

Linear Equations

Math 4A – Scharlemann

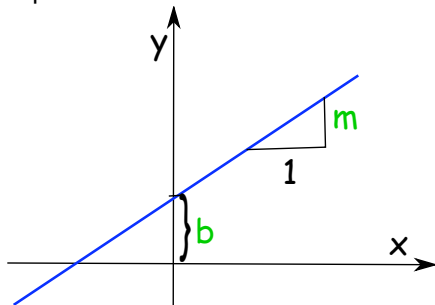
UCSB

5 January 2015



CELL PHONES OFF

Equation of a line:

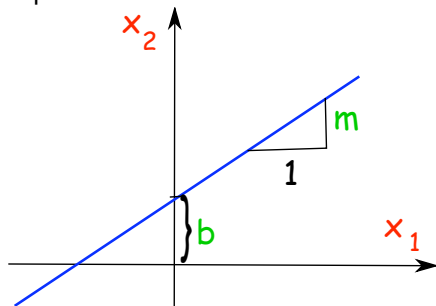


$$y = mx + b$$



$$y - mx = b$$

Equation of a line:



$$y = mx + b$$

$$\Leftrightarrow$$

$$y - mx = b$$

$$\Leftrightarrow$$

$$x_2 - mx_1 = b$$

$$\Leftrightarrow$$

$$-mx_1 + x_2 = b$$

$$\Downarrow (\Uparrow a_2 \neq 0)$$

$$a_1x_1 + a_2x_2 = b'$$

iClicker questions

In the plane above, what is meant by the x_1 axis?

- A) The line where $x_1 = 0$
- B) The line where $x_2 = 0$
- C) The place where y_1 goes for a spin.

Let L be the line given by the equation $5x_1 + 3x_2 = 7$.

At what value of x_1 does L intersect the x_1 axis?

- A) $x_1 = 5$
- B) $x_2 = 7$
- C) $x_2 = 3/7$
- D) $x_1 = 7/5$
- E) Um, what?

Linear equation in variables $x_1, \dots, x_n$:

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

where $a_1, \dots, a_n$ and maybe b are all known in advance.

Solution is a list $s_1, \dots, s_n$ of numbers so

$$a_1s_1 + a_2s_2 + \dots + a_ns_n = b$$

Example 1: For the equation $a_1x_1 + a_2x_2 = b$ of a line, given above:

The pair s_1, s_2 is a solution $\iff$ the point (s_1, s_2) is on the line.

Example 2 Converting grades to the standard scale:

$F = 0 \leq \text{grade} \leq 4 = A$, let

- x_1 be your first midterm grade
- x_2 be your second midterm grade
- x_3 be your grade on the final
- x_4 be your homework & quiz grade
- x_5 be your i-clicker grade

Then

before then.

Grades:	Midterms @ 20%	40%
	Final	40%
	Homework and Quizzes	15%
	Class participation (via iClicker use)	5%

See "Discussion Sections" above for major penalty clause.

If this sounds too straightforward, consult the course [Randomly Asked Questions](#) page.

translates to

$$0.2x_1 + 0.2x_2 + 0.4x_3 + 0.15x_4 + 0.05x_5 = \text{your course grade}$$

A system of linear equations is a bunch of linear equations:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

A **solution** to the system is a list

$$s_1, \dots, s_n \in \mathbb{R}$$

that is simultaneously a solution to **all** **m** equations.

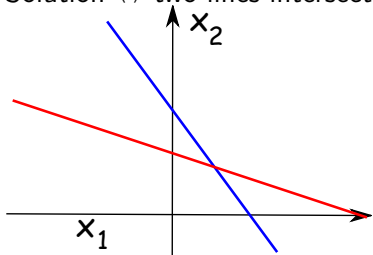
That is, **all** **m** equations are true when $x_1 = s_1, x_2 = s_2, \dots, x_n = s_n$.

Conceptual example:

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

Solution $\Leftrightarrow$ two lines intersect:



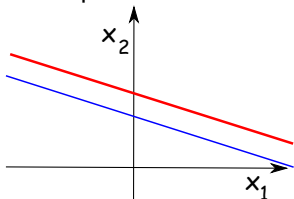
Example:

$$1x_1 + 2x_2 = 3$$

$$2x_1 + 1x_2 = 3$$

$$\Leftrightarrow (x_1, x_2) = (1, 1)$$

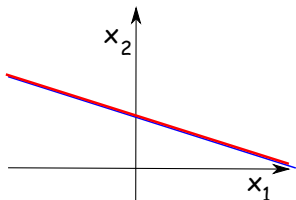
Other possibilities:



$$1x_1 + 2x_2 = 3$$

$$1x_1 + 2x_2 = 4$$

inconsistent



$$1x_1 + 2x_2 = 3$$

$$2x_1 + 4x_2 = 6$$

redundant

Upshot: There are exactly three possibilities

- There could be no solution
- There could be exactly one solution
- There could be infinitely many solutions

Some goals:

- Figure out which possibility applies.
- Write down the solution if it's unique.
- if there are infinitely many solutions, figure out a way to describe them all.

The critical information is in the a_{ij}, b_i .

So extract just those numbers into a matrix

$$a_{11}x_1 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + \dots + a_{mn}x_n = b_m$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$m \times n$ coefficient matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right]$$

$m \times (n + 1)$ augmented matrix

These don't change the set of solutions for a system of linear equations:

- Reorder the equations
- Multiply an equation by $c \neq 0$
- Replace one equation by itself plus a multiple of another equation

Matrix operations:

- Reorder the rows (interchange)
- Multiply a row by $c \neq 0$ (scaling)
- Replace one row by itself plus a multiple of another row (replacement)

Grand strategy: Do this until the equations are easy to solve.

$$1x_1 + 2x_2 = 3$$

$$2x_1 + 1x_2 = 3$$

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 1 & 3 \end{array} \right]$$

Subtract twice 1st from 2nd:

$$1x_1 + 2x_2 = 3$$

$$-3x_2 = -3$$

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -3 & -3 \end{array} \right]$$

Add $\frac{2}{3}$ second to first:

$$1x_1 = 1$$

$$-3x_2 = -3$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & -3 & -3 \end{array} \right]$$

Multiply second by $-\frac{1}{3}$

$$x_1 = 1$$

$$x_2 = 1$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array} \right]$$

Step one:

Use first x_1 term [upper left matrix entry] to eliminate all other x_1 terms [change rest of first column to 0]:

$$\begin{array}{lcl} L1 : & x_1 - 3x_2 - 2x_3 = & 6 \\ L2 : & 2x_1 - 4x_2 - 3x_3 = & 8 \\ L3 : & -3x_1 + 6x_2 + 8x_3 = & -5 \end{array} \quad \left[\begin{array}{ccc|c} 1 & -3 & -2 & 6 \\ 2 & -4 & -3 & 8 \\ -3 & 6 & 8 & -5 \end{array} \right]$$

Subtract $2 \times$ first from second

Add $3 \times$ first to third:

$$\begin{array}{lcl} L1 : & x_1 - 3x_2 - 2x_3 = & 6 \\ L2 - 2L1 : & 2x_2 + x_3 = & -4 \\ 3L1 + L3 : & -3x_2 + 2x_3 = & 13 \end{array} \quad \left[\begin{array}{ccc|c} 1 & -3 & -2 & 6 \\ 0 & 2 & 1 & -4 \\ 0 & -3 & 2 & 13 \end{array} \right]$$

Move on to **Step two!** - the second column

Step two:

$$L1: x_1 - 3x_2 - 2x_3 = 6$$

$$L2: 2x_2 + x_3 = -4$$

$$L3: -3x_2 + 2x_3 = 13$$

Add $\frac{3}{2} \times L2$ to L1 and L3:

$$L1 + \frac{3}{2}L2: x_1 - \frac{1}{2}x_3 = 0$$

$$L2: 2x_2 + x_3 = -4$$

$$L3 + \frac{3}{2}L2: \frac{7}{2}x_3 = 7$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & 6 \\ 0 & 2 & 1 & -4 \\ 0 & -3 & 2 & 13 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 2 & 1 & -4 \\ 0 & 0 & \frac{7}{2} & 7 \end{array} \right]$$

Multiply L3 by $\frac{2}{7}$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 2 & 1 & -4 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

Move on to **Step three!** - the third column

Step three:

$$L1: \quad x_1 - \frac{1}{2}x_3 = 0$$

$$L2: \quad 2x_2 + x_3 = -4$$

$$L3: \quad x_3 = 2$$

Subtract L3 from L2;
add $\frac{1}{2}$ L3 to L1:

$$L1 + \frac{1}{2}L3: \quad x_1 = 1$$

$$L2 - \frac{1}{2}L3: \quad 2x_2 = -6$$

$$L3: \quad x_3 = 2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 2 & 1 & -4 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & -6 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

Multiply L2 by $\frac{1}{2}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

We're done: $(x_1, x_2, x_3) = (1, -3, 2)$

What can go wrong?

$$x_1 + 2x_2 = 3$$

$$x_1 + 2x_2 = 4$$

Subtract L1 from L2:

$$x_1 + 2x_2 = 3$$

$$0 = 1$$

inconsistent
NO SOLUTION

$$x_1 + 2x_2 = 3$$

$$2x_1 + 4x_2 = 6$$

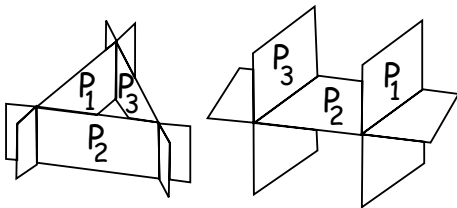
Subtract 2L1 from L2:

$$x_1 + 2x_2 = 3$$

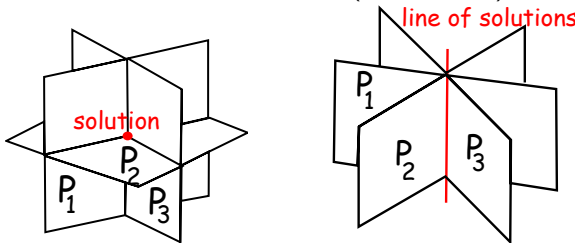
$$0 = 0$$

redundant
SOLUTIONS INFINITE

If the system has 3 variables, an equation determines a plane in $\mathbb{R}^3$.
So a solution will be the intersection of 3 planes: P_1, P_2, P_3 :



inconsistent equations (no solution)



single solution

redundant

iClicker question

George tells you the system of equations

$$5x_1 + 2x_2 - 3x_3 = 4$$

$$12x_1 - 7x_2 + 2x_3 = 8$$

$$-3x_1 + 4x_2 + 5x_3 = 10$$

has exactly **three solutions**.

- A) George is probably right, since he's Honest George.
- B) George is probably wrong.
- C) George is definitely right.
- D) George is definitely wrong.
- E) My brain is full.