

Matrix inverse

Math 4A – Scharlemann

2 February 2015



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An $n \times n$ matrix (i. e. a square matrix) A is called **invertible** if there is a matrix C so that

$$AC = CA = I_n$$

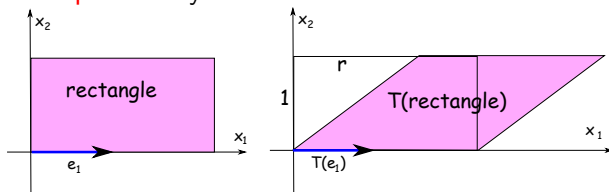
Note: I_n often abbreviated I .

There may be no such matrix C , but if one exists it is the only such matrix. It is called the **inverse** of A and written $C = A^{-1}$.

To show that C is unique, suppose also $AB = BA = I$ then

$$B = BI = B(AC) = (BA)C = IC = C$$

Example 1: Any **shear** is invertible:



Let $A = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}$ and try $C = \begin{bmatrix} 1 & -r \\ 0 & 1 \end{bmatrix}$

In other words, let C be the matrix that shears the same amount, but in the opposite direction.

Then

$$AC = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -r \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & r - r \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and

$$CA = \begin{bmatrix} 1 & -r \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & r - r \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

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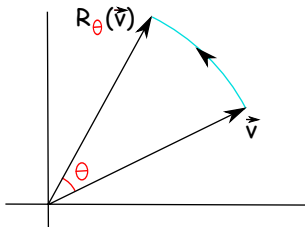
Question: What is the inverse of the matrix $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$?

- A) $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ B) $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ C) $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ D) $\begin{bmatrix} \frac{1}{3} & 0 \\ -0 & 1 \end{bmatrix}$
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Question: What is the inverse of the matrix $\begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}$?

- A) $\begin{bmatrix} -\frac{1}{3} & 0 \\ 3 & 1 \end{bmatrix}$ B) $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ C) $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ D) $\begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$

Example 2: Any **rotation** is invertible:



Let $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ and try $C = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$

C is matrix that rotates θ in **opposite** (clockwise) direction.

Remember that **$\sin(-\theta) = -\sin \theta$** whereas **$\cos(-\theta) = \cos \theta$** .

Hence $C = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$

Then

$$\begin{aligned} AC &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = \\ &\begin{bmatrix} \cos^2 \theta + \sin^2 \theta & \cos \theta \sin \theta - \sin \theta \cos \theta \\ \sin \theta \cos \theta - \cos \theta \sin \theta & \sin^2 \theta + \cos^2 \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \end{aligned}$$

Similarly, $CA = I_2$. Since $AC = CA = I$, C is the inverse of I .

Example 3: Suppose E is a matrix obtained from I_n by one of the elementary operations:

- Exchange two rows
- Multiply row i by $r \neq 0$
- Add a multiple of row i to row j

These special matrices are called **elementary matrices**.

Remarkable property: For any A the matrix EA comes from doing the same elementary operation to A .

Hence can get E^{-1} from I_n by the **opposite** move (or moves) in the opposite order:

- Subtract the same multiple of row i from row j
- Multiply row i by $\frac{1}{r}$
- Exchange the same two rows

since multiplying by this matrix undoes the move done by E

Example: Let

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \pi & 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & -3 \\ -2 & 7 & 4 \end{bmatrix}$$

Then

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \pi & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & -3 \\ -2 & 7 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & -3 \\ 2\pi - 2 & 2\pi + 7 & 4\pi + 4 \end{bmatrix}$$

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$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \pi \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & -3 \\ -2 & 7 & 4 \end{bmatrix} = ??$$

$$\text{A) } \begin{bmatrix} 2 & 2 & 4 \\ -2\pi & 1 + 7\pi & -3 + 4\pi \\ -2 & 7 & 4 \end{bmatrix} \quad \text{B) } \begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & -3 \\ 2\pi - 2 & 2\pi + 7 & 4\pi + 4 \end{bmatrix}$$

$$\text{C) } \begin{bmatrix} 2 & 2 & 4 + 2\pi \\ 0 & 1 & -3 + \pi \\ -2 & 7 & 4 + 7\pi \end{bmatrix} \quad \text{D) } \begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & -3 \\ -2 & \pi + 7 & 4 - 3\pi \end{bmatrix}$$

Discussion

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \pi \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & -3 \\ -2 & 7 & 4 \end{bmatrix} = ??$$

Multiplication by an elementary matrix E on the **left**, always corresponds to a **row** operation. So think about $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \pi \\ 0 & 0 & 1 \end{bmatrix}$

as the matrix obtained from $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ by adding π times the third row to the second.

So the answer is: A) $\begin{bmatrix} 2 & 2 & 4 \\ -2\pi & 1 + 7\pi & -3 + 4\pi \\ -2 & 7 & 4 \end{bmatrix}$

Side note:

If we multiplied on the **right** by E , the same sort of thing happens to the **columns**!

So, for example, since $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \pi \\ 0 & 0 & 1 \end{bmatrix}$ is **also** obtained from I_3 by adding $\pi \times$ second **column** to the third **column**,

$$AE = \begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & -3 \\ -2 & 7 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \pi \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 4 + 2\pi \\ 0 & 1 & -3 + \pi \\ -2 & 7 & 4 + 7\pi \end{bmatrix}$$

This was option C)

Theorem

Let A be a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

- A is *invertible* if and only if $ad - bc \neq 0$
- If A is invertible, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Demonstration:

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right) &= \frac{1}{ad - bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \\ \frac{1}{ad - bc} \begin{bmatrix} ad - bc & -ab + ab \\ dc - dc & ad - bc \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \end{aligned}$$

Example: Find inverse of $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

- **Step 1:** Check if A is invertible:
 $ad - bc = 1 \cdot 4 - 3 \cdot 2 = -2 \neq 0$ so **Yes** 😊.
- **Step 2:** Find A^{-1} :

$$\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$$

You'd be **nuts** not to check it:

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -2 + 3 & \frac{3}{2} - 3 \cdot \frac{1}{2} \\ -2 \cdot 2 + 4 & 2 \cdot \frac{3}{2} - 4 \cdot \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Here's one advantage of inverting a matrix A :

Theorem

Suppose A is an $n \times n$ invertible matrix. Then for any $\vec{b} \in \mathbb{R}^n$ the equation

$$A\vec{x} = \vec{b}$$

has the unique solution $\vec{x} = A^{-1}\vec{b}$.

Demonstration: Just try it – $A\vec{x} = A(A^{-1}\vec{b}) = (AA^{-1})\vec{b} = I_n\vec{b} = \vec{b}$

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Question: We saw that the inverse of $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ is $\begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$.

Find the solution to the system of equations

$$x_1 + 3x_2 = 1$$

$$2x_1 + 4x_2 = 2$$

A) $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ B) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ C) $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ D) $\begin{bmatrix} \frac{1}{2} & 0 \\ -0 & 1 \end{bmatrix}$

Answer: The matrix representation is $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ so the

answer is B): $\begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Some general properties of the inverse:

Suppose A and B are invertible matrices and the scalar $r \neq 0$.

- $(rA)^{-1} = \frac{1}{r}A^{-1}$
- $(AB)^{-1} = B^{-1}A^{-1}$ Order is reversed!
- $(A^T)^{-1} = (A^{-1})^T$

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- $(\frac{1}{r}A^{-1})rA = (r(\frac{1}{r}A^{-1}))A = A^{-1}A = I$
 - $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$
 - $(A^{-1})^T A^T = (AA^{-1})^T = I^T = I.$

There is a general way to find the inverse of a square matrix A :

- 1 Write down the super-augmented matrix $n \times 2n$ matrix $\left[A \mid I_n \right]$
- 2 Reduce it to reduced echelon form.
- 3 If the reduced row echelon form of the first half (namely A) is not I_n , then A is not invertible.
- 4 If the reduced row echelon form is $\left[I_n \mid B \right]$, then $A^{-1} = B$.

Example: Find the inverse of the matrix

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$$

Write down the super-augmented matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array} \right]$$

Then reduce:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array} \right] \rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & -1 & 0 & 4 & 0 & -1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array} \right]$$

Since the left half is I_3 , we are done & we know the matrix was invertible. The inverse is the second half:

$$A^{-1} = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$$

Check this:

$$AA^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix} \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$