolds. Describe the key automaticity results about Riemannian manifolds (in arbitary dimensions) with strictly negative sectional curvature. Include the finite volume hyperbolic and euclidean orbifold results, and mention the Neumann and Shapiro [34] paper that classifies *all* automatic and biautomatic structures on geometrically finite hyperbolic groups.

Piecewise Euclidean Complexes. In modern terminology, two of the main results that Gersten and Short proved in [26] and [25] are that groups that act freely, cellularly and cocompactly on either (1) CAT(0) square complexes or (2) CAT(0) triangle complexes, are biautomatic. A triangle complex is a simplicial 2-complex in which every edge has unit length and the 2-cells are isometric to equilateral triangles. Since Świątkowski [38] recently proved a theorem (using what he calls regular path systems) that allows many previous theorems proving biautomaticity that assumed the group actions on various complexes were free to be weakened to actions that are merely proper, the Gersten and Short results can be strengthened to assert that all groups acting geometrically on CAT(0) square complexes or on CAT(0) triangle complexes are biautomatic.

Both of these results have been generalized to higher dimensions. The square complex result was generalized to cube complexes by Niblo and Reeves [35]. Combined with Świątkowski's results, we now know that groups that act geometrically on CAT(0) cube complexes are biautomatic. The triangle complex result have been generalized by Januszkiewicz and Świątkowski through their theory of simplicial non-positive curvature [28]. It is still an open question whether the straightforward CAT(0) generalization of the triangle complex result is true.

Conjecture 29. *If G acts geometrically on a* CAT(0) *piecewise Euclidean simplicial complex with unit edge lengths, then G is biautomatic.*

Recently, Rena Levitt has shown [29] that any group that acts geometrically on a CAT(0) triangle-square complex is biautomatic. By a triangle-square complex, I mean a piecewise Euclidean complex with unit edge lengths in which each 2-cell is a triangle or a square. This leads to the following conjecture that would encompass cube complexes and Conjecture 29 as special cases. Call a piecewise Euclidean cell complex a *simplicial product complex* if every cell is isometric to a direct product of regular Euclidean simplices with edges of unit length.

Conjecture 30. *If G acts geometrically on a* CAT(0) *simplicial product complex, then G should be biautomatic.*

Alternatively, find conditions similar to Januszkiewicz and Świątkowski simplicial nonpositive curvature that ensure biautomaticity.

11. Relations with Standard Classes of Groups

Much of the recent work on automaticity and biautomaticity has been on directed towards clarifying how these concepts interact with well-known classes of groups.

3-Manifold Groups. Describe the geometrization-automatic results from [21]. Thurston's geometries and the no nil or sol pieces implies automatic.

Mapping Class Groups. In 1995, Lee Mosher proved that all mapping class groups are automatic [32], a result that can be viewed as a vast generalization of the automaticity of the braid groups. This leads to an obvious question that remains wide open.

Question 31 (Mod($\mathcal{S}$) biautomaticity)**.** *Are mapping class groups biautomatic?*

Based on Mosher's result and the usual analogy between mapping class groups and automorphisms of free groups, one might conjecture that $\text{Aut}(\mathbb{F}_n)$ and $\text{Out}(\mathbb{F}_n)$ would at the least be automatic, but Martin Bridson and Karen proved in [12] that this is not the case.

Theorem 5. *For $n \geq 3$ the groups $\text{Aut}(\mathbb{F}_n)$ and $\text{Out}(\mathbb{F}_n)$ do not admit a bounded bicombing of subexponential length. As a consequence, they are not biautomatic, or semihyperbolic, or* CAT(0)*.*

Question 32. *Is $\text{Aut}(\mathbb{F}_n)$ or $\text{Out}(\mathbb{F}_n)$ automatic?*

Free-by-$\mathbb{Z}$ Groups. A related class of groups are those that can be constructed from a single free group automorphism.

Problem 33 (Free-by-$\mathbb{Z}$ Groups)**.** *For a finitely generated free group $\mathbb{F}$ and each $\phi \in \text{Aut}(\mathbb{F})$, let $G = \mathbb{F} \rtimes_\phi \mathbb{Z}$ be the corresponding mapping torus. When is G automatic? when is G biautomatic?*

Martin and Noel have an example showing that not all are biautomatic and then Martin and Lawrence have shown that the same example is not automatic. The example is the $\mathbb{F}_3 \rtimes_\phi \mathbb{Z}$ defined by the automorphism ϕ sending $a \mapsto a$, $b \mapsto ba$, and $c \mapsto ca^2$. Gersten previously proved that this example is not CAT(0). add cites

Note that Martin Bridson and Daniel Groves have recently been able to show that all finitely generated free-by-$\mathbb{Z}$ groups have a quadratic isoperimetric inequality. add cites and other partial results

Lattices in Lie Groups. Two questions from the original problem lists asked about the automaticity of lattices in specific Lie groups. In particlular, he asked whether every lattice in $\mathrm{SL}_3(\mathbb{R})$ is automatic and whether every cocompact irreducible lattice in $\mathrm{Isom}(\mathbb{H}^2) \times \mathrm{Isom}(\mathbb{H}^2)$ is automatic. Notice that this is a completely explicit class of groups. Cite Dave Witte-Morris These questions can be placed in a larger context.

Problem 34 (Lattices)**.** *Classify which lattices in Lie groups are automatic and which ones are biautomatic.*

Ask Benson

The original questions are still open. Of course many non-cocompact lattices can be immediately ruled out by the strong restrictions on nilpotent subgroups.

Tubular Groups. Tubular groups include several particular examples that are potential counterexamples to other open questions. In particular, the groups

$$G_{p,p} = \langle a, b, s, t \mid ab = ba, s^{-1}a^p s = a^p b, t^{-1}a^p t = a^p b^{-1}\rangle$$

have a quadratic Dehn function [5] but are not CAT(0) [24] or biautomatic (Brady and Bridson, unpublished), and $G_{1,1}$ at least is asynchronously automatic [20].

Question 35. *Are the tubular groups $G_{p,p}$ automatic?*

The tubular group $W = \langle a, b, s, t \mid ab = ba, s^{-1}as = (ab)^2, t^{-1}bt = (ab)^2\rangle$ is known as *Wise's group*. Dani Wise has shown [41] that it is CAT(0) and non-Hopfian, and Murray Elder has found an asynchronous automatic structure on W [20]. It is unknown whether or not it is automatic.

Question 36 (Wise's group)**.** *Is Wise's tubular group automatic?*

Resolving this Question 36 in either direction would answer an open question about automatic groups. It either provides an example of a CAT(0) group that is not automatic (Question 16), or it provides an example of an automatic group that is not Hopfian (Question 21).

Coxeter groups. As is the case for mapping class groups, the automaticity of Coxeter groups has been known for a while now. Brink and Howlett proved that all Coxeter groups are automatic in 1993 [13]. Progress on biautomaticity has been slower.

Question 37 (Coxeter Groups)**.** *Are all Coxeter groups biautomatic?*

Significant progress on this question occured when Graham Niblo and Lawrence Reeves were able to show in [36] that all Coxeter groups act on properly discontinuously by isometries on CAT(0) cube complexes, and in many cases, these actions are cocompact. In particular, they were able to prove the following result.

Theorem 6. *If G is a finitely generated Coxeter group that contains only finitely many conjugacy classes of subgroups isomorphic to triangle groups, then G is biautomatic.*

The analysis of the conditions under which the action is cocompact was completed by Niblo's Ph.D. student Ben William in his dissertation [40]. On a separate note, Bill Cassell has written a survey article that describes how to efficiently carry out computations in Coxeter groups using the GAP package called CHEVIE [14]. Several of the algorithms he describes are based on the work of du Cloux [19] and the automatic structures on Coxeter groups figure heavily in these algorithms. See also, the webpages he has set up based on the 2002 CRM conference on compuatations in Coxeter groups [15].

Artin groups and Garside structures. In the same paper where Garside groups were first defined, Patrick Dehornoy showed that every group with a Garside structure is biautomatic [18]. One consequence of this theorem is that it shows that every Artin group of finite type is biautomatic (a result first proved by Ruth Charney in [17]). In particular, this reproves once again the basic result that the braid groups are biautomatic.

Problem 38 (Artin groups)**.** *Determine which Artin groups are biautomatic, automatic, or asynchronously automatic.*

This problem remains wide open, predominantly because for most Artin groups, we do not even know how to solve the word problem. First things, first. Eventually, this may become an interesting question.

One relator groups. The class of one-relator groups have an uneasy relationship with the classes of groups under discussion since it clearly contains many hyperbolic examples (such as any one-relator group with torsion), but it also contains groups such as $BS(1,2)$ that are asynchronously automatic but not automatic, and groups such as Baumslag's group that are not even asynchronously combable. In particular, it is not yet clear which one-relator groups belong to which classes.

Problem 39 (One-relator groups)**.** *Determine which one-relator groups are hyperbolic, biautomatic, automatic, or asynchronously automatic.*

Although Problem 39 is wide open, some answers have been conjectured.

Conjecture 40. *A one-relator group is hyperbolic iff it contains no Baumslag-Solitar subgroups, and it is automatic iff it contains no metabelian subgroups.*

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