ARTIN GROUPS OF EUCLIDEAN TYPE

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ABSTRACT. This article resolves several long-standing conjectures about Artin groups of euclidean type. Specifically we prove that every irreducible euclidean Artin group is a torsion-free centerless group with a decidable word problem and a finite-dimensional classifying space. We do this by showing that each of these groups is isomorphic to a subgroup of a group with an infinite-type Garside structure. The Garside groups involved are introduced here for the first time. They are constructed by applying semi-standard procedures to crystallographic groups that contain euclidean Coxeter groups but which need not be generated by the reflections they contain.

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Arbitrary Coxeter groups are groups defined by a particularly simple type of presentation, but the central motivating examples that lead to the general theory are the irreducible groups generated by reflections that act geometrically (i.e. properly discontinuously and cocompactly by isometries) on spheres and euclidean spaces. Presentations for these spherical and euclidean Coxeter groups are encoded in the well-known Dynkin diagrams and extended Dynkin diagrams, respectively.

Arbitrary Artin groups are groups defined by a modified version of these simple presentations, a definition designed to describe the fundamental group of a space constructed from the complement of the hyperplanes in a complexified version of the reflection arrangement for the corresponding Coxeter group.

The spherical Artin groups, i.e. the Artin groups corresponding to Coxeter groups that act geometrically on spheres, have been well understood ever since Artin groups themselves were introduced in 1972 by Pierre Deligne [Del72] and by Egbert Brieskorn and Kyoji Saito [BS72] in adjacent articles in the *Inventiones.* Given the centrality of euclidean Coxeter groups in Coxeter theory and Lie theory more generally, it has been somewhat surprising that the structure of most euclidean Artin groups has remained mysterious for the past forty plus years.

In this article we clarify the structure of all euclidean Artin groups by showing that they are isomorphic to subgroups of a new class of Garside groups that we believe to be of independent interest. More specifically we prove four main results. The first establishes the existence of a new class of Garside groups based on intervals in crystallographic groups closely related to the irreducible euclidean Coxeter groups.

Theorem A (Crystallographic Garside groups)**.** *Let* $W = \textsc{Cox}(\widetilde{X}_n)$ *be an irreducible euclidean Coxeter group and let* R *be its set of reflections. For each Coxeter element* $w \in W$ *there exists a set of translations* T *and a crystallographic group* $\textsc{Cryst}(\widetilde{X}_n, w)$ *containing* W *with generating set* $R \cup T$ *so that the weighted factorizations of* w *over this expanded generating set form a balanced lattice. As a consequence, this collection of factorizations define a group* $\textsc{Gar}(\widetilde{X}_n, w)$ *with a Garside structure of infinite-type.*

The second shows that these crystallographic Garside groups contain subgroups that we call dual euclidean Artin groups.

$$\begin{array}{ccccc} \text{ART}(\widetilde{X}_n) & \cong & \text{ART}^*(\widetilde{X}_n, w) & \hookrightarrow & \text{GAR}(\widetilde{X}_n, w) \\ \downarrow & & \downarrow & & \downarrow \\ \text{COX}(\widetilde{X}_n) & \cong & \text{COX}^*(\widetilde{X}_n, w) & \hookrightarrow & \text{CRYST}(\widetilde{X}_n, w) \end{array}$$

FIGURE 1. For each Coxeter element w in an irreducible euclidean Coxeter group of type $\widetilde{X}_n$ we define several related groups.

Theorem B (Dual Artin Subgroups)**.** *For each irreducible euclidean Coxeter group* $\text{COX}(\widetilde{X}_n)$ *and for each choice of Coxeter element* w*, the Garside group* $\text{GAR}(\widetilde{X}_n, w)$ *is an amalgamated free product of explicit groups with the dual Artin group* $\text{ART}^*(\widetilde{X}_n, w)$ *as one of its factors. In particular, the dual Artin group* $\text{ART}^*(\widetilde{X}_n, w)$ *injects into the Garside group* $\text{GAR}(\widetilde{X}_n, w)$*.*

The third shows that this dual euclidean Artin group is isomorphic to the corresponding Artin group.

Theorem C (Naturally isomorphic groups)**.** *For each irreducible euclidean Coxeter group* $W = \text{COX}(\widetilde{X}_n)$ *and for each choice of Coxeter element* w *as the product of the standard Coxeter generating set* S*, the Artin group* $A = \text{ART}(\widetilde{X}_n)$ *and the dual Artin group* $W_w = \text{ART}^*(\widetilde{X}_n, w)$ *are naturally isomorphic.*

And finally, our fourth main result uses the Garside structure of the crystallographic Garside supergroup to derive structural consequences for its euclidean Artin subgroup.

Theorem D (Euclidean Artin groups)**.** *Every irreducible euclidean Artin group* $\text{ART}(\widetilde{X}_n)$ *is a torsion-free centerless group with a solvable word problem and a finite-dimensional classifying space.*

The relations among these groups are shown in Figure 1. The notations in the middle column refer to the Coxeter group and the Artin group as defined by their dual presentations. These dual presentations facilitate the connection between the Coxeter group $\text{COX}(\widetilde{X}_n)$ and the crystallographic group $\text{CRYST}(\widetilde{X}_n, w)$ and between the Artin group $\text{ART}(\widetilde{X}_n)$ and the crystallographic Garside group $\text{GAR}(\widetilde{X}_n, w)$.

Theorem D represents a significant advance over what was previous known. In 1987, Craig Squier analyzed the euclidean Artin groups with three generators: $\text{ART}(\widetilde{A}_2)$, $\text{ART}(\widetilde{C}_2)$ and $\text{ART}(\widetilde{G}_2)$ [Squ87]. His main technique was to analyze the presentations as amalgamated products and HNN extensions of known groups, a technique that does not appear to generalize to the remaining groups. The ones of type A have been understood via a semi-classical embedding $\textsc{Art}(\widetilde{A}_{n-1}) \hookrightarrow \textsc{Art}(B_n)$ into a type B spherical Artin group [All02, CP03, KP02, tD98].

More recently François Digne used dual Garside structures to successfully analyze the euclidean Artin groups of types A and C [Dig06, Dig12]. This article is the third in a series which continues the investigation along these lines. The first two papers are [BM15] and [McC15] and there also is a survey article [McCb] that discusses the results in all three papers. The main result of [McC15] was a negative one: types A, C and G are the only euclidean types whose dual presentations are Garside. The results in this article show how to overcome the deficiencies that arise in types B, D, E and F.

Overview: The article is divided into four parts. Part I contains basic background definitions for posets, Coxeter groups, intervals and Garside structures. Part II introduces an interesting discrete group generated by coordinate permutations and translations by integer vectors whose structure is closely related by the Coxeter and Artin groups of type B. These "middle groups" and the structure of their intervals play a major role in the proofs of the main results. Part III shifts attention to intervals in arbitrary irreducible euclidean Coxeter groups and introduces various new groups including the crystallographic groups and crystallographic Garside groups mentioned above. Part IV contains the proofs of our four main results.

Part 1. Background

This part contains background material with one section focusing on posets and Coxeter groups, another on intervals and Garside structures.

1. Posets and Coxeter groups

This section reviews some basic definitions for the sake of completeness. Our conventions follows [Hum90], [Sta97], and [DP02].

Definition 1.1 (Coxeter groups)**.** A *Coxeter group* is any group W that can be defined by a presentation of the following form. It has a standard finite generating set S and only two types of relations. For each $s \in S$ there is a relation $s^2 = 1$ and for each unordered pair of distinct elements $s, t \in S$ there is at most one relation of the form $(st)^m = 1$ where $m = m(s,t) > 1$ is an integer. When no relation involving s and t occurs we consider $m(s,t) = \infty$. A *reflection* in W is any conjugate of an element of S and we use R to denote the set of all reflections in W. In other words, $R = \{wsw^{-1} \mid s \in S, w \in W\}$. This presentation is usually encoded in a labeled graph Γ called a *Coxeter*

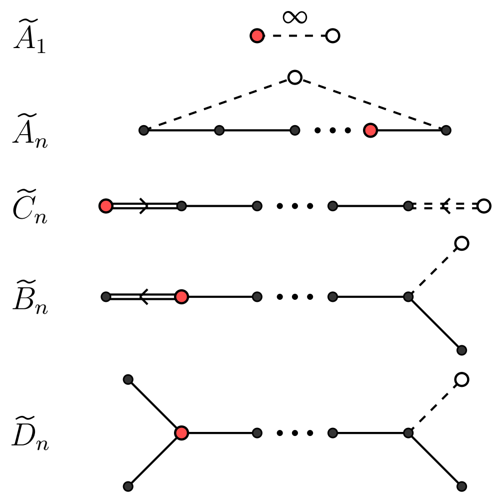

FIGURE 2. Diagrams for the four infinite families.

diagram with a vertex for each $s \in S$, an edge connecting s and t if $m(s,t) > 2$ and a label on this edge if $m(s,t) > 3$. When every $m(s,t)$ is contained in the set $\{2,3,4,6,\infty\}$ the edges labeled 4 and 6 are replaced with double and triple edges, respectively. The group defined by the presentation encoded in Γ is denoted $W = \text{Cox}(\Gamma)$. A Coxeter group is *irreducible* when its diagram is connected.

Definition 1.2 (Artin groups)**.** For each Coxeter diagram Γ there is an *Artin group* $\text{Art}(\Gamma)$ defined by a presentation with a relation for each two-generator relation in the standard presentation of $\text{Cox}(\Gamma)$. More specifically, if $(st)^m = 1$ is a relation in $\text{Cox}(\Gamma)$ then the presentation of $\text{Art}(\Gamma)$ has a relation that equates the two length m words that strictly alternate between s and t. Thus $(st)^2 = 1$ becomes $st = ts$, $(st)^3 = 1$ becomes $sts = tst$, $(st)^4 = 1$ becomes $stst = tsts$, etc. There is no relation when $m(s,t)$ is infinite.

The general theory of Coxeter groups is motivated by those which act *geometrically* (i.e. properly discontinuously and cocompactly by isometries) on spheres and euclidean spaces and they are classified by the famous Dynkin diagrams and extended Dynkin diagrams, respectively.

Definition 1.3 (Extended Dynkin diagrams)**.** There are four infinite families and five sporadic examples of irreducible euclidean Coxeter groups. The extended Dynkin diagrams for the infinite families, including the unusual $\widetilde{A}_1$ diagram, are shown in Figure 2 and the five

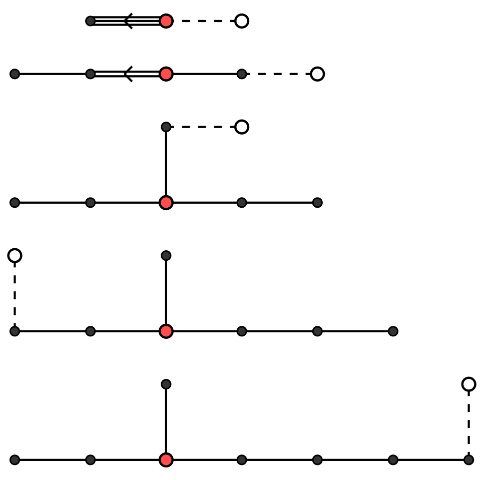

FIGURE 3. Diagrams for the five sporadic examples.

sporadic examples are shown in Figure 3. The large white dot connected to the rest of the diagram by dashed lines is the *extending root* and the diagram with this dot removed is the ordinary Dynkin diagram for the corresponding spherical Coxeter group. The large shaded dot is called the *vertical root* of the diagram. Its definition and meaning are discussed in Section 6.

Since we do not need most of the heavy machinery developed to study euclidean Coxeter groups, it suffices to loosely introduce some standard terminology.

Definition 1.4 (Simplices and tilings)**.** One way to understand the meaning of the extended Dynkin diagrams is that they encode the geometry of a euclidean simplex σ in which each dihedral angle is $\frac{\pi}{m}$ for some integer m. The vertices correspond to facets of σ and the integer m associated to a pair of vertices encodes the dihedral angle between these facets. The reflections that fix the facets of σ then generate a group of isometries which tile euclidean space with copies of σ. For example, the $\widetilde{G}_2$ diagram corresponds to a triangle in the plane with dihedral angles $\frac{\pi}{2}$, $\frac{\pi}{3}$, and $\frac{\pi}{6}$ and the corresponding tiling is shown in Figure 11 on page 27. The top dimensional simplices in this tiling are called *chambers*.

Definition 1.5 (Roots and reflections)**.** The *root system* Φ_{X_n} associated with the $\widetilde{X}_n$ tiling is a collection of pairs of antipodal vectors called *roots* which includes one pair $\pm\alpha$ normal to each infinite family of parallel hyperplanes and the length of α encodes the consistent

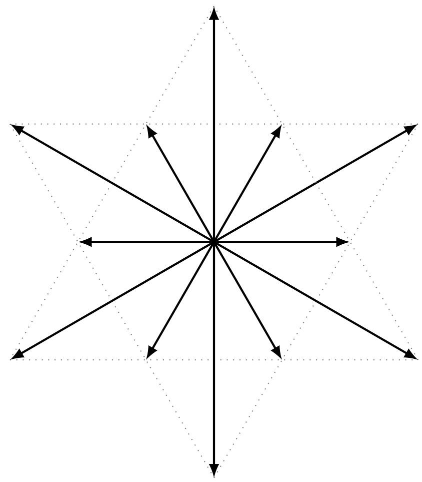

FIGURE 4. The G_2 root system.

spacing between these hyperplanes. The G_2 root system is shown in Figure 4. The $\widetilde{X}_n$ tiling can be reconstructed from Φ_{X_n} root system as follows. For each $\alpha \in \Phi_{X_n}$ and for each $k \in \mathbb{Z}$, let $H_{\alpha,k}$ be a hyperplane $\{x \mid x \cdot \alpha = k\}$ orthogonal to α and let $r_{\alpha,k}$ be the reflection that fixes $H_{\alpha,k}$ pointwise. The chambers of the tiling are the connected components of the complement of the union of all such hyperplanes and the set $R = \{r_{\alpha,k}\}$ is the full set of reflections in the irreducible euclidean Coxeter group $W = \textsc{Cox}(\widetilde{X}_n)$. The reflections $S \subset R$ that reflect in the facets of a single chamber σ are a minimal generating set corresponding to the vertices of the $\widetilde{X}_n$ Dynkin diagram.

Definition 1.6 (Coroots and translations)**.** One consequence of these definitions is that a longer root corresponds to a family of hyperplanes that are more closely spaced. Let t_λ be the translation produced by multiplying reflections associated with adjacent and parallel hyperplanes such as $r_{\alpha,k+1}$ and $r_{\alpha,k}$. The translation vector is a multiple of α and one can compute $\lambda = \left(\frac{2}{\alpha\cdot\alpha}\right)\alpha$. This vector is called the *coroot* $\alpha^\vee$ corresponding to α. In other words, $t_{\alpha^\vee} = r_{\alpha,k+1}r_{\alpha,k}$.

We also record basic terminology for lattices and posets.

Definition 1.7 (Posets)**.** Let P be a partially ordered set. If P contains both a minimum element and a maximum element then it is *bounded*. For each $Q \subset P$ there is an induced *subposet* structure on Q by restricting the partial order on P. A subposet C in which any two elements are comparable is called a *chain* and its *length* is $|C|-1$. Every finite chain is bounded and its maximum and minimum elements are

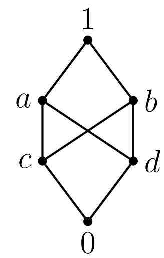

FIGURE 5. A bounded graded poset that is not a lattice.

its *endpoints*. If a finite chain C is not a subposet of a strictly larger finite chain with the same endpoints, then C is *saturated*. Saturated chains of length 1 are called *covering relations*. If every saturated chain in P between the same pair of endpoints has the same finite length, then P is *graded*. There is also a weighted version where one defines a weight or length to each covering relation and calls P *weighted graded* when every saturated chain in P between the same pair of endpoints has the same total weight. When varying weights are introduced they shall always be *discrete* in the sense that the set of all weights is a discrete subset of the positive reals bounded away from zero. The *dual* P^* of a poset P has the same underlying set but the order is reversed, and a poset is *self-dual* when it and its dual are isomorphic.

Definition 1.8 (Lattices)**.** Let Q be any subset of a poset P. A lower bound for Q is any $p \in P$ with $p \leq q$ for all $q \in Q$. When the set of lower bounds for Q has a unique maximum element, this element is the *greatest lower bound* or *meet* of Q. Upper bounds and the *least upper bound* or *join* of Q are defined analogously. The meet and join of Q are denoted $\bigwedge Q$ and $\bigvee Q$ in general and $u \wedge v$ and $u \vee v$ if u and v are the only elements in Q. When every pair of elements has a meet and a join, P is a *lattice* and when every subset has a meet and a join, it is a *complete lattice*.

Definition 1.9 (Bowties)**.** Let P be a poset. A *bowtie* in P is a 4-tuple of distinct elements $(a, b : c, d)$ such that a and b are minimal upper bounds for c and d and c and d are maximal lower bounds for a and b. The name reflects the fact that when edges are drawn to show that a and b are above c and d, the configuration looks like a bowtie. See Figure 5.

In [BM10, Proposition 1.5] Tom Brady and the first author noted that a bounded graded poset P is a lattice if and only if P contains no bowties. The same result holds, with the same proof, when P is graded with respect to a discrete weighting of its covering relations. We reproduce the proof for completeness.

Proposition 1.10 (Lattice or bowtie). *If P is a bounded poset that is graded with respect to a set of discrete weights, then P is a lattice if and only if P contains no bowties.*

Proof. If P contains a bowtie $(a, b : c, d)$, then c and d have no join and P is not a lattice. In the other direction, suppose P is not a lattice because x and y have no join. An upper bound exists because P is bounded, and a minimal upper bound exists because P is weighted graded. Thus x and y must have more than one minimal upper bound. Let a and b be two such minimal upper bounds and note that x and y are lower bounds for a and b. If c is a maximal lower bound of a and b satisfying $c \geq x$ and d is a maximal lower bound of a and b satisfying $d \geq y$, then $(a, b : c, d)$ is a bowtie. We know that a and b are minimal upper bounds of c and d and that c and d are distinct since either failure would create an upper bound of x and y that contradicts the minimality of a and b. When x and y have no meet, the proof is analogous. ☐

We conclude with a remark about subposets. Notice that bowties remain bowties in induced subposets. Thus if P is not a lattice because it contains a bowtie $(a, b : c, d)$ and Q is any subposet that contains all four of these elements, then Q is also not a lattice since it contains the same bowtie.

2. Intervals and Garside structures

As mentioned in the introduction, attempts to understand euclidean Artin groups of euclidean type directly have only met with limited success. The most promising progress has been by François Digne and his approach is closely related to the dual presentations derived from an interval in the corresponding Coxeter group [McC15]. We first recall how a group with a fixed generating set naturally acts on a graph and how assigning discrete weights to its generators turns this graph into a metric space invariant under the group action.

Definition 2.1 (Marked groups). A *marked group* is a group G with a fixed generating set S which, for convenience, we assume is symmetric and injects into G. The (right) *Cayley graph of G with respect to S* is a labeled directed graph denoted $\textsc{Cay}(G, S)$ with vertices indexed by G and edges indexed by $G \times S$. The edge $e_{(g,s)}$ has *label s*, it starts at v_g and ends at $v_{g'}$ where $g' = g \cdot s$. There is a natural faithful, vertex-transitive, label and orientation preserving left action of G on its right Cayley graph and these are the only graph automorphisms that preserve labels and orientations.

Definition 2.2 (Weights)**.** Let S be a generating set for a group G. We say S is a *weighted generating set* if its elements are assigned positive weights bounded away from 0 that form a discrete subset of the positive reals. The elements s and s^{-1} should, of course, have the same weight. For finite generating sets discreteness and the lower bound are automatic but these are important restrictions when S is infinite. One can always use a *trivial weighting* which assigns the same weight to each generator. When G is generated by a weighted set S, its Cayley graph can be made into a metric space where the length of each edge is its weight. The length of a combinatorial path in the Cayley group is then the sum of the weights of its edges and the distance between two vertices is the minimum length of such a combinatorial path. For infinite generating sets the lower bound on the weights can be used to bound on the number of edges involved in a minimum length path and the discreteness condition ensures that the infimum of these path lengths is actually achieved by some path.

In any metric space, one can define the notion of an interval.

Definition 2.3 (Intervals in metric spaces)**.** Let x, y and z be points in a metric space (X, d). We say z is *between* x and y if the triangle inequality is an equality: $d(x, z) + d(z, y) = d(x, y)$. The *interval* $[x, y]$ is the collection of points between x and y, and note that this includes both x and y. Intervals can also be endowed with a partial ordering by defining $u \leq v$ when $d(x, u) + d(u, v) + d(v, y) = d(x, y)$.

We are interested in intervals in groups.

Definition 2.4 (Intervals in groups)**.** Let G be a group with a fixed symmetric discretely weighted generated set and let $d(g, h)$ denote the distance between v_g and v_h in the corresponding metric Cayley graph. Note that the symmetry assumption on the generating set allows us to restrict attention to directed paths. From this metric on G we get bounded intervals with a weighted grading: for $g, h \in G$, the *interval* $[g, h]^G$ is the poset of group elements between g and h with $g' \in [g, h]^G$ when $d(g, g') + d(g', h) = d(g, h)$ and $g' \leq g''$ when $d(g, g') + d(g', g'') + d(g'', h) = d(g, h)$. In this article we include the superscript G as part of the notation since we often consider similar intervals in closely related groups.

Remark 2.5 (Intervals in Cayley graphs)**.** The interval $[g, h]^G$ is a bounded poset with discrete levels whose Hasse diagram is embedded as a subgraph of the weighted Cayley graph $\textsc{Cay}(G,S)$ as the union of all minimal length directed paths from v_g to v_h. This is because $g' \in [g,h]^G$ means $v_{g'}$ lies on some minimal length path from v_g to v_h and $g' < g''$ means that $v_{g'}$ and $v_{g''}$ both occur on a common minimal length path from v_g to v_h with $v_{g'}$ occurring before $v_{g''}$. Because the structure of such a poset can be recovered from its Hasse diagram, we let $[g,h]^G$ denote the edge-labeled directed graph that is visible inside $\textsc{Cay}(G,S)$. The left action of a group on its right Cayley graph preserves labels and distances. Thus the interval $[g,h]^G$ is isomorphic (as a labeled oriented directed graph) to the interval $[1,g^{-1}h]^G$. In other words, every interval in the Cayley graph of G is isomorphic to one that starts at the identity. We call $g^{-1}h$ the *type* of the interval $[g,h]^G$ and note that intervals are isomorphic if and only if they have the same type.

Intervals in groups can be used to construct new groups.

Definition 2.6 (Interval groups)**.** Let G be a group generated by a weighted set S and let g and h be distinct elements in G. The *interval group* $G_{[g,h]}$ is defined as follows. Let S_0 be the elements of S that actually occurs as labels of edges in $[g,h]^G$. The group $G_{[g,h]}$ has S_0 as its generators and we impose all relations that are visible as closed loops inside the portion of the Cayley graph of G that we call $[g,h]^G$. The elements in $S \setminus S_0$ are not included since they do not occur in any relation. More precisely, if they were included as generators, they would generate a free group that splits off as a free factor. Thus it is sufficient to understand the group defined above. Next note that this group structure only depends on the type of the interval so it is sufficient to consider interval groups of the form $G_{[1,g]}$. For these groups we simplify the notation to G_g and say that G_g is the interval group obtained by *pulling* G *apart at* g.

The interval $[1,g]^G$ implicitly encodes a presentation of G_g and various explicit presentations can be found in [McC15] and [McCa]. Dual Artin groups are examples of interval groups.

Definition 2.7 (Dual Artin groups)**.** Let $W = \textsc{Cox}(\Gamma)$ be a Coxeter group with standard generating set S and let R be the full set of reflections with a trivial weighting. For any fixed total ordering of the elements of S, the product of these generators in this order is called a *Coxeter element* and for each Coxeter element w there is a dual Artin group defined as follows. Let $[1,w]^W$ be the interval in the left Cayley graph of W with respect to R and let $R_0 \subset R$ be the subset of reflections that actually occur in some minimal length factorizations of w.

The *dual Artin group with respect to* w is the group $W_w = \text{Art}^*(\Gamma, w)$ generated by R_0 and subject only to those relations that are visible inside the interval $[1, w]^W$.

An explicit presentation for the dual $\widetilde{G}_2$ Artin group is given at the end of Section 5.

Remark 2.8 (Artin groups and dual Artin groups)**.** In general the relationship between the Artin group $\text{Art}(\Gamma)$ and the dual Artin group $\text{Art}^*(\Gamma, w)$ is not yet completely clear. It is straightforward to show using the Tits representation that the product of the elements in S that produce w is a factorization of w into reflections of minimum length which means that this factorization describes a directed path in $[1, w]^W$. As a consequence S is a subset of R_0. Moreover, the standard Artin relations are consequences of relations visible in $[1, w]^W$ (as illustrated in [BM00] and in Example 2.9) so that the injection of S into R_0 extends to a group homomorphism from $\text{Art}(\Gamma)$ to $\text{Art}^*(\Gamma, w)$. See also Proposition 10.1. When this homomorphism is an isomorphism, we say that the interval $[1, w]^W$ encodes a *dual presentation* of $\text{Art}(\Gamma)$.

To date, every dual Artin group that has been successfully analyzed is isomorphic to the corresponding Artin group and as a consequence its group structure is independent of the Coxeter element w used in its construction. In particular, this is known to hold for all spherical Artin groups [Bes03, BW02] and we prove it here for all euclidean Artin groups as our third main result. It is precisely because this assertion has not been proved in full generality that dual Artin groups deserve a separate name. The following example illustrates the relationship between Artin group presentations and dual presentations.

Example 2.9 (Dihedral Artin groups)**.** The spherical Coxeter groups with two generators are the dihedral groups. Let W be the dihedral group of order 10 with Coxeter presentation $\langle a, b \mid a^2 = b^2 = (ab)^5 = 1\rangle$ where a and b are reflections of $\mathbb{R}^2$ through the origin with an angle of $\pi/5$ between their fixed lines. The corresponding Artin group has presentation $\langle a, b \mid ababa = babab\rangle$. The set $S = \{a, b\}$ is a standard generating set for W and the set $R = \{a, b, c, d, e\}$ is its full set of reflections where these are the five reflections in W in cyclic order. The Coxeter element $w = ab$ is a $2\pi/5$ rotation and its minimum length factorizations over R are ab, bc, cd, de and ea. The dual Artin group has presentation $\langle a, b, c, d, e \mid ab = bc = cd = de = ea\rangle$. Systematically eliminating c, d and e recovers the original Artin group presentation.

The dual presentations for the spherical Artin groups were introduced and studied by David Bessis [Bes03] and by Tom Brady and

Colum Watt [BW02]. Here we pause to record one technical fact about Coxeter elements in the corresponding spherical Coxeter groups.

Proposition 2.10 (Spherical Coxeter elements)**.** *Let w_0 be a Coxeter element for a spherical Coxeter group $W_0 = \mathrm{Cox}(X_n)$ and let R be its set of reflections. For every $r \in R$ there is a chamber in the corresponding spherical tiling and an ordering on the reflections fixing its facets so that (1) the product of these reflections in this order is w_0 and (2) the leftmost reflection in the list is r.*

One reason that dual presentations of Artin groups are of interest is that they satisfy almost all of the requirements of a Garside structure. In fact, there is only one property that they might lack.

Proposition 2.11 (Garside structures)**.** *Let G be a group with a symmetric discretely weighted generating set that is closed under conjugation. If for some element g the weighted interval $[1,g]^G$ is a lattice, then the group G_g is a Garside group. In particular, if $W = \mathrm{Cox}(\Gamma)$ is a Coxeter group generated by its full set of reflections with Coxeter element w and the interval $[1,w]^W$ is a lattice, then the dual Artin group $W_w = \mathrm{Art}^*(\Gamma, w)$ is a Garside group.*

The reader should note that we are using "Garside structure" and "Garside group" in the expanded sense of Digne [Dig06, Dig12] rather than the original definition that requires the generating set to be finite. The discreteness of the grading of the interval substitutes for finiteness of the generating set. In particular, the discreteness of the grading forces the standard Garside algorithms to terminate. The standard proofs are otherwise unchanged. Proposition 2.11 was stated by David Bessis in [Bes03, Theorem 0.5.2], except for the shift from finite to infinite discretely weighted generating sets. For a more detailed discussion see [Bes03] and particularly the book [Deh15]. Interval groups appear in [DDM13] and in [Deh15, Chapter VI] as the "germ derived from a groupoid". The terminology is different but the translation is straightforward. When an interval such as $[1,w]$ is a lattice and it is used to construct a Garside group, the interval $[1,w]$ itself embeds in the Cayley graph of the new group G and the element w, viewed as an element in G, is called a *Garside element*. Being a Garside group has many consequences.

Theorem 2.12 (Consequences)**.** *If G is a group with a Garside structure in the expanded sense of Digne, then its elements have normal forms and it has a finite-dimensional classifying space whose dimension is equal to the length of the longest chain in its defining interval. As a consequence, G has a decidable word problem and it is torsion-free.*

Proof. The initial consequences follow from [DP99] and [CMW04] with minor modifications to allow for infinite discretely weighted generating sets, and the latter ones are immediate corollaries. □

A detailed description of the Garside normal form is never needed, but we give a coarse description sufficient to state a key property of elements that commute with the Garside element.

Definition 2.13 (Normal forms)**.** Let G be a Garside group in the expanded sense used here and with Garside element w. The elements in the interval $[1,w]$ are called *simple elements*. For every $u \in G$ there is an integer n and simple elements u_i such that $u = w^n u_1 u_2 \cdots u_k$. If we impose a few additional conditions, the integer n and the simples u_i are uniquely determined by u and this expression is called its (left-greedy) *normal form* $NF(u)$. Note that the integer n might be negative. When this happens, it indicates that the word u does not belong to the positive monoid generated by the simple elements. The value of n is the smallest integer such that $w^{-n}u$ lies in this positive monoid.

One consequence of being a Garside group is that the set of simples is closed under conjugation by w. In fact, conjugation by w is a lattice isomorphism (but one that typically does not preserve edge-labels) sending each simple to the left complement of its left complement. In particular, the simple by simple conjugation of the normal form for u remains in normal form, its product is u^w and by the uniqueness of normal forms, this must be the normal form for u^w. In particular, this proves the following.

Proposition 2.14 (Normal forms)**.** *Let G be a Garside group with Garside element w. For each $u \in G$, the Garside normal form of u^w is obtained by conjugating each simple in the Garside normal form for u. In other words, if $NF(u) = w^n u_1 u_2 \cdots u_k$ then $NF(u^w) = w^n u_1^w u_2^w \cdots u_k^w$. In particular, an element in G commutes with w if and only if its normal form is built out of simples that commute with w.*

There is one final fact about Garside structures that we need in the later sections, and that is an elementary observation about nicely situated sublattices of lattices and how they relate to normal forms.

Proposition 2.15 (Injective maps)**.** *Let G' be a subgroup of G and let S' and S be their conjugacy closed generating sets with $S' \subset S$. If w is an element of G' and there is a weighting on S such that (1) both $[1,w]^{G'}$ and $[1,w]^G$ are lattices and (2) the inclusion map $[1,w]^{G'} \hookrightarrow [1,w]^G$ is a lattice homomorphism preserving meets and joins, then the interval groups G'_w and G_w are both Garside groups and the natural map $G'_w \to G_w$ is an injection.*

Proof. First note that there is a natural map from G'_w to G_w because the relations defining G'_w are included among the relations that define G_w. For injectivity, let u be a nontrivial element of G'_w with normal form $NF(u) = w^n u_1 u_2 \cdots u_k$. Suppose we view this as an expression representing an element of G_w. The fact that the inclusion of the smaller interval into the larger one preserves meets and joins means that this expression remains in normal form in this new context. Therefore the image of u in G_w is also nontrivial and the map is an injection. □

Part 2. Middle groups

This part focuses on a series of elementary groups that we call "middle groups" and it establishes their key properties.

3. Permutations and translations

The discrete group of euclidean isometries generated by all coordinate permutations and all translations by vectors with integer coordinates is a group that plays an important role in the proofs of our main results. In this section we record its basic properties and relate it to the spherical Coxeter group $\text{Cox}(B_n)$ which encodes the symmetries of the n-cube.

Definition 3.1 (Cubical symmetries)**.** Let $[-1,1]^n$ denote the points in $\mathbb{R}^n$ where every coordinate has absolute value at most 1. This n-dimensional cube of side length 2 centered at the origin has isometry group $\text{Cox}(B_n)$ also called the *signed symmetric group*. It has n^2 reflection symmetries. We give these reflections nonstandard names based on an alternative realization of this group described below. Let r_{ij} be the reflection which switches the i-th and j-th coordinates fixing the hyperplane $x_i = x_j$ and let t_i denote the reflection which changes the sign of the i-th coordinate fixing the hyperplane $x_i = 0$. We call these collections $\mathcal{R}$ and $\mathcal{T}$ respectively. Together they generate $\text{Cox}(B_n)$, but they are neither a minimal generating set nor all of the reflections. The remaining reflections are obtained by conjugation. Conjugating r_{ij} by t_i, for example, produces an isometry which switches the i-th and j-th coordinates and changes both signs fixing the hyperplane $x_i = -x_j$.

The unusual names for the reflections are explained by an alternative geometric realization of $\text{Cox}(B_n)$ as isometries of an n-torus.

Definition 3.2 (Toroidal symmetries)**.** Let T^n be the n-dimensional torus formed by identifying opposite sides of the n-cube $[-1,1]^n$ and note that the previously defined action of action of $\text{Cox}(B_n)$ the n-cube descends to T^n and it permutes the 2^n special points with every coordinate equal to $\pm\frac{1}{2}$. In fact, the action of $\textsc{Cox}(B_n)$ on these 2^n special points is faithful. A new action of $\textsc{Cox}(B_n)$ on T^n is obtained by leaving the action of the $r_{ij} \in \mathcal{R}$ unchanged and by replacing the "reflection" $t_i \in \mathcal{T}$ with a "translation" which adds 1 to the i-th coordinate mod 2. This has the net effect of switching the sign of the i-th coordinate of each special point because the translation $x \mapsto x+1$ in $\mathbb{R}/2\mathbb{Z}$ switches $\frac{1}{2}$ and $-\frac{1}{2}$. Since the elements in $\mathcal{R} \cup \mathcal{T}$ act on the 2^n special points as before, they generate the same group up to isomorphism.

The group we wish to discuss is generated by lifts of these toriodal isometries to all of $\mathbb{R}^n$.

Definition 3.3 (Permutations and translations)**.** Let r_{ij} act on $\mathbb{R}^n$ as before and let t_i denote the translation which adds 1 to the i-th coordinate leaving the others unchanged. The reflections $\mathcal{R} = \{r_{ij}\}$ and the translations $\mathcal{T} = \{t_i\}$ generate a group $\textsc{Mid}(B_n)$ that we call the *middle group* or more formally the *annular symmetric group*. The names are explained below. The isometries in $\mathcal{R}$ generates the symmetric group $\textsc{Sym}_n$ and the isometries in $\mathcal{T}$ generate a free abelian group $\mathbb{Z}^n$. Moreover, because the $\mathbb{Z}^n$ subgroup generated by the translations is normalized by the permutations in $\textsc{Sym}_n$ with trivial intersection, the full group $\textsc{Mid}(B_n)$ has the structure of a semidirect product $\mathbb{Z}^n \rtimes \textsc{Sym}_n$. Every element of $\textsc{Mid}(B_n)$ can be written uniquely in the form $t_\lambda r_\pi$ where $\lambda \in \mathbb{Z}^n$ is a vector with integer entries and $\pi \in \textsc{Sym}_n$ is a permutation.

Definition 3.4 (Reflections)**.** As in $\textsc{Cox}(B_n)$ there are other reflections in $\textsc{Mid}(B_n)$ obtained by conjugation. The basic translations in $\mathcal{T}$ are closed under conjugation but infinitely many new reflections are added to $\mathcal{R}$ when we close this set under conjugation. For example, $t_1 r_{12} t_1^{-1}$ is a reflection whose fixed hyperplane is parallel to that of r_{12}. We call this reflection $r_{12}(1)$. More generally, for each integer k we define $r_{ij}(k) \coloneqq t_i^k r_{ij} t_i^{-k} = t_j^{-k} r_{ij} t_j^k$. The original reflections are $r_{ij} = r_{ij}(0)$. Let $\mathcal{R}'$ denote the set of all these reflections. The set $\mathcal{R}' \cup \mathcal{T}$ is called the *full generating set* of $\textsc{Mid}(B_n)$.

The center of a middle group is easy to compute.

Proposition 3.5 (Center)**.** *The center of the middle group* $\textsc{Mid}(B_n)$ *is an infinite cyclic subgroup generated by the pure translation* $t_{\mathbf{1}} = \prod_{i=1}^n t_i$.

Proof. Let $u = t_\lambda r_\pi$ be an element in the center. If π is a nontrivial permutation and i is an index that is moved by π then u conjugates t_i to $t_{\pi(i)}$, contradiction. Thus u must be a pure translation t_λ. In order for t_λ to be central λ must be orthogonal to all of the roots of the

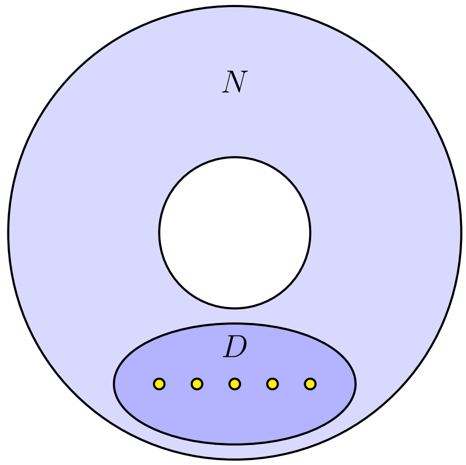

FIGURE 6. The configuration that explains the name annular symmetric group.

reflections and thus in the direction $\mathbf{1} = \langle 1^n \rangle$. In particular, the center is contained in the infinite cyclic subgroup generated by $t_{\mathbf{1}} = \prod_{i=1}^{n} t_i$ which adds 1 to every coordinate. Conversely, $t_{\mathbf{1}}$ commutes with every element of M. □

We call the spherical Artin group $\text{ART}(B_n)$ an *annular braid group* because it is the *braid group of the annulus* in the sense of Birman [Bir74, p.11]. The analogous definition of an *annular symmetric group* leads to an alternative perspective on the group $\text{MID}(B_n)$.

Definition 3.6 (Annular symmetric groups)**.** Let N be an annulus, let D be a disk contained in N and let p_i, $i \in \{1, 2, \ldots, n\}$ be a set of n distinct points in D. See Figure 6. The annular braid group is defined as the fundamental group of the configuration space of n unordered distinct points in the annulus with this configuration as its base point. It keeps track of the way in which the points braid around each other well as how much they wind around the annulus. An *annular symmetric group* ignores the braiding and only keeps track of how the points permute and wind around the annulus. More concretely, if we define r_{ij} as the motion which swaps p_i and p_j without leaving the disk D and define t_i as the motion which wraps the point p_i once around the annulus N in the direction considered positive in its fundamental group, then the elements of this group can be identified with the elements of $\text{MID}(B_n)$ and its normal form $t_\lambda r_\pi$ can be recovered as follows. The permutation π records the permutation of the points and the vector λ is a tuple of winding numbers obtained by viewing the path of the

$$\begin{array}{ccccc}
\textsc{Art}(\widetilde{A}_{n-1}) & \hookrightarrow & \textsc{Art}(B_n) & \twoheadrightarrow & \mathbb{Z} \\
\downarrow & & \downarrow & & \| \\
\textsc{Cox}(\widetilde{A}_{n-1}) & \hookrightarrow & \textsc{Mid}(B_n) & \twoheadrightarrow & \mathbb{Z} \\
 & & \downarrow & & \\
 & & \textsc{Cox}(B_n) & &
\end{array}$$

FIGURE 7. Middle groups and their relatives.

point p_i as a (near) loop that starts and ends in the disk D and letting its winding number be the i-th coordinate of λ.

The name "middle group" refers to its close connections with various other groups as shown in Figure 7.

Remark 3.7 (Affine braid groups)**.** Since the groups $\textsc{Cox}(A_{n-1})$ and $\textsc{Art}(A_{n-1})$ are symmetric groups and braid groups, we call their natural euclidean extensions, the *euclidean symmetric group* $\textsc{Cox}(\widetilde{A}_{n-1})$ and the *euclidean braid group* $\textsc{Art}(\widetilde{A}_{n-1})$. The name "affine braid group" often appears in the literature but its meaning is not stable. For geometric group theorists it refers to the euclidean braid group $\textsc{Art}(\widetilde{A}_{n-1})$ [CP03] but for representation theorists it refers to the annular braid group $\textsc{Art}(B_n)$ [OR07]. Our alternative names aim to limit this potential confusion. The adjective "euclidean" also highlights that the Coxeter group preserves lengths and angles.

The maps in Figure 7 are easy to describe.

Definition 3.8 (Maps)**.** The map from $\textsc{Mid}(B_n)$ onto $\textsc{Cox}(B_n)$ can be seen geometrically. The squares of the basic translations in $\mathcal{T}$ generate a normal subgroup $K \cong (2\mathbb{Z})^n$ in $\textsc{Mid}(B_n)$ and if we quotient $\mathbb{R}^n$ by the action of K the result is the n-torus T^n. The kernel of the induced action of $\textsc{Mid}(B_n)$ on T^n is K and the action itself is easily seen to be the toroidal action of $\textsc{Cox}(B_n)$ on T^n. To understand the horizontal map from $\textsc{Cox}(\widetilde{A}_{n-1})$ to $\textsc{Mid}(B_n)$ we note that the reflections in $\textsc{Mid}(B_n)$ acting on the hyperplane in $\mathbb{R}^n$ perpendicular to the vector $\mathbf{1} = \langle 1^n \rangle$ is the standard realization of the euclidean symmetric group $\textsc{Cox}(\widetilde{A}_{n-1})$. Its image in $\textsc{Mid}(B_n)$ is normal and the quotient sends $u \in \textsc{Mid}(B_n)$ to the sum of the coordinates of the image of the origin under u. We call the map from $\textsc{Mid}(B_n) \twoheadrightarrow \mathbb{Z}$ the *vertical displacement map*. The map from $\textsc{Art}(B_n)$ to $\textsc{Mid}(B_n)$ is clear when these are viewed as the annular braid group and annular symmetric group and the horizontal maps along the top row are well-known [CP03]. More precisely, the map from $\textsc{Art}(B_n)$ to $\mathbb{Z}$ sends elements to the sum of the winding numbers of the various paths from the disk to itself and the

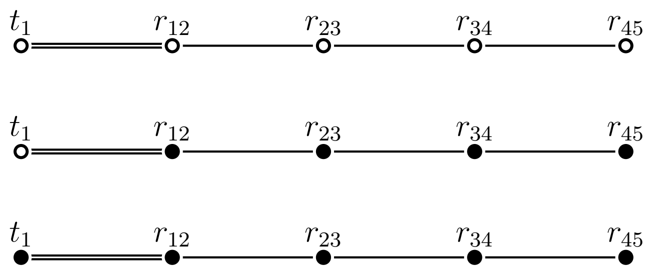

FIGURE 8. Dynkin-diagram style presentations for the three groups $\text{ART}(B_5)$, $\text{MID}(B_5)$ and $\text{COX}(B_5)$. In these diagrams solid circles indicate generators of order 2 and empty circles indicate generators of infinite order.

kernel of this map, the set of annular braids with global winding number 0 is the group $\text{ART}(\widetilde{A}_{n-1})$. We call the map from $\text{ART}(B_n) \twoheadrightarrow \mathbb{Z}$ the *global winding number map.*

The middle column of Figure 7 can be understood via presentations.

Definition 3.9 (Presentation)**.** A standard minimal generating set for $\text{MID}(B_n)$ consists of adjacent transpositions $S = \{r_{ij} \mid j = i+1\} \subset \mathcal{R}$ and the single translation t_1. The set S generates all coordinate permutations and the other basic translations can be obtained by conjugating t_1 by a permutation. There are a number of obvious relations among these generators in addition to the standard Coxeter presentation for the symmetric group. For example, t_1 commutes with r_{ij} for $i, j > 1$ and $t_1 r_{12} t_1 r_{12} = r_{12} t_1 r_{12} t_1$ since both motions are translations which add 1 to the first two coordinates leaving the others unchanged. These relations can be summarized in a diagram following the usual conventions: generators label the vertices, vertices not connected by an edge indicate generators which commute, vertices connect by a single edge indicate generators a and b which "braid" (i.e. $aba = bab$) and vertices connected by a double edge indicate generators a and b which satisfy the relation $abab = baba$. For Coxeter groups, the generators have order 2 and for Artin groups they have infinite order. The middle groups are a mixed case: t_1 has infinite order but the adjacent transpositions have order 2. See Figure 8. It is an easy exercise to show that the relations encoded in the diagram for $\text{MID}(B_n)$ are a presentation and as a consequence the surjections from $\text{ART}(B_n)$ to $\text{MID}(B_n)$ to $\text{COX}(B_n)$ become clear. Also note that the composition of these maps is the standard projection map from $\text{ART}(B_n)$ to $\text{COX}(B_n)$.

And finally we record a slightly more general context where groups isomorphic to middle groups arise. These are the exact conditions which occur in the later sections.

Proposition 3.10 (Recognizing middle groups)**.** *Suppose the symmetric group* $\textsc{Sym}_n$ *acts faithfully by isometries on an* m*-dimensional euclidean space with root system* Φ *and* $m \geq n$*. In addition, let* r *be an element of a Coxeter generating set* S *for* $\textsc{Sym}_n$ *representing one of two ends of the corresponding Dynkin diagram (so that* r *does not commute with exactly one element of* S*). If* $t = t_\lambda$ *is a translation such that* t *does not commute with* r*,* t *does commute with the rest of* S*, and* λ *is not in the span of root system* Φ*, then the group of isometries generated by* $S \cup \{t\}$ *is isomorphic to* $\textsc{Mid}(B_n)$*.*

Proof. Let G be the group generated by these elements. First pick a point fixed by $\textsc{Sym}_n$ to serve as our origin and consider the n-dimensional subspace of $\mathbb{R}^m$ through this point spanned by the vectors $\Phi \cup \{\lambda\}$. Because $S \cup \{t\}$ preserves this subspace and fixes its orthogonal complement, the same is true for the group G that these isometries generate. Thus we may restrict our attention to this subspace. Next establish a coordinate system on this $\mathbb{R}^n$ so that $\textsc{Sym}_n$ is acting by coordinate permutations with the elements of S switching adjacent coordinates and $r = r_{12}$. From the conditions imposed on t we know that in this coordinate system $\lambda = (a, b, b, \ldots, b)$ with $a \neq b$. Finally, note that the generators of G (and thus every element of G) commute with the linear maps which fix the codimension one subspace spanned by Φ and rescale the vectors perpendicular to this subspace, i.e. the vectors with all coordinates equal. After conjugating G by the appropriate such map, we get the standard realization of $\textsc{Mid}(B_n)$ and because this conjugation is reversible the groups are isomorphic. □

4. Intervals and noncrossing partitions

Section 3 discussed the groups $\textsc{Cox}(B_n)$, $\textsc{Mid}(B_n)$ and $\textsc{Art}(B_n)$, the maps between them, and a consistently labeled minimal generating set $\{t_1\} \cup S$ with $S = \{r_{ij} \mid j = i + 1\}$. In this section we investigate intervals in these groups and relate them to noncrossing partitions.

Definition 4.1 (Special elements)**.** Recall that a *Coxeter element* is a element obtained by multiplying together the elements of some Coxeter generating set in a Coxeter group in some order and that for spherical Coxeter groups, or more generally for Coxeter groups whose Dynkin diagram is a tree, all Coxeter elements belong to a single conjugacy

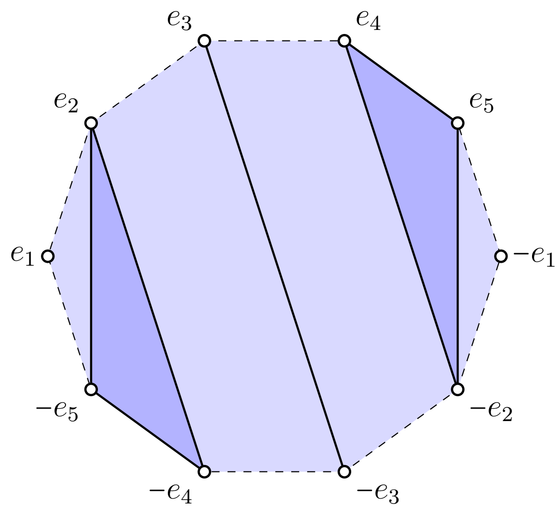

FIGURE 9. A centrally symmetric noncrossing partition in a regular convex decagon.

class. For the group $\text{Cox}(B_n)$ we pick as our *standard Coxeter element* w the product on the standard minimal generating set in the order they appear in the Dynkin diagram: t_1, r_{12}, r_{23}, and so on. Thus, for $\text{Cox}(B_5)$, shown in Figure 8, we have $w = t_1 r_{12} r_{23} r_{34} r_{45}$ and since we compose these as functions (from right to left) the element w sends the point $(x_1, x_2, x_3, x_4, x_5)$ to the point $(-x_5, x_1, x_2, x_3, x_4)$. The groups $\text{Mid}(B_n)$ and $\text{Art}(B_n)$ have analogues of the standard Coxeter element obtained by multiplying the corresponding generators together in the same fashion. In $\text{Mid}(B_5)$ the resulting euclidean isometry $w = t_1 r_{12} r_{23} r_{34} r_{45}$ sends the point $(x_1, x_2, x_3, x_4, x_5)$ to the point $(x_5 + 1, x_1, x_2, x_3, x_4)$ which is consistent with the reinterpretation of t_1 as a translation. We call w the *special element* in all three contexts but in $\text{Cox}(B_n)$ it is more properly called a Coxeter element and in $\text{Art}(B_n)$ it is a *dual Garside element*.

Definition 4.2 (Noncrossing partitions)**.** A *noncrossing partition* is a partition of the vertices of a regular convex polygon so that the convex hulls of distinct blocks are disjoint. A *noncrossing partition of type B* is a noncrossing partition of an even-sided polygon whose blocks are symmetric with respect to a π-rotation about its center. Figure 9 shows a type B noncrossing partition. One partition is below another if every block of the first is contained in some block of the second. Thus the partition where every block is a singleton is the minimum element and the partition with only one block is the maximum element.

It is well-known, at this point, that the type B noncrossing partitions correspond to the special interval in the type B Coxeter group [Rei97].

Lemma 4.3 (Type B intervals)**.** *If $W = \textsc{Cox}(B_n)$ is the type B Coxeter group generated by all of its reflections and w is its special element, then there is a natural identification of the interval $[1,w]^W$ and the type B noncrossing partitions of a $2n$-gon.*

Proof. Every element in W is determined by how it permutes the $2n$ unit vectors $\{\pm e_i\}$ on the coordinate axes and under the element w these form a single cycle of length $2n$: it sends $\pm e_i$ to $\pm e_{i+1}$ for $i < n$ and $\pm e_n$ to $\mp e_1$. If we label the vertices of a $2n$-gon with the vectors in $\{\pm e_i\}$ so that w permutes them in a clockwise fashion (the case $n = 5$ is shown in Figure 9), then the interval $[1,w]^W$ is isomorphic as a poset to the type B noncrossing partitions of this $2n$-gon. The identification goes as follows: associate to each type B noncrossing partition the unique element of W which sends the vector e_i to the vector which occurs next in clockwise order in the boundary of the block to which e_i belongs. For example, the element u corresponding to the partition shown in Figure 9 sends e_1 to e_1, e_2 to $-e_4$, e_3 to $-e_3$ and e_4 to e_5 and e_5 to $-e_2$. Conversely, $u \in W$ lies in the interval $[1,w]^W$ if and only if the orbits of these vectors under u form noncrossing blocks which u rotates in a clockwise manner. □

The following is thus only a minor extension of known results.

Theorem 4.4 (Special intervals)**.** *Let $W = \textsc{Cox}(B_n)$, $M = \textsc{Mid}(B_n)$ and $A = \textsc{Art}(B_n)$ be the Coxeter group of type B, the middle group, and the Artin group of type B with their standard full generating sets and let w denote the special element in all three contexts. The intervals $[1,w]^W$, $[1,w]^M$ and $[1,w]^A$ are isomorphic as labeled posets and their common underlying poset structure is that of the type B noncrossing partition lattice. As a consequence, the group obtained by pulling a middle group apart at its special element is an annular braid group.*

Proof. It is well known that the intervals $[1,w]^W$ and $[1,w]^A$ are isomorphic as labeled posets, that their common underlying poset is the type B noncrossing partition lattice and that A is the group whose presentation is encoded in $[1,w]^W$. This is essentially what is meant when we say that spherical Artin groups have dual Garside presentations. In particular, the final assertion is immediate once we show that $[1,w]^M$ is isomorphic to the others as a labeled poset. Showing that all the factorizations in $[1,w]^W$ lift from a factorization of an isometry of the n-torus T^n to a factorization of the corresponding isometry of $\mathbb{R}^n$ and that no new factorizations arise is an easy exercise. □

These noncrossing diagrams make it easy to show that very few simples commute with the special element.

Proposition 4.5 (Commuting with w)**.** *Let G be the group* $\text{Cox}(B_n)$*,* $\text{Mid}(B_n)$ *or* $\text{Art}(B_n)$ *with its standard generating set and let w be its special element. The only elements in $[1,w]^G$ that commute with w are the bounding elements* 1 *and w.*

Proof. If an element u in $[1,w]^G$ commutes with w then it must correspond to a centrally symmetric noncrossing partition of a $2n$-gon that is invariant under a $\frac{\pi}{n}$-rotation since this is how conjugation by w acts on the type B noncrossing partitions. The only noncrossing partitions left invariant under this action are the partition in which every vertex belongs to a distinct block and the partition in which all the vertices belong to single block, and these correspond to 1 and w respectively. ☐

Remark 4.6 (Generators and relations)**.** The generators and relations visible inside $[1,w]^M$ can be given more explicitly. The edge labels in the interval $[1,w]^W$ are exactly the n^2 reflection generators of $W = \text{Cox}(B_n)$ but this finite set is far from the full (infinite) generating set of $M = \text{Mid}(B_n)$. In fact, the only elements which appear are $\mathcal{T} = \{t_i\}$, $\mathcal{R} = \{r_{ij}\}$ and the set $\mathcal{R}(1) = \{r_{ij}(1)\}$. The element t_i corresponds to the diagonal edge connecting e_i and $-e_i$, the element r_{ij} corresponds to the pair of edges connecting $\pm e_i$ to $\pm e_j$ and the element $r_{ij}(1)$ corresponds to the pair of edges connecting $\pm e_i$ to $\mp e_j$. If two generators correspond to edges which are completely disjoint, then there exists a commutation relation visible in the interval $[1,w]^M$. For example, the length four cycle $t_1 r_{23} = r_{23} t_1$ can be found inside $[1,w]^M$. If two generators correspond to pairs of nondiagonal edges with only one endpoint in common, then dual braid relations are visible inside $[1,w]^M$ of the following form: there are three generators a, b and c with $ab = bc = ca$ visible in the interval. To illustrate, the generators r_{45} and $r_{25}(1)$ share an endpoint and we have relations $r_{45} r_{25}(1) = r_{25}(1) r_{24}(1) = r_{24}(1) r_{45}$ and so r_{45} and $r_{25}(1)$ braid in the corresponding interval group. Finally, the generators t_i, r_{ij}, t_j and $r_{ij}(1)$ (with $i < j$) satisfy a dual Artin relation of length 4: $t_i r_{ij} = r_{ij} t_j = t_j r_{ij}(1) = r_{ij}(1) t_i$. These relations, taken together, are a complete presentation for the spherical Artin group $\text{Art}(B_n)$.

Our final result in this section gives a new perspective on the horizontal maps between the first two columns of Figure 7.

Proposition 4.7 (Horizontal maps)**.** *If $M = \text{Mid}(B_n)$ is a middle group with special element w, then* (1) *the reflections labeling edges in*

$[1,w]^M$ *generate a copy of* $\textsc{Cox}(\widetilde{A}_{n-1})$ *inside* M*,* (2) *the group generated by these elements and subject only to the relations among them visible in* $[1,w]^M$ *is isomorphic to* $\textsc{Art}(\widetilde{A}_{n-1})$*, and* (3) *the natural projection map from this group to* M *factors through and injects into the annular braid group* $\textsc{Art}(B_n)$*.*

Proof. The reflections labeling an edge in $[1,w]^M$ are $\{r_{ij}\} \cup \{r_{ij}(1)\}$ and the subset $\{r_{ij} \mid j = i+1\} \cup \{r_{1n}(1)\}$ is already sufficient to generate the $\textsc{Cox}(\widetilde{A}_{n-1})$ subgroup of M since they bound a chamber in the $\widetilde{A}_{n-1}$ tiling of the hyperplane orthogonal to the vector $\mathbf{1} = \langle 1^n \rangle$. Next, notice that these elements correspond to the boundary edges of the $2n$-gon for w. As such they never cross and either braid or commute depending on whether or not they have endpoints in common. Thus the group defined by just these generators and relations is isomorphic to the group $\textsc{Art}(\widetilde{A}_{n-1})$. It is now straightforward to check the other generators and relations are consistent with this identification and what we have described is the standard copy of $\textsc{Art}(\widetilde{A}_{n-1})$ inside $\textsc{Art}(B_n)$. □

Part 3. New Groups

In this part we introduce several new groups closely related to each irreducible euclidean Coxeter group and its corresponding Artin group.

5. Intervals in euclidean Coxeter groups

Let $W = \textsc{Cox}(\widetilde{X}_n)$ be an irreducible euclidean Coxeter group with reflections R and Coxeter element w. The coarse structure of the interval $[1,w]^W$ was determined in the earlier articles [BM15] and [McC15] and in this section we recall the revelant definitions and results. The first article, by Noel Brady and the first author characterized the set of all possible minimum length factorizations of a fixed euclidean isometry into arbitrary reflections and the second showed that Coxeter intervals in irreducible euclidean Coxeter groups are subposets of the unrestricted intervals analyzed in the first article. We begin by recalling the distinction between points and vectors.

Definition 5.1 (Points and vectors)**.** Let V denote an n-dimensional real vector space with the standard positive definite inner product and let E be the corresponding euclidean analogue where the location of the origin has been forgotten leaving only a simply transitive action of V on E. The elements of V are called *vectors* and the elements of E are called *points*. Ordered pairs of points in E determine a vector in V.

In [BM15] euclidean isometries are analyzed in terms of their two basic invariants: min-sets in E and move-sets in V.

Definition 5.2 (Basic invariants)**.** Let u be an isometry of E. If λ is the vector from x to $u(x)$ then we say that x is *moved by* λ *under* u. The collection $\textsc{Mov}(u) = \{\lambda \mid x + \lambda = u(x)\} \subset V$ of all such vectors is the *move-set* of u. The subset $\textsc{Mov}(u)$ is an affine subspace of V and for each $\lambda \in \textsc{Mov}(u)$ the points of E moved by λ form an affine subspace of E [BM15, Proposition 3.2]. In particular, there is a unique vector μ in $\textsc{Mov}(u)$ of minimal length and the corresponding points in E form the *min-set* of u, $\textsc{Min}(u)$. An isometry u is *elliptic* under the equivalent conditions that the vector μ is trivial, $\textsc{Mov}(u)$ contains the origin in V and there are points fixed by u. For elliptic isometries we sometimes write $\textsc{Fix}(u)$ instead of $\textsc{Min}(u)$. Isometries that are not elliptic are called *hyperbolic*.

Let $L = \textsc{Isom}(E)$ be the Lie group of all euclidean isometries. The main results in [BM15] analyze the structure of the intervals in $\textsc{Isom}(E)$ with all reflections as its (trivially weighted) generating set. We call the interval $[1, w]^L$ an *ellptic* or *hyperbolic interval* depending on the nature of w. In both cases, the elements of the intervals and the ordering can be precisely described in terms of their basic invariants. See [BM15] for details. In this article we only need the coarse structure of hyperbolic intervals where w has maximal reflection length, and in this context we define horizontal and vertical directions.

Definition 5.3 (Horizontal and vertical)**.** If w is a hyperbolic isometry whose min-set is a line, then the direction this line is translated is declared to be *vertical* and the orthogonal directions are *horizontal*. A reflection, or more generally an elliptic isometry is called *horizontal* if every point moves in a horizontal direction and it is vertical otherwise. Thus a vertical elliptic isometry merely needs to have some vertical component to the motion of some point.

A Coxeter element for the $\widetilde{G}_2$ tiling is a glide reflection and thus an isometry of this type (see Figure 11). Of the 6 families of parallel reflections there is one family of horizontal reflections and five families of vertical reflections. These can be distinquished by whether or not their fixed hyperplanes cross the glide axis.

Definition 5.4 (Coarse structure)**.** Let $L = \textsc{Isom}(E)$ be the Lie group of all euclidean isometries and let w be a hyperbolic euclidean isometry whose min-set is a line. For each element u in the interval $[1, w]^L$ we consider the pair (u, v) where $uv = w$. There are exactly three possible cases: (1) u is a horizontal elliptic isometry and v is hyperbolic, (2) both u and v are vertical elliptic isometries, and (3) u is hyperbolic and v is horizontal elliptic. These form the three rows of the *coarse structure* of

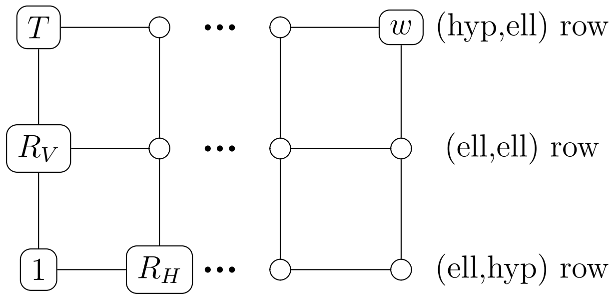

FIGURE 10. Coarse structure of a hyperbolic interval.

the interval arranged from bottom to top and shown in Figure 10. The bottom row is graded by the dimension of the fixed set of u from the identity element on the left to the elliptics fixing only a vertical line on the right. The middle row has a similar grading: from those that fix a non-vertically invariant hyperplane on the left to those fixing only a single point on the right. Alternatively, we could focus on v instead of u. The v on the left end of the middle row fix only a point and the v on the right fix a non-vertically invariant hyperplane. Finally, the top row is also graded by the fixed set of v: from v fixing a vertical line on the left to v equal to the identity on the right. For every affine subspace of E there is exactly one elliptic u in one of the bottom two rows whose fix-set is this subspace. Similarly, there is exactly one elliptic v in one of the top two rows whose fix-set is this subspace. Covering relations correspond to one horizontal or one vertical step in this grid. Elements higher in the poset order are above and/or to the right while those lower down are down and/or to the left. Finally, note that the second box on the bottom row contains the horizontal reflections, the first box in the middle row contains the vertical reflections, and the first box on the top row contains pure translations.

It turns out that for any irreducible euclidean Coxeter group $W = \text{Cox}(\widetilde{X}_n)$ with reflections R and Coxeter element w, the *Coxeter interval* $[1,w]^W$ is a subposet of the corresponding hyperbolic interval in the full euclidean isometry group [McC15]. In particular, it has the same basic structure.

Definition 5.5 (Coxeter intervals)**.** As described in [McC15], the min-set of the Coxeter element w is a line ℓ called the *Coxeter axis*. Every point on this line is contained in the interior of some top-dimensional simplex, except for a discrete set of equally spaced points x_i for $i \in \mathbb{Z}$. The simplices through which ℓ passes are called *axial simplices* and the vertices of these simplices are *axial vertices*. The reflections which occur

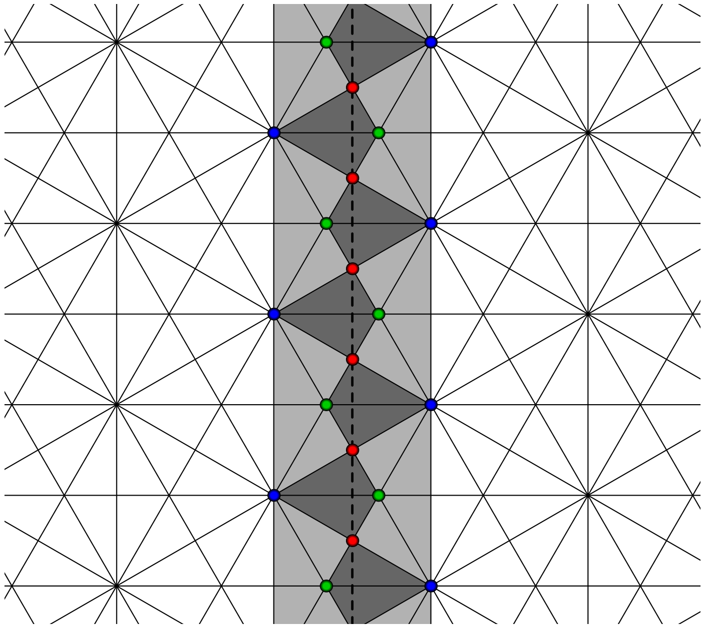

FIGURE 11. The $\widetilde{G}_2$ tiling of the plane with annotations corresponding to a particular Coxeter element w.

as edge labels in the interval $[1,w]^W$ are precisely those that contain an axial vertex in its fixed hyperplane [McC15, Theorem 9.6]. This includes all of the vertical reflections in W but only a finite number of the horizontal ones. We call these sets R_V and R_H respectively. Since the Coxeter axis passes through the interior of top-dimensional simplices, it does not lie on the hyperplane of any horizontal reflection. For each family of parallel horizontal reflections, the only ones in the interval are the ones determined by the adjacent pair of hyperplanes which contain the Coxeter axis between them. In other words, there are precisely two horizontal reflections in the interval for each antipodal pair of horizontal roots in the root system.

The next lemma records a slightly technical fact about roots and axial vertices that generalizes the observation above about horizontal reflections. It was verified by computer for the sporadic types and by hand for the infinite families.

Lemma 5.6 (Convexity)**.** *Let $W = \text{Cox}(\widetilde{X}_n)$ be an irreducible euclidean Coxeter group with Coxeter element w and let r be a reflection that contains at least one axial vertex in its fixed hyperplane H. If α is a root in the type X_n root system such that α has a positive dot product with the direction of the Coxeter axis and the image of α under the*

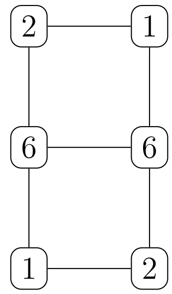

FIGURE 12. Coarse structure of the $\widetilde{G}_2$ interval.

reflection r has a negative dot product with the direction of the Coxeter axis, then the convex hull of the axial vertices contained in H lies between two consecutive hyperplanes in the Coxeter complex with normal vector α.

Heuristically, the reason why Lemma 5.6 is true is that there are Coxeter elements whose axial vertices overlap with this set of axial vertices in the hyperplane H and in this alternative world, the consecutive reflections with normal vector α are horizontal with respect to the other Coxeter element and bound its column of axial vertices.

Example 5.7 ($\widetilde{G}_2$ interval)**.** The $\widetilde{G}_2$ tiling of the plane is shown in Figure 11 with various aspects highlighted. The Coxeter element w is a glide reflection whose glide axis is its min-set. This is shown as a dashed line. The heavily shaded triangles are the axial simplices of w and the large dots indicate the axial vertices. The lightly shaded vertical strip is the convex hull of the axial vertices and it is bounded by the only two horizontal reflections which occur in the Coxeter interval. The coarse structure of the interval is shown in Figure 12. The numbers along the top and bottom rows represent the finite number of elements of each type in the interval. Thus, R_H contain two horizontal reflections and T contains two pure translations. The middle row requires a more detailed explanation. The convex hull has a structure which repeats vertically and the numbers in the middle row record how many distinct local situations there are in each box. For example, there are infintely many vertical reflections in the interval but only six different types and there are infinitely many elliptic isometries in the interval that fix a single point but only six different types. In the former case the reflections are mostly distinguished by their slope but there are two with horizontal fixed lines that have distinct local neighborhoods. Similarly, in the latter case the rotations are mostly distinguished by the horizontal displacement of their fixed point except that there are

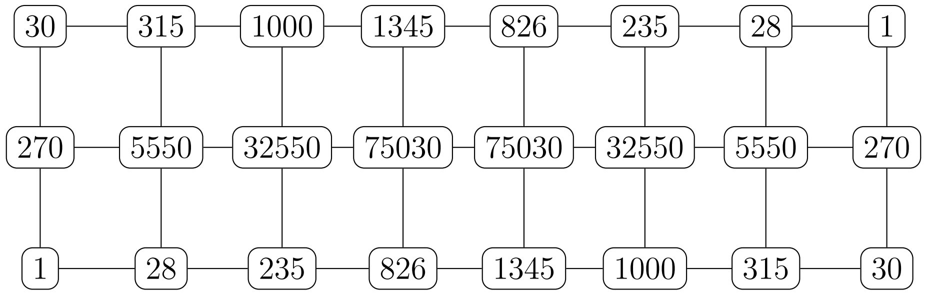

FIGURE 13. Coarse structure of the $\widetilde{E}_8$ interval.

two distinct types of fixed points along the Coxeter axis itself. Both of these are π-rotations about their fixed point but they have distinct local neighborhoods and thus decompose into distinct types of reflections.

The coarse structure of the Coxeter interval in the largest of the sporadic euclidean Coxeter groups offers a more substantial illustration.

Example 5.8 ($\widetilde{E}_8$ interval)**.** The coarse structure of the Coxeter interval $[1,w]^W$ for the group $W = \textsc{Cox}(\widetilde{E}_8)$ is shown in Figure 13. From the figure we see that it contains 28 horizontal reflections, 30 pure translations and 270 infinite families of similarly situated vertical reflections. In general, the numbers along the top and bottom refer to the number of individual elements in that box and the numbers in the middle row refer to number of infinite families of similarly situated elliptic elements. We should note representatives of the roughly quarter-million types summarized in the figure were computed by a program `euclid.sage` written by the first author and available upon request.

We conclude this section by reviewing an explicit presentation for the dual euclidean Artin group derived from the Hurwitz action of the braid group on factorizations of w.

Definition 5.9 (Hurwitz action)**.** Because reflections in W are closed under conjugation, factorizations in $[1,w]^W$ can be rewritten in many ways and, in fact, there is an action of the braid group on the minimal length factorizations of w called the *Hurwitz action*. The i-th standard braid generator replaces the two letter subword ab in positions i and $i+1$ with the subword ca where $c = aba^{-1}$ and it leaves the letters in the other positions unchanged. It is easy to check that this action satisfies the relations in the standard presentation of the braid group.

When a standard braid generator replaces ab with ca inside a minimal length factorization of w, the relation $ab = ca$ is visible in $[1,w]^W$. Such a relation is called a *Hurwitz relation* or a *dual braid relation*. When the Hurwitz action is transitive on factorizations, these relations are sufficient to define the interval group W_w [McC15, Proposition 3.2] and we call this the *Hurwitz presentation*. In 2010 Igusa and Schiffler proved transitivity of the Hurwitz action on reflection factorizations of Coxeter elements in Coxeter groups in complete generality [IS10] and in 2014 a short proof of this general fact was posted by Baumeister, Dyer, Stump and Wegener [BDSW14]. As an illustration, we give the Hurwitz presentation of the dual $\widetilde{G}_2$ Artin group. We start with the generators. The dual generators are closely connected to the Coxeter axis of w and we introduce a notation that reflects this fact.

Definition 5.10 (Dual $\widetilde{G}_2$ generators)**.** In the case of $\widetilde{G}_2$ we use the letters a through f to indicate the slope of its fixed line in the ascending order: $-\sqrt{3}$, $\frac{-1}{\sqrt{3}}$, 0, $\frac{1}{\sqrt{3}}$, $\sqrt{3}$ and ∞, respectively. See Figure 11. Next, recall that the hyperplanes of the vertical reflections intersect the axis in an equally spaced set of points x_i for $i \in \mathbb{Z}$ [McC15, Section 8]. We use subscripts on the vertical reflections that indicates which x_i its hyperplane contains. Note that not every combination of letter and subscript actually occurs. For $\widetilde{G}_2$ we let x_0 be the intersection of one of the horizontal lines with the axis, specifically one which intersects an axial vertex on the lefthand side of the shaded vertical strip. There are only two horizontal reflections in the interval $[1,w]^W$ and we call these f_- and f_+. Putting this all together, the dual generators of $\textsc{Art}^*(\widetilde{G}_2, w)$ are the set $\{a_i, b_j, c_k, d_i, e_j, f_\ell\}$ where $i = 1 \mod 4$, $j = 3 \mod 4$, $k = 0 \mod 2$ and $\ell \in \{+,-\}$.

The periodicity of the subscripts corresponds to the fact that there is a power of w which acts as a pure translation in the direction of the Coxeter axis. In $\textsc{Cox}(\widetilde{G}_2)$ this power is w^2 and the action of w^2 on the plane shifts the point x_i to x_{i+4}.

Definition 5.11 (Dual $\widetilde{G}_2$ relations)**.** The dual braid relations in the $\widetilde{G}_2$ case are obtained by factoring the elements in the interval $[1,w]^W$ of reflection length 2. In the coarse structure, the elements to be factored belong to the third box in the bottom row, the second box in the middle row and the first box in the top row. The first type does not occur in $\widetilde{G}_2$. The third type are the pure translations and they have infinitely many factorizations. In the case of $\widetilde{G}_2$ there are exactly two translations in the interval and their factorizations are as follows.

$$\begin{aligned} \cdots = a_9a_5 = a_5a_1 = a_1a_{-3} = a_{-3}a_{-7} = \cdots \\ \cdots = e_{11}e_7 = e_7e_3 = e_3e_{-1} = e_{-1}e_{-5} = \cdots \end{aligned} \tag{1}$$

It only remains to list the factorizations of the 6 infinite families of elliptic elements that correspond to the second box in the middle row. In the $\widetilde{G}_2$ case, these are rotations that fix a single point. Representative sets of equations are as follows.

$$\begin{gathered}
a_1 d_1 = d_1 a_1 \\
b_3 e_3 = e_3 b_3 \\
c_2 a_1 = e_3 c_2 = a_1 e_3 \\
a_1 c_0 = e_{-1} a_1 = c_0 e_1 \\
a_{-3} f_- = b_{-1} a_{-3} = c_0 b_{-1} = d_1 c_0 = e_3 d_1 = f_- e_3 \\
e_{-1} f_+ = d_1 e_{-1} = c_2 d_1 = b_3 c_2 = a_5 b_3 = f_+ a_5
\end{gathered} \tag{2}$$

To get all of the equations in the six infinite families, one should pick an arbitrary multiple of 4 and consistently add it to each of the subscripts in each of six lines of equations above. This corresonds to the vertical shift which conjugation by w^2 produces. The $+/-$ subscripts remain underchanged since these reflections are invariant under vertical translation.

6. Horizontal roots and factored translations

In this section we describe the roots that are horizontal with respect to the axis of a Coxeter element and we use their geometry to define a series of crystallographic groups acting geometrically on euclidean space. Although Coxeter elements are usually defined as a product of the reflections fixing the facets of a chamber in the Coxeter tiling, there are other factorizations and one in particular where most of the reflections are horizontal with respect to its axis.

Definition 6.1 (Horizontal roots)**.** If w is a Coxeter element for the irreducible euclidean Coxeter group $W = \textsc{Cox}(\widetilde{X}_n)$, then w has a factorization into a pure translation and $n-1$ horizontal reflections. To see this we start with a standard factorization such as $w = r_{\alpha,1} w_0$ where $r_{\alpha,1}$ is the reflection corresponding to the root α used to extend the Dynkin diagram X_n shifted so it does not fix the origin and w_0 is a Coxeter element of the spherical Coxeter group $W_0 = \textsc{Cox}(X_n)$. By Proposition 2.10 we can find an alternative factorization of w_0 as the product of a Coxeter generating set whose leftmost reflection is r_α. Thus we can write $w_0 = r_\alpha w_h$ where w_h is a Coxeter element of a maximal parabolic subgroup of W_0. This means that $w = r_{\alpha,1} r_\alpha w_h = t_{\alpha^\vee} w_h$. Since the element w_h is an elliptic isometry fixing a line and $t = t_{\alpha^\vee}$ is a pure translation, the fixed line of w_h must be parallel to the Coxeter axis of w. As a consequence the $n-1$ reflections multiplied together to produce w_h are horizontal with respect to the axis of w. Moreover, since the fixed line of w_h passes through the fixed point of w_0 and every family of parallel hyperplanes contains one which passes through this fixed point, these $n-1$ horizontal reflections generate a group W_h with one representative from every parallel family of horizontal reflections in W. In other words, all reflections in W_h are horizontal and every horizontal reflection is parallel to one in W_h. We call W_h the *horizontal Coxeter group* and $w = t_{\alpha^\vee} w_h$ a *horizontal factorization of* w. The horizontal roots associated to these reflections are a root system described by the diagram for W_h, and this diagram is the diagram for W_0 with an additional vertex removed, the one shown in Figures 2 and 3 as a large shaded dot. We call the corresponding root the *vertical root*.

Remark 6.2 (Finding vertical roots)**.** The vertical roots were first found in [McC15] on a case-by-case basis but once the principles are clear they can be easily spotted. Because the simple system for W_0 used to create the horizontal factorization spans a positive cone, the vertical root should be as close to horizontal as possible. This favors branch points and vertices involved in multiple bonds, specifically the end corresponding to the longer root. This rule uniquely determines the vertice root in all cases except in type A where there are distinct conjugacy classes of Coxeter element that lead to distinct choices of vertical root.

The next proposition records some basic facts about the pure translations that occur in the interval $[1,w]^W$ of an irreducible euclidean Coxeter group W. These are easily checked by hand for the infinite families and by computer for the sporadic types.

Proposition 6.3 (Pure translations below w)**.** *If* $W = \textsc{Cox}(\widetilde{X}_n)$ *is an irreducible euclidean Coxeter group with Coxeter element* w*, then every pure translation* t *contained in the interval* $[1,w]^W$ *is the translation part of some horizontal factorization of* w*. Moreover, if* $t = r'r$ *is a factorization of* t *into a pair of reflections, then* $r' = (w^p)r(w^{-p})$ *where* w^p *is the smallest power of* w *which acts on the Coxeter complex as a pure translation. In fact, all factorizations of* t *in* $[1,w]^W$ *are of the form* $t = r_{i+1}r_i$ *where* $r_i = (w^{ip})r(w^{-ip})$ *for some integer* i*.*

Definition 6.4 (Components)**.** The structure of the horizontal root system is listed in Table 1 for each irreducible type and note that the number of irreducible components varies from one to three. The groups of type $\widetilde{C}_n$, $\widetilde{A}_n$ (with $q = 1$) and $\widetilde{G}_2$ have a single component, the groups of type $\widetilde{B}_n$, $\widetilde{A}_n$ (with $q \geq 2$) and $\widetilde{F}_4$ have two components, and the groups of type $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$ and $\widetilde{E}_8$ have three components. We orthogonally decompose the space V of vectors into components as

Type	Horizontal root system
A_n	$\Phi_{A_{p-1}} \cup \Phi_{A_{q-1}}$
C_n	$\Phi_{A_{n-1}}$
B_n	$\Phi_{A_1} \cup \Phi_{A_{n-2}}$
D_n	$\Phi_{A_1} \cup \Phi_{A_1} \cup \Phi_{A_{n-3}}$
G_2	Φ_{A_1}
F_4	$\Phi_{A_1} \cup \Phi_{A_2}$
E_6	$\Phi_{A_1} \cup \Phi_{A_2} \cup \Phi_{A_2}$
E_7	$\Phi_{A_1} \cup \Phi_{A_2} \cup \Phi_{A_3}$
E_8	$\Phi_{A_1} \cup \Phi_{A_2} \cup \Phi_{A_4}$

TABLE 1. The structure of the horizontal root system for each irreducible euclidean Coxeter group $\text{Cox}(\widetilde{X}_n)$. For the group of type A_n we list the structure of system of roots horizontal with respect to the axis of the (p,q)-bigon Coxeter elements defined in [McC15].

follows: $V = V_0 \oplus \cdots \oplus V_k$ where V_0 is the line spanned by the direction of the Coxeter axis and the components V_i for $1 \leq i \leq k$ correspond to the subspaces spanned by the irreducible components of the horizontal root system. Since every horizontal reflection corresponds to a root in exactly one of component V_i, we can partition any minimal Coxeter generating set S_H for W_h and the full set of reflections R_H into disjoint subsets $S_H^{(i)}$ and $R_H^{(i)}$.

It was an early hope that every dual Artin group would be a Garside group, but it was shown in [McC15] that this is not always the case, even when attention is restricted to Artin groups of euclidean type. It turns out that the number of components of the horizontal root system is crucial.

Remark 6.5 (Garside structures)**.** In [McC15] the first author proved that the unique dual presentation of $\text{Art}(\widetilde{X}_n)$ is a Garside structure when X is C or G and it is not a Garside structure when X is B, D, E or F. When the group has type A there are distinct dual presentations and the one investigated by Digne is the only one that is a Garside structure. The positive results for types A and C are due to Digne. The negative results are a direct consequence of horizontal root systems with more than one irreducible component. The reducibility leads directly to a failure of the lattice condition [McC15, Theorem 10.3]. Knowing explicitly how and why the lattice property fails led to the groups we introduce below. The second author worked out the structure of the

Name	Symbol	Generating set
Coxeter	W	$R_H \cup R_V$ $(\cup T)$
Horizontal	H	R_H
Diagonal	D	$R_H \cup T$
Factorable	F	$R_H \cup T_F$ $(\cup T)$
Crystallographic	C	$R_H \cup R_V \cup T_F$ $(\cup T)$

TABLE 2. Five euclidean isometry groups.

Artin group of type $\widetilde{B}_3$ along the lines presented here in his dissertation under the supervision of the first author and it is these arguments that have now been generalized to arbitrary Artin groups of euclidean type [Sul10].

Definition 6.6 (Diagonal translations)**.** Let w be a Coxeter element in an irreducible euclidean Coxeter group $W = \textsc{Cox}(\widetilde{X}_n)$ and let $w = t_\lambda w_h$ be a horizontal factorization of w. We call the translation t_λ a *diagonal translation* because λ projects nontrivially to each of the components V_i $0 \leq i \leq k$. The vector λ projects nontrivially to V_0, the direction of the Coxeter axis, because w translates the axis vertically but the element w_h only moves points horizontally. And λ projects nontrivially to each horizontal component V_i with $i > 0$ because the vertical root is connected by an edge to each component of the horizontal root system in the diagram X_n. Also note that t_λ is not orthogonal to exactly one reflection in each horizontal component.

Definition 6.7 (Factored translations)**.** Let w be Coxeter element in an irreducible euclidean Coxeter group $W = \textsc{Cox}(\widetilde{X}_n)$ with a fixed horizontal factorization of w and let k be the number of horizontal components. Let t_λ be the corresponding vertical root translation and let $\lambda_i = \text{proj}_{V_i}(\lambda)$ denote the nontrivial projection vectors to each subspace V_i. Finally let $t_i = t_{\lambda_i} + \frac{1}{k} t_{\lambda_0}$ so that $t = \prod_{i=1}^{k} t_i$. The translations t_i are called *factored translations*. If we do this for every translation in the interval $[1, w]^W$ then we get a collection T_F of all factored translations. Like the horizontal reflections, they can be partitioned into subsets $T_F^{(i)}$ based on the particular component of the horizontal root system involved so that $T_F^{(i)}$ contains the factored translations whose displacement vector lies in $V_0 \oplus V_i$.

For each Coxeter element in an irreducible euclidean Coxeter group there are five closely related euclidean isometry groups that are involved in our proofs.

Definition 6.8 (Five euclidean isometry groups)**.** Let $W = \text{Cox}(\widetilde{X}_n)$ be an irreducible euclidean Coxeter group. For each choice of Coxeter element w, we have defined four sets of euclidean isometries: the horizontal reflections R_H and the vertical reflections R_V labeling edges in the interval $[1,w]^W$, the translations T and the factored translations T_F. Various combinations of these sets generate five euclidean isometry groups as shown in Table 2. The *horizontal group* H is the euclidean isometry group generated by the set R_H of horizontal reflections below w. It contains but is bigger than the group W_h because it contains two horizontal reflections for each horizontal root. The *diagonal group* D is the euclidean isometry group generated by $R_H \cup T$, the horizontal reflections and the pure translations below w. The *factorable group* F is the euclidean isometry group generated by $R_H \cup T_F$. And the *crystallographic group* $C = \text{Cryst}(\widetilde{X}_n, w)$ is the group generated by the union of all four sets. Since every diagonal translation can be written either as a product of two parallel vertical reflections or as a product of k factored translations, the set T can be optionally included in the generating sets for W, F and C without altering the group.

The crystallographic group $C = \text{Cryst}(\widetilde{X}_n, w)$ and the Coxeter group $W = \text{Cox}(\widetilde{X}_n)$ have a very similar structure.

Remark 6.9 (Crystallographic)**.** Recall that a group action on a metric space is *geometric* when the group acts properly discontinuously and cocompactly by isometries and that a group acting geometrically on a finite dimensional euclidean space is a *crystallographic group.* This category includes but is larger than the class of euclidean Coxeter groups since crystallographic groups do not need to be generated by reflections. For example, most of the 17 distinct wallpaper groups acting geometrically on the plane are not euclidean Coxeter groups. The group $C = \text{Cryst}(\widetilde{X}_n, w)$ is crystallographic because its structure is essentially the same as that of the Coxeter group $W = \text{Cox}(\widetilde{X}_n, w)$. In particular, it has a normal translation subgroup with quotient spherical Coxeter group $W_0 = \text{Cox}(X_n)$. The only difference is that the new translation subgroup is slightly bigger: the old translation subgroup is finite index in the new one.

7. Intervals in the new groups

In this section we define and analyze intervals in four of the groups introduced in the previous section. We begin by extending our system of weights to the larger generating sets.

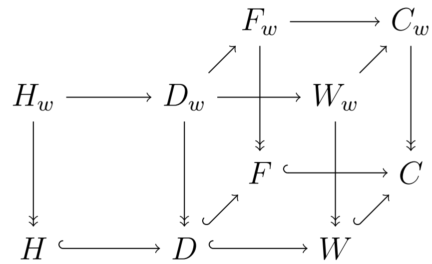

FIGURE 14. Ten groups defined for each choice of a Coxeter element in an irreducible euclidean Coxeter group and some of the maps between them.

Definition 7.1 (Weights)**.** We extend the trivial weighting on the full set R of reflections so that the new factorizations preserve length. The natural weights assign 1 to each horizontal and vertical reflection, 2 to each diagonal translation and $\frac{2}{k}$ to each factored translation where k is the number of components of the horizontal root system.

With this system of weights, the intervals behave as expected. There are inclusions among the intervals $[1,w]^X$ where X is D, F, W or C, that mimic the relations between the groups as shown in Figure 14. The next lemma records additional relations among these intervals.

Lemma 7.2 (Interval relations)**.** *For each choice of a Coxeter element w in an irreducible euclidean Coxeter group, the intervals described above are related as follows:*

$$[1,w]^C = [1,w]^W \cup [1,w]^F$$
$$[1,w]^D = [1,w]^W \cap [1,w]^F$$

Proof. The second equality is an immediate consequence of the relations among the generating sets, as is the fact that $[1,w]^C \supset [1,w]^W \cup [1,w]^F$. It only remains to show that there does not exist a minimal length factorization of w in C that includes both a factored translation and a vertical reflection. To see this consider the map from C to W_0 obtained by quotienting out its normal subgroup of pure translations. The image of w under this map is a Coxeter element for the horizontal Coxeter group W_h. It fixes a line parallel to the Coxeter axis through the unique point fixed by all of W_0. Since its move-set is $(n-1)$-dimensional, its minimal reflection length is $n-1$, and this length is only possible if each of the $n-1$ reflections in the product contain the fixed line in their fixed hyperplane. In other words, this happens only when they are all horizontal reflections. When this minimum is not

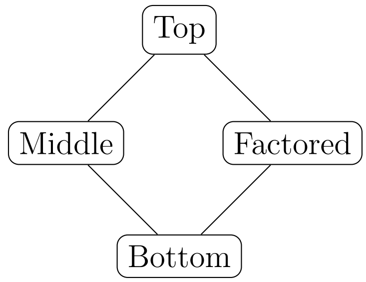

FIGURE 15. A very coarse overview of the structure of the interval $[1,w]^C$.

achieved, at least $n+1$ reflections are involved because of parity issues. If we start with a factorization of w that contains a factored translation, then its image in W_0 has length strictly less than $n+1$, and as a consequence all of the reflections involved are horizontal. □

Using Lemma 7.2 we extend the notion of a coarse structure to these new intervals.

Remark 7.3 (Coarse structure)**.** The crystallographic interval $[1,w]^C$ is obtained by adding additional elements to the original three rows in the coarse structure of the Coxeter interval $[1,w]^W$. This is schematically shown in Figure 15 but the reader should note that the box labeled Factored is not a single row but rather it includes all factorization pairs (u,v) with $uv = w$ where both u and v require a factored translation in their construction. The original Coxeter interval $[1,w]^W$ is the subposet containing the top, middle and bottom portion, the diagonal interval $[1,w]^D$ is the poset containing only the top and bottom rows, and the factor interval $[1,w]^F$ is the subposet containing only the top, bottom and factored portions. One consequence of this is that the groups D (and the pulled apart group D_w defined below) have alternate generating sets. Instead of using $R_H \cup T$ we could instead use $R_H \cup \{w\}$. This is because every element in the bottom row is a product of horizontal reflections and every element in the top row differs from w by a product of horizontal reflections.

There are other properties that are nearly immediate.

Proposition 7.4 (Balanced and self-dual)**.** *For each choice of a Coxeter element w in an irreducible euclidean Coxeter group, the interval between* 1 *and w in each of D, W, F and C is a balanced and self-dual poset.*

Proof. Each interval is balanced because the generating sets are closed under local conjugations. This also means that the map sending u to its left complement is an order-reversing poset isomorphism. □

Using these intervals we can create new groups.

Definition 7.5 (Five groups via presentations)**.** Four of the groups on the top level of Figure 14 are interval groups obtained by pulling apart the corresponding groups on bottom level. The exception is H_w. We define this group as the group generated by the horizontal reflections R_H in the interval $[1,w]^W$ and subject only to the relations among them that are visible there. There is not a natural interval group here because w itself is not an element of H; it is merely the horizontal portion of the other groups on the top level. Finally, we should note that the groups C_w and W_w turn out to be the Garside group described in the introduction and the Artin group $\textsc{Art}(\widetilde{X}_n)$ respectively.

The inclusion relations among the various generating sets suffice to establish the injections shown on the lower level of Figure 14 and inclusions among the sets of relations induce the homomorphisms on the top level. It turns out that all the maps on the top level are also injective but this is not immediately clear. Several of these groups are easily identified.

Proposition 7.6 (Products)**.** *If $W = \textsc{Cox}(\widetilde{X}_n)$ is an irreducible euclidean Coxeter group with Coxeter element w and k horizontal components, then the interval $[1,w]^F$ is a direct product of k type B noncrossing partition lattices and F is a central product of k middle groups. As a consequence:*

1. *F_w is a direct product of k annular braid groups,*
2. *H_w is a direct product of k euclidean braid groups, and*
3. *H is a direct product of k euclidean symmetric groups.*

Proof. The group F is minimally generated by the set $S_H \cup \{t_i\}$ contained inside $R_H \cup T_F$ (with the t_i being the factors of the diagonal translation t_λ as described in Definition 6.7) and note that both S_H and $\{t_i\}$ can be partitioned based on the unique component of the horizontal root system involved in each motion. By Proposition 3.10 the elements associated with each component generate a middle group. Moreover, since generators associated to different components commute and w can be factored into a product of special elements for these middle groups, the interval $[1,w]^F$ is a direct product of special intervals in middle groups. By Theorem 4.4 each of these is a type B noncrossing partition lattice. This also means that F is almost, but not quite, a direct product of these middle groups because these groups have a nontrivial intersection. They overlap in elements whose motions lie solely in the V_0 direction, a description which only applies to the pure translations that form their centers. Thus F is a central product rather than a direct product. On the other hand, since $[1,w]^F$ is a direct product of lattices with disjoint edge labels, F_w is a direct product of annular braid groups. The group H_w and H are identified by applying Proposition 4.7 to each factor. □

We illustrate Proposition 7.6 with a concrete example.

Example 7.7 ($\widetilde{E}_8$ groups)**.** Since the horizontal E_8 root system decomposes as $\Phi_{A_1} \cup \Phi_{A_2} \cup \Phi_{A_4}$ (Table 1), the group F is a central product of $\textsc{Mid}(B_2)$, $\textsc{Mid}(B_3)$ and $\textsc{Mid}(B_5)$. In addition,

- $[1,w]^F \cong NC_{B_2} \times NC_{B_3} \times NC_{B_5}$,
- $F_w \cong \textsc{Art}(B_2) \times \textsc{Art}(B_3) \times \textsc{Art}(B_5)$,
- $H_w \cong \textsc{Art}(\widetilde{A}_1) \times \textsc{Art}(\widetilde{A}_2) \times \textsc{Art}(\widetilde{A}_4)$, and
- $H \cong \textsc{Cox}(\widetilde{A}_1) \times \textsc{Cox}(\widetilde{A}_2) \times \textsc{Cox}(\widetilde{A}_4)$.

Part 4. Main Theorems

In this final part we prove our four main results.

8. Proof of Theorem A: Crystallographic Garside groups

In this section we prove our first main result, that for every choice of a Coxeter element w in an irreducible euclidean Coxeter group $W = \textsc{Cox}(\widetilde{X}_n)$, the group $C_w = \textsc{Gar}(\widetilde{X}_n, w)$ is a Garside group. The most difficult step is to establish the lattice property and we begin with a lemma which show that in discretely graded posets, it is sufficient to work inductively and to establish that all pairs of atoms have a well-defined join.

Lemma 8.1 (Atoms and subintervals)**.** *Let P be a bounded poset that is graded with respect to a discrete weighting. If all pairs of atoms in P have well-defined joins and P is not a lattice, then P contains a proper subinterval that is not a lattice.*

Proof. Since P is not a lattice, it contains a bowtie $(a,b : c,d)$ by Proposition 1.10. Let e and f be atoms in P below c and d respectively. By assumption atoms e and f have a join $g = e \vee f$ and since a and b are upper bounds for e and f, we have $a \geq g$ and $b \geq g$ by definition of being a join. Finally, let h a maximal lower bound for a and b that is above g. See Figure 16 and note that such e, f and h exist because of the discreteness of the grading. If $h \neq c$, then $(a,b : c,h)$ is a bowtie in

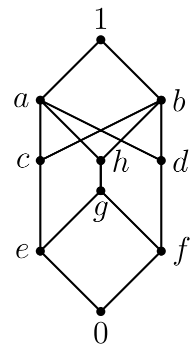

FIGURE 16. Posets elements used in the proof of Lemma 8.1.

the proper subinterval $[e,1]$, if $h \neq d$, then $(a,b:h,d)$ is a bowtie in the proper subinterval $[f,1]$, and one of these conditions holds because c and d are distinct. □

The following corollary restates Lemma 8.1 as a positive assertion.

Corollary 8.2 (Lattice induction). *If P is a discretely graded bounded poset in which all atoms have joins and all proper subintervals are lattices, then P itself is a lattice.*

In order to help investigate the lattice question in this context, the first author wrote a program `euclid.sage` which is available upon request. Using this program we verified that these intervals are lattices up through dimensions 9 and we record this fact as a proposition.

Proposition 8.3 (Low rank). *Let w be a Coxeter element in an irreducible euclidean Coxeter group $\textsc{Cox}(\widetilde{X}_n)$. If $n \leq 9$ then the interval $[1,w]^C$ is a lattice in the corresponding crystallographic group.*

Since all five sporadic examples of irreducible euclidean Coxeter groups are covered by Proposition 8.3, we may turn our attention to the four infinite families. Before considering joins of atoms in the intervals for the infinite euclidean families, it might be useful to consider the properties of atomic joins in the Coxeter intervals of the most classical spherical family.

Remark 8.4 (Atomic joins in the symmetric group). If W is the symmetric group, i.e. the spherical Coxeter group of type A, then its Coxeter element is an n-cycle and the interval $[1,w]^W$ is the lattice of noncrossing partitions. The atoms in this case are the transpositions and these are represented as boundary edges or diagonals in the corresponding convex n-gon. Notice that the join of two atoms always has very low rank: it is reflection length 2 or 3 regardless of n. It has length 2 when the edges are noncrossing or share an endpoint and it has length 3 when they cross. In all three situations the join is below the element that corresponds to the triangle or square which is the convex hull of the union of their endpoints.

The situation in the infinite euclidean families is very similar in the sense that joins of atoms are of uniformly low rank and they live in subposets defined by the endpoints, or equivalently the coordinates, involved. The first crucial fact is that there is a well-defined projection from the middle row to the top and from the middle row to the bottom row.

Lemma 8.5 (Projection)**.** *Let w be a Coxeter element in an irreducible euclidean Coxeter group $W = \mathrm{Cox}(\widetilde{X}_n)$. For each element u in the middle row of the coarse structure of $[1,w]^W$, the set of elements in the top row that are above u have a unique minimum element. Similarly, the set of elements in the bottom row that are below u have a unique maximum element.*

Proof. For the five sporadic examples and the beginnings of the infinite families, we verified these assertions using the program `euclid.sage`. Next we consider the elements in the first box of the middle row, the ones corresponding to vertical reflections. Because of the explicit and regular nature of the infinite families (as illustrated by the computations given in [McC15, Section 11]), the list of top row elements above each vertical reflection can be explicitly written down and a unique minimal top element identified. In type A, regardless of choice of Coxeter element, each vertical reflection is below a unique top row element in first box (i.e. a pure translation). In type C, some vertical translations project upwards to elements in the first box of the row and other to the second. In type B, each vertical translation projects upwards to a unique element in either the first, the second or the third box in the top row. And in type D, each vertical translation projects upwards to a unique element in either the first or the fourth box in the top row.

Finally, let u be an arbitrary element of the middle row and let a be one of the vertical reflections below u. Such a reflection must exists in any factorization of u because, by definition of the middle row, some point experiences a vertical motion under u. We claim that the unique minimum top row element above u is the join of u and the projection of a to the top row inside the interval $[a,w]^W$. Because a is a vertical reflection, its complement is also a vertical elliptic isometry and the interval $[a,w]^W$ is that of spherical type, thus a lattice, and so the join of these two elements is well-defined. This element is clearly in the top row (because it is above the upward projection of a) and above u. It is the minimum such element because any v in the top row that is above u is also above a, thus above the upward projection of a, and so above the join of u and the upward projection of a. The second assertion follows immediately from the first because these posets are self-dual (Proposition 7.4). □

Using Lemma 8.5 we define an *upward projection map* from $[1,w]^C$ to $[1,w]^F$ which is the identity on $[1,w]^F$ and sends elements in the middle row to the elements described in the lemma. It can be used to show that the meets and joins that exist in the factor interval $[1,w]^F$ remain meets and joins inside the crystallographic interval $[1,w]^C$.

Lemma 8.6 (Factor meets and joins)**.** *For each choice of Coxeter element w in an irreducible euclidean Coxeter group $W = \text{Cox}(\widetilde{X}_n)$, the inclusion of the factor lattice $[1,w]^F$ into the crystallographic interval $[1,w]^C$ preserves meets and joins. In particular, any two elements in $[1,w]^F$ have a well-defined meet in $[1,w]^C$ that agrees with their meet in $[1,w]^F$ and a well-defined join in $[1,w]^C$ that agrees with their join in $[1,w]^F$*

Proof. Let $P = [1,w]^C$ be the crystallographic interval, let $Q = [1,w]^F$ be the factor subposet and suppose that u and v are elements in Q with a maximal lower bound a in P that is not their meet $b = u \wedge_Q v$ in Q. If a is in Q then $a = b$ because Q is a lattice, in particular a product of type B noncrossing partiition lattices. Thus a is not in Q and must lie in the middle row of the coarse structure. This means that u and v, being both above a and in Q, must both lie in the top row. By Lemma 8.5 there is a unique minimum top row element c above a which would, by definition, be below both u and v, contradicting the maximality of a as a lower bound for these elements. Thus no such u and v exist. The assertion involving joins is true by duality. □

Lemma 8.5 can also be used to show that joins with factored translations are well-defined.

Lemma 8.7 (Translation joins)**.** *Let w be a Coxeter element in an irreducible euclidean Coxeter group $W = \text{Cox}(\widetilde{X}_n)$. If a and b are atoms in the crystallographic interval $[1,w]^C$ and one of them is a factored translation then their join is well-defined.*

Proof. Let $b \in T_F$ be the factored translation. If a is in F then by Lemma 8.6 the join of a and b is well-defined. The only remaining case is where a is in the middle row of the coarse structure and we claim that the join of b with the upward projection of a to the top row (Lemma 8.5) is the join of a and b. In this case, the only upper bounds for a and b are to be found in the top row of the coarse structure and any such element is above the projection of a by definition and thus above its join with b. This completes the proof. ☐

And finally, we consider the case where both atoms are reflections.

Lemma 8.8 (Reflection joins)**.** *Let w be a Coxeter element in an irreducible euclidean Coxeter group $W = \mathrm{Cox}(\widetilde{X}_n)$. If a and b are reflections in the interval $[1,w]^C$ and a is a vertical reflection then their join is well-defined.*

Proof. If a and b have no upper bounds in the middle row of the coarse structure then their join is the join of their images under the upward projection map by Lemma 8.5 and Lemma 8.6. If W is of sporadic type then the join of a and b exists by Proposition 8.3. And finally, if W belongs to one of the infinite euclidean families, one can use properties of the noncrossing partition lattices in the spherical infinite families, and properties of the upward projection map to show that every possible minimal upper bound for a and b is below a low-rank top row element solely defined by the set of coordinates involved in the roots of a and b and the type of W. This is the euclidean analogue of the situation described in Remark 8.4. In other words, if there is a pair of reflection atoms in a crystallographic interval for one of the infinite families that has no well-defined join, then there is such a pair in such an interval where the rank is low and uniformly bounded. And since no such pair exists in low rank (Proposition 8.3), no such pair exists at all. ☐

Combining these lemmas establishes the following.

Theorem 8.9 (Lattice)**.** *For each choice of Coxeter element w in an irreducible euclidean Coxeter group $W = \mathrm{Cox}(\widetilde{X}_n)$, the crystallographic interval $[1,w]^C$, in the corresponding crystallographic group $C = \mathrm{Cryst}(\widetilde{X}_n)$, is a lattice.*

Proof. Proposition 8.3 covers the five sporadic examples. and for the four infinite families we proceed by induction. The base cases are again covered by Proposition 8.3, so suppose by induction that X is A, B, C or D and that all crystallographic intervals are lattices for $k < n$. Atoms in $[1,w]^C$ correspond to elements in $R_H \cup R_V \cup T_F$ and all possible combinations of pairs of atoms are covered by Lemma 8.6, Lemma 8.7, or Lemma 8.8. Thus all pairs of atoms have well-defined joins and the interval is a lattice by Corollary 8.2. ☐

Theorem 8.9 and Proposition 7.4 show that Proposition 2.11 can be applied and this immediately proves the following slightly more explicit version of Theorem A.

Theorem 8.10 (Crystallographic Garside groups)**.** *Let w be a Coxeter element in an irreducible euclidean Coxeter group $W = \mathrm{Cox}(\widetilde{X}_n)$ and let $C = \mathrm{Cryst}(\widetilde{X}_n, w)$ be the corresponding crystallographic group with its natural weighted generating set. The interval $[1,w]^C$ is a balanced lattice and, as a consequence, it defines an interval group $C_w = \mathrm{Gar}(\widetilde{X}_n, w)$ with a Garside structure of infinite type.*

9. Proof of Theorem B: Dual Artin subgroups

In this section we prove Theorem B by showing that the Garside group $\mathrm{Gar}(\widetilde{X}_n, w)$ is an amalgamated free product with the dual Artin group $\mathrm{Art}^*(\widetilde{X}_n, w)$ as one of its factors. The proof begins by noting the immediate consequences of Lemma 7.2 on the level of presentations.

Lemma 9.1 (Presentation)**.** *For each choice of Coxeter element w in an irreducible euclidean Coxeter group $W = \mathrm{Cox}(\widetilde{X}_n)$, the Garside group $C_w = \mathrm{Gar}(\widetilde{X}_n, w)$ has a presentation whose generators and relations are obtained as a union of the generators and relations for presentations for D_w, F_w and W_w.*

Proposition 9.2 (Pushout)**.** *For each irreducible euclidean Coxeter group and for each choice of Coxeter element w, the Garside group C_w is the pushout of the diagram $F_w \leftarrow D_w \rightarrow W_w$. If the maps from D_w to F_w and W_w are both injective, then C_w is an amalgamated free product of F_w and W_w over D_w and, in particular, W_w injects into C_w.*

We now show that these maps are injective.

Lemma 9.3 ($H_w \hookrightarrow F_w$)**.** *For each irreducible euclidean Coxeter group and for each choice of Coxeter element w, the horizontal group H_w injects into the factorable interval group F_w. As a consequence, the horizontal group H_w also injects into the diagonal interval group D_w.*

Proof. The first assertion is a consequence of Proposition 4.7 applied to each factor and the second assertion follows immediately since $H_w \hookrightarrow F_w$ factors through D_w. □

Lemma 9.4 ($D_w \hookrightarrow F_w$)**.** *For each irreducible euclidean Coxeter group and for each choice of Coxeter element w, the diagonal interval group D_w injects into the factorable interval group F_w.*

Proof. Recall that $R_H \cup \{w\}$ is one possible generating set for the diagonal interval group D_w (Remark 7.3) and let U be a word in these generators that represents an element $u \in D_w$. If u is in the kernel of the map $D_w \to F_w$, then u is also in the kernel of the composite map $D_w \to F_w \to F \to \mathbb{Z}$ where the middle map is the natural projection and the final map to $\mathbb{Z}$ is the vertical displacement map. Since this composition sends each horizontal reflection to 0 and each w to a nonzero integer, we conclude that the exponent sum of the w's inside U is zero. Using the relations in D_w which describe how w conjugates the elements of R_H, we can then find a word U' with no w's which still represents u in D_w. This means that u is in the subgroup generated by elements of R_H which by Lemma 9.3 we can identify with H_w. In particular, u is in the kernel of the map $H_w \to F_w$ which is trivial by Lemma 9.3 proving $D_w \hookrightarrow F_w$. □

Lemma 9.5 ($D_w \hookrightarrow W_w$)**.** *For each irreducible euclidean Coxeter group and for each choice of Coxeter element w, the factorable interval group F_w injects into the Garside group C_w. As a consequence, the diagonal interval group D_w injects into the Artin group W_w.*

Proof. The interval $[1,w]^F$ is a lattice because it is a product of type B partition lattices and $[1,w]^C$ is a lattice by Theorem 8.10. That they are balanced follows immediately from the fact that the corresponding generating sets in F and C are closed under conjugation. Finally, by Lemma 8.6 the inclusion of the former into the latter preserves meets and joins. Thus by Proposition 2.15 the induced map from F_w to C_w is injective. Since $D_w \hookrightarrow F_w$ by Lemma 9.4, the composition injects D_w into C_w. But $D_w \hookrightarrow C_w$ factors through W_w, so the map $D_w \to W_w$ is also one-to-one. □

Proposition 9.2 combined with Lemmas 9.4 and 9.5 immediately prove the following slightly more explicit version of Theorem B.

Theorem 9.6 (Amalgamated free product)**.** *For each irreducible euclidean Coxeter group* $\textsc{Cox}(\widetilde{X}_n)$ *and for each choice of Coxeter element w, the Garside group* $G = \textsc{Gar}(\widetilde{X}_n, w)$ *can be written as an amalgamated free product of W_w and F_w amalgamated over D_w where W_w is the dual Artin group* $\textsc{Art}^*(\widetilde{X}_n, w)$, *$F_w$ is the factorable inteval group, and D_w is the diagonal interval group. As a consequence, the dual Artin group W_w injects into the Garside group G.*

Note that when the horizontal root system has only a single component, $T \cong T_F$, $D_w \cong F_w$ and $W_w \cong C_w$. This occurs in types C and G and in type A when $q = 1$.

10. Proof of Theorem C: Naturally isomorphic groups

In this section we prove that the dual Artin group $\textsc{Art}^*(\widetilde{X}_n, w)$ is isomorphic to the Artin group $\textsc{Art}(\widetilde{X}_n)$. The first step is to find homomorphisms between them. In one direction this is easy to do.

Proposition 10.1 ($A \twoheadrightarrow W_w$)**.** *For every irreducible euclidean Coxeter group $W = \textsc{Cox}(\widetilde{X}_n)$ and for each choice of Coxeter element w as the product of the standard Coxeter generating set S, there is a natural map from the Artin group $A = \textsc{Art}(\widetilde{X}_n)$ onto the dual Artin group $W_w = \textsc{Art}^*(\widetilde{X}_n, w)$ which extends the identification of the generators of A with the subset of generators of W_w indexed by S.*

Proof. For every pair of elements in S, there is a rewritten factorization of w where they occur successively and then the Hurwitz action on this pair produces the dual dihedral Artin relations corresponding to the angle between these two facets of σ (Example 2.9). Systematically eliminating the other variables shows that these two elements in the dual Artin group satisfy the appropriate Artin relation. This shows that the function injecting the generating set of the Artin group into the dual Artin group extends to a group homomoprhism. The fact that it is onto is a consequence of the transitivity of the Hurwitz action. □

Remark 10.2 (A $\widetilde{G}_2$ map)**.** As an example of such a homomorphism, consider the simplex in the $\widetilde{G}_2$ tiling bounded by the lines c_0, a_1 and d_1 with bipartite Coxeter element $w = a_1 d_1 c_0$ in the notation of Definition 5.10. Proposition 10.1 gives a homomorphism from the Artin group $\textsc{Art}(\widetilde{G}_2)$ with generators that we call a, c and d satisfying the relations $aca = cac$, $ad = da$ and $cdcdcd = dcdcdc$ to the dual Artin group $\textsc{Art}^*(\widetilde{G}_2, w)$ extending the map sending a, c and d to a_1, c_0 and d_1, respectively.

Defining a homomorphism in the other direction is more difficult because we need to describe where the infinitely many generators are to be sent and we need to check that infinitely many dual braid relations are satisfied. The first step is to describe certain portions of the Cayley graph of an irreducible Artin group that are already well understood. These are portions of the Coxeter group Cayley graph that lift to the Artin group.

Definition 10.3 (Cayley graphs and Coxeter groups)**.** The standard way to view the right Cayley graph of an irreducible euclidean Coxeter group with respect to a Coxeter generating set S is to consider the cell complex dual to the Coxeter complex. The dual complex for the $\widetilde{A}_2$ Coxeter group, for example, is a hexagonal tiling of $\mathbb{R}^2$. The dual complex has one vertex for each chamber of the Coxeter complex (and thus one vertex for each element of W) and it is convenient to place this vertex at the center of the insphere of this simplex so that it is equidistant from each facet. Once labels are added to the edges of the 1-skeleton of the dual cell complex, this becomes either the full right Cayley graph of W with respect a simple system S, or it is a portion of the left Cayley graph with respect to the set of all reflections. To get the full right Cayley graph we label the edges leaving a particular chamber σ and then propigate the labels so that they are invariant under the group action. To get a portion of the left Cayley graph we label the edges dual to the facets of the simplices by the unique hyperplane the facet determines.

Converting between left Cayley graph labels and right Cayley graph labels is a matter of conjugation.

Remark 10.4 (Converting Labels)**.** Suppose that we have picked a vertex corresponding to a chamber as our basepoint and indexed the vertices by the unique group element in W which takes our base vertex to this vertex and suppose further that a, c and d are part of the standard generating set leaving our base vertex v_1. In the right Cayley graph the edge connecting the adjacent vertices v_{ac} and v_{acd} is labeled by d but in the corresponding portion of the left Cayley graph, its label is the reflection $(ac)d(ac)^{-1}$. This is because this is the reflection we multiply by on the left to get from ac to acd. Geometrically we are conjugating the label in the right Cayley graph by the path in the right Cayley graph from v_1 to its starting vertex.

There are a variety of ways that the unoriented right Cayley graph for an irreducible euclidean Coxeter group can be converted into a portion of the right Cayley graph for the corresponding Artin group. We describe two such procedures.

Definition 10.5 (Standard flats)**.** Let $W = \textsc{Cox}(\widetilde{X}_n)$ be an irreducible euclidean Coxeter group and let $A = \textsc{Art}(\widetilde{X}_n)$ be the corresponding Artin group. If we pick a vector γ that is generic in the sense that none of the roots of the hyperplanes of W are orthogonal to γ, then we can orient the edges of the right Cayley graph of W (which are transverse to the hyperplanes) according to the direction that forms an acute angle with γ. Such a Morse function turns the boundary of every 2-cell in the dual cell complex into an Artin relation. In particular, the 2-skeleton of the dual cell complex is simply connected and its labeled oriented 1-skeleton is a portion of the right Cayley graph of A that we call a *standard flat*. The terminology reflects the fact that the polytopes in the dual cell complex with labelled oriented edges can be added to the presentation complex for the Artin group A without changing its fundamental group. The universal cover of the result is known as the *Salvetti complex* [Sal87, Sal94]. If each polytope is given the natural euclidean metric that it inherits, then a standard flat represents the 1-skeleton of a metric copy of $\mathbb{R}^n$ inside the Salvetti complex.

An easy way to create a standard flat is to let γ be a generic perturbation of the direction of the Coxeter axis. What we really need is a slight variation of this procedure.

Definition 10.6 (Axial flats)**.** Let $W = \textsc{Cox}(\widetilde{X}_n)$ be an irreducible euclidean Coxeter group with Coxeter element w and let $A = \textsc{Art}(\widetilde{X}_n)$ be the corresponding Artin group. Orient the edges of the dual cell complex as follows. For hyperplanes that cross the Coxeter axis, orient the transverse edges so that their direction vector forms an acute angle with the direction of the Coxeter axis. For the other hyperplanes with horizontal normal vectors, orient the transverse edges to point to the side that does not contain the Coxeter axis. We call such an oriented 1-skeleton an *axial flat*. As before every 2-cell in the dual cell complex has a boundary labelled by an Artin relation so this simply-connected 2-complex lives in the Salvetti complex for A. In fact, it is easy to see that it can be constructed by assembling sectors of standard flats around the column containing the Coxeter axis.

Next we use axial flats to define reflections in euclidean Artin groups.

Definition 10.7 (Facets and reflections)**.** Let $W = \textsc{Cox}(\widetilde{X}_n)$ be an irreducible euclidean Coxeter group with Coxeter element w and fix a simplex σ in the Coxeter complex or, equivalently, fix a vertex in the dual cell complex. For each facet of each simplex in the Coxeter complex we define a reflection in the corresponding Artin group $A = \textsc{Art}(\widetilde{X}_n)$ as follows. Orient the edges of the dual cell complex so that it is the axial flat for w and then conjugate the labelled oriented edge transverse to the specified facet by a path in the axial flat from the fixed basepoint to the start of the transverse edge.

Many of the facets belonging to a common hyperplane determine the same reflection in the Artin group but describing which ones are equal is slightly subtle.

Lemma 10.8 (Consistency)**.** *Let $W = \textsc{Cox}(\widetilde{X}_n)$ be an irreducible euclidean Coxeter group with Coxeter element w and a fixed base simplex. Let H be a hyperplane in the Coxeter complex, let P be convex hull of the axial vertices in H and suppose that P contains at least one facet* *of a chamber. If σ_1 and σ_2 are simplices on the same side of H and $P \cap \sigma_i$ is a facet of σ_i for $i = 1, 2$, then the reflections r_1 and r_2 that they define in the axial flat are equal in the Artin group $A = \mathrm{ART}(\widetilde{X}_n)$.*

Proof. The idea of the proof is straightforward. Let p be a path in the axial flat from the fixed base simplex to σ_1 and let q be a path from σ_1 to σ_2 (also in the axial flat) that is as short as possible. By construction $r_1 = (p)s_1(p)^{-1}$ and $r_2 = (pq)s_2(pq)^{-1}$ for appropriate standard generators s_1 and s_2. Because q is as short as possible, it only crosses the hyperplanes that separate σ_1 from σ_2 and by Lemma 5.6 this only includes hyperplanes whose normal vectors do not change sign in the axial flat when reflected across the hyperplane H. In particular, the path $qs_2q^{-1}s_1^{-1}$ is visible as a closed loop in the axial flat. As a consequence it is trivial in A and this relation shows that the elements r_1 and r_2 are equal. □

The necessity of the specificity given in Lemma 10.8 can be seen even in the $\widetilde{G}_2$ case. We continue to use the notation of Definition 5.10.

Remark 10.9 (Consistency)**.** Consider the four line segments of the hyperplane e_3 inside the lightly shaded strip of Figure 11. The reflections in $\mathrm{ART}(\widetilde{G}_2)$ that they determine are $(d)c(d)^{-1}$, $(dac)a(dac)^{-1}$, $(dacd)a(dacd)^{-1}$ and $(dacdca)c(dacdca)^{-1}$. All four belong to the convex hull of the axial vertices in the e_3 hyperplane and it is straightforward to show that all four expressions represent the same group element in $\mathrm{ART}(\widetilde{G}_2)$. On the other hand, consider the two line segments of the c_2 hyperplane inside the lightly shaded strip. The reflections in $\mathrm{ART}(\widetilde{G}_2)$ that they determine are $(ad)c(ad)^{-1}$ and $(dc)a(dc)^{-1}$. The first is bounded by two axial vertices, the second is not and these two reflections are not equal in the Artin group.

Fortunately the level of consistency available is sufficient to establish the homomorphism we require.

Definition 10.10 (Dual reflections in the Artin group)**.** Let $W = \mathrm{COX}(\widetilde{X}_n)$ be an irreducible euclidean Coxeter group with Coxeter element w and a fixed base simplex. For each reflection labeling an edge in the interval $[1, w]^W$ we define an element of the Artin group $A = \mathrm{ART}(\widetilde{X}_n)$ as follows. When the axial vertices in the fixed hyperplane of a reflection r have a convex hull which contains a facet of a simplex, we define the corresponding reflection in A as described in Definition 10.7. By Lemma 10.8 the element defined is independent of the facet in the convex hull that we use. This applies to all vertical reflections and to those horizontal reflections which contain a facet of the boundary of the convex hull of all axial vertices. We call these the *standard horizontal reflections*. For the nonstandard horizontal reflections we proceed as follows. By Proposition 7.6 the subgroup H_w generated by the horizontal reflections can be identified with a product of k euclidean braid groups. From this identification it is clear that the standard horizontal reflections generate. Next, in the axial flat we can see that the reflections in A corresponding to the standard horizontal reflections satisfy the Artin relations associated with the dihedral angles between their hyperplanes. This means that there is a natural homomorphism from the subgroup of H_w to the subgroup generated by the images of the standard horizontal reflections in A. We use this map to define the images of the nonstandard horizontal reflections in A.

Proposition 10.11 (Pure Coxeter element)**.** *Let $W = \textsc{Cox}(\widetilde{X}_n)$ be an irreducible euclidean Coxeter group with Coxeter element w and a fixed base simplex. If w^p is the smallest power of w which acts on the Coxeter complex as a pure translation and r is a standard horizontal reflection in the Artin group $A = \textsc{Art}(\widetilde{X}_n)$, then w^p and r (viewed as elements in A) commute. As a consequence, w^p centralizes the full subgroup of A generated by these standard horizontal reflections.*

Proof. This follows immediately from Lemma 10.8. The convex hull of all axial vertices is, metrically speaking, a product of simplices cross the reals and the convex hull P of the axial vertices contained in the fixed hyperplane of r is one facet of this product of simplices cross the reals. The entire configuration in the axial flat is invariant under the vertical translation induced by w^p and thus r and $(w^p)r(w^p)^{-1}$ define the same reflection in A. □

We are now ready to define a homomorphism from the dual Artin group to the Artin group.

Proposition 10.12 ($W_w \twoheadrightarrow A$)**.** *For every irreducible euclidean Coxeter group $W = \textsc{Cox}(\widetilde{X}_n)$ for every choice of Coxeter element w as the product of the standard Coxeter generating set S, the map on generators described above extends to a group homomorphism from the dual Artin group $W_w = \textsc{Art}^*(\widetilde{X}_n, w)$ onto the Artin group $A = \textsc{Art}(\widetilde{X}_n)$.*

Proof. Let σ be the chamber in the Coxeter complex of W bounded by the fixed hyperplanes of the reflections indexed by S and consider the function from the reflections in $[1, w]^W$ to A which sends each reflection to the reflection in A as defined in Definition 10.10. We only need to show that this function extends to a homomorphism. As mentioned in Definition 5.11 there are three types of dual braid relations in the interval $[1,w]^W$. The ones indexed by the third box in the bottom row are relations among horizontal reflections and their satisfaction was described in Definition 10.10.

The ones indexed by the second box in the middle row are vertical elliptics which rotate around a codimension 2 subspace. Since its right complement is also vertical elliptic, all the reflections in the factorization fix an axial vertex v which belongs to some axial simplex σ'. The reflections in the Artin group A that fix the facets of σ' form an alternative simple system S' for A. Using an old result from van der Lek's thesis, the subset of elements of S' that fix v generate an Artin group which injects into A, in this case an Artin group of spherical type [vdL83]. Using the known equivalence of dual and standard presentations for spherical Artin groups we see that these dual braid relations are satisfied by their images in A.

Finally, the ones indexed by the first box in the top row are the various ways to factor a pure translation t in W and these are described in Proposition 6.3. Using the Hurwitz action there is a factorization of w in A that maps to a horizontal factorization of w in W. In particular, there is an element t in A that differs from w by a product of (the images of) horizontal reflections and which has a factorization $t = r'r$ in A into reflections where r and $r' = (w^p)r(w^{-p})$ are defined by vertically shifted facets of simplices. The first observation combined with Proposition 10.11 shows that this t commutes with w^p inside A. If we define reflections $r_i = (w^{ip})r(w^{-ip})$ as the reflections in A defined by the various vertical shifts of the facet that defines r, then $t = (w^{ip})t(w^{-ip}) = (w^{ip})r_1r_0(w^{-ip}) = r_{i+1}r_i$ shows that all of factorizations of t in the interval $[1,w]^W$ are also satisfied in A. Since all three types of dual braid relations are satisfied, the function on reflections extends to a homomorphism, and this homomorphism is onto because its image includes a generating set for the Artin groups A. □

Our third main result now follows as a easy corollary.

Theorem C (Naturally isomorphic groups)**.** *For each irreducible euclidean Coxeter group $W = \textsc{Cox}(\widetilde{X}_n)$ and for each choice of Coxeter element w as the product of the standard Coxeter generating set S, the Artin group $A = \textsc{Art}(\widetilde{X}_n)$ and the dual Artin group $W_w = \textsc{Art}^*(\widetilde{X}_n, w)$ are naturally isomorphic.*

Proof. Let σ be the chamber in the Coxeter complex for W whose facets index the reflections in S. Because w is obtained as a product of the elements in S, every vertex of σ is an axial vertex and all of σ is contained in the convex hull of the axial vertices. By composing the surjective homomorphisms described in Propositions 10.12 and 10.1 we find a map from A to itself which must be the identity homomorphism since it fixes each element of the generating set S. This means the first map in the composition from A to W_w is injective as well as surjective and thus an isomorphism. □

11. Proof of Theorem D: Euclidean Artin groups

In a recent survey article Eddy Godelle and Luis Paris highlighted how little we know about general Artin groups by stating four basic conjectures that remain open [GP12]. Their four conjectures are:

(A) All Artin groups are torsion-free.
(B) Every non-spherical irreducible Artin group has a trivial center.
(C) Every Artin group has a solvable word problem.
(D) All Artin groups satisfy the $K(\pi,1)$ conjecture.

Godelle and Paris also remark that these conjectures remain open and are a "challenging question" even in the case of the euclidean Artin groups. These are precisely the conjectures that we set out to resolve. In this section we prove our final main result, Theorem D, which resolves the first three of these questions for euclidean Artin groups. Most of the structural properties follows from the existence of a classifying space which is itself an easy corollary of Theorems B and C.

Proposition 11.1 (Classifying space)**.** *Every irreducible Artin group of euclidean type is the fundamental group of a finite dimensional classifying space.*

Proof. By Theorem 2.12, the Garside group $\text{Gar}(\widetilde{X}_n, w)$ has a finite-dimensional classifying space and the cover of this space corresponding to the subgroup $\text{Art}(\widetilde{X}_n)$ is a classifying space for the Artin group. □

Remark 11.2 (Finite-dimensional)**.** The reader should note that the spaces involved are finite-dimensional but not finite. More precisely, because the interval $[1,w]^C$ has infinitely many elements, the natural classifying space constructed for $\text{Gar}(\widetilde{X}_n, w)$ has infinitely many simplices, but their dimension is nevertheless bounded above by the combinatorial length of the longest chain.

To compute the center of $\text{Art}(\widetilde{X}_n)$ we recall an elementary observation about euclidean isometries which quickly leads to the well-known fact that irreducible euclidean Coxeter groups are centerless.

Lemma 11.3 (Coxeter groups)**.** *Let $W = \text{Cox}(\widetilde{X}_n)$ be an irreducible euclidean Coxeter group, let $u \in W$ be an elliptic isometry and let $v \in W$*

be a hyperbolic isometry. If λ is the translation vector of v on $\textsc{Min}(v)$ *and* $\textsc{Fix}(u)$ *is not invariant under λ, then u and v do not commute.*

Proof. Because v lives in W, there is a power v^m that is a pure translation with translation vector $m\lambda$. If u commutes with v then u commutes with v^m but the fixed set of the conjugation of u by v^m is the translation of the fixed set of u by $m\lambda$, contradiction. □

Corollary 11.4 (Trivial center)**.** *Every irreducible euclidean Coxeter group has a trivial center.*

Proof. Using the criterion of Lemma 11.3, it is easy to find a noncommuting hyperbolic for each elliptic in W and a noncommuting elliptic for each hyperbolic in W. □

We note one quick consequence for Artin groups.

Lemma 11.5 (Powers of w)**.** *For each irreducible euclidean Coxeter group and for each choice of Coxeter element w, the nontrivial powers of w are not central in the Artin group W_w.*

Proof. For each nonzero integer m, the element w^m projects to a nontrivial hyperbolic element in W. By Corollary 11.4 there is an element u in W that does not commute with w^n and because the projection map $W_w \twoheadrightarrow W$ is onto, it has preimages that do not commute with w^n in W_w. □

We also derive a second more substantial consequence.

Lemma 11.6 ($C_w \rightsquigarrow F_w$)**.** *For each irreducible euclidean Coxeter group and for each choice of Coxeter element w, the simples in C_w which commute with w are simples in F_w. As a consequence, the elements of C_w which commute with w are contained in the subgroup F_w.*

Proof. If u is a simple in C_w which commutes with w then the image of u as a euclidean isometry commutes with the power w^m whose image as a euclidean isometry in C is a pure translation in the direction of the Coxeter axis. When u is elliptic, by Lemma 11.3 it has a vertically invariant fixed set, it does not belong to the middle row of the coarse structure, and thus $u \in [1,w]^F$. The extension from simples to elements follows from Proposition 2.14. □

Lemma 11.7 ($F_w \rightsquigarrow \mathbb{Z}^k$)**.** *For each irreducible euclidean Coxeter group and for each choice of Coxeter element w, the centralizer of w in F_w is the group $\mathbb{Z}^k \cong \langle w_i \rangle$ where the w_i are the special elements in the factors whose product is w.*

Proof. By Proposition 7.6, the group F_w and the interval $[1,w]^{F_w}$ split as direct products. Thus the simples that commute with w are products of the simples in each factor that commute with the factor w_i. Since by Proposition 4.5 such a simple in each factor must be 1 or w_i, there are exactly 2^k simples that commute with w. And since the w_i commute with each other in F_w they generate a subgroup isomorphic to $\mathbb{Z}^k \cong \langle w_i \rangle$, with one $\mathbb{Z}$ from each factor. All of these commute with w and by Proposition 2.14 these are the only elements that commute with w. □

Lemma 11.8 ($\mathbb{Z}^k \rightsquigarrow \mathbb{Z}$)**.** *For each irreducible euclidean Coxeter group and for each choice of Coxeter element w, the intersection of D_w and the group $\mathbb{Z}^k$ (generated by the w_i factors of w) is an infinite cyclic subgroup generated by w. In symbols $D_w \cap \langle w_i \rangle \cong \langle w \rangle$.*

Proof. Combining the global winding number maps for each factor (Definition 3.8) produces a map $F_w \to \mathbb{Z}^k$ which sends w_i to e_i, the i-th unit vector in $\mathbb{Z}^k$ which restricts to an isomorphism on the subgroup $\langle w_i \rangle$ in F_w. The image of D_w under the composition $D_w \hookrightarrow F_w \to \mathbb{Z}^k$ is the set of k-tuples with all coordinates equal. To see this we view an element of D_w as a product of simples thought of as elements of D rather than D_w. The relevant maps are now vertical displacement maps rather than global winding number maps. From this perspective it is clear that the horizontal reflections are sent to the zero vector under this composition and that every diagonal translation t is sent to the vector with all coordinates equal to 1. Thus the only elements in the intersection are those with the same number of w_i's for each i. Using the fact that they commute with each other, we can thus rewrite this expression as a power of $w = \prod_i^k w_i$. □

And finally we put all the pieces together.

Proposition 11.9 (Center)**.** *An irreducible euclidean Artin group has a trivial center.*

Proof. Let W_w be a dual euclidean Artin group with special element w. If u is central in W_w then u commutes with w and by Proposition 2.14, u, viewed as an element of C_w, has a Garside normal form built out of simples that commute with w. By Lemma 11.6 the only such simples are simples in F_w, so $u \in F_w$ and by Lemma 11.7 the element u in fact belongs to the subgroup $\mathbb{Z}^k \cong \langle w_i \rangle$ generated by the special factors w_i of w. This means that u is in $F_w \cap W_w$ and thus in D_w by the amalgamated free product structure of C_w (Theorem 9.6). But by Lemma 11.8 the only portion of the $\mathbb{Z}^k \cong \langle w_i \rangle$ contained in D_w is the subgroup $\mathbb{Z} \cong \langle w \rangle$. In particular, $u = w^n$ for some n. And finally, by Lemma 11.5 the nontrivial powers of w are not central in W_w, so the center of W_w is trivial. □

These combine to give our main result.

Theorem D (Euclidean Artin groups)**.** *Every irreducible euclidean Artin group* $\mathrm{ART}(\widetilde{X}_n)$ *is a torsion-free centerless group with a solvable word problem and a finite-dimensional classifying space.*

Proof. Because $\mathrm{ART}(\widetilde{X}_n)$ is isomorphic to W_w which is a subgroup of a Garside group $C_w = \mathrm{GAR}(\widetilde{X}_n, w)$, the standard solution to the word-problem in C_w gives a solution to the word problem in W_w and by Proposition 11.1 it has a finite dimensional classifying space. Groups with finite-dimensional classifying spaces are torsion-free and by Proposition 11.9 its center is trivial. □

The fourth question of Godelle and Paris, the $K(\pi, 1)$ conjecture, would have a positive resolution if one could establish the following.

Conjecture 11.10 (Homotopy equivalence)**.** *The classifying space for each irreducible Artin group of euclidean type constructed here, should be homotopy equivalent to the standard topological space with this fundamental group constructed from the action of the corresponding Coxeter group on its complexified hyperplane complement.*

And finally, there is another obvious question to ask at this point, although we suspect that it may have a negative answer.

Question 11.11. Is there a natural way to extend the definitions of $\mathrm{CRYST}(\widetilde{X}_n, w)$ and $\mathrm{GAR}(\widetilde{X}_n, w)$ to other infinite Coxeter groups so that they retain their key properties? In particular, is every Artin group isomorphic to a subgroup of a suitably-defined Garside group?

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