

Characterizations of invertible matrices

Math 4A – Scharlemann

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Recall from last time: An $n \times n$ matrix A is **invertible** if there is a matrix C so that

$$AC = CA = I_n.$$

Here I_n (sometimes written just I) is the identity matrix

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

If A is invertible, then the matrix C is unique and is called the **inverse** of A , sometimes written A^{-1} .

A given matrix may or may not be invertible, but if it is invertible, then there are an amazing number of equivalent facts true about the matrix, as we shall see.

Theorem

Suppose A is an $n \times n$ matrix A ; then these are equivalent:

- ① A is invertible.
- ② When put in echelon form, A has n pivot positions.
- ③ When put in reduced echelon form, A becomes I_n .
- ④ The **only** solution of the equation $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.
- ⑤ For each $\vec{b} \in \mathbb{R}^n$ there is a solution of the equation $A\vec{x} = \vec{b}$.
- ⑥ Linear transformation $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is one-to-one (injective).
- ⑦ Linear transformation $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is onto (surjective).
- ⑧ The columns of A are linearly independent.
- ⑨ The columns of A span $\mathbb{R}^n$.
- ⑩ There is an $n \times n$ matrix C so that $CA = I_n$.
- ⑪ There is an $n \times n$ matrix D so that $AD = I_n$.
- ⑫ A^T is invertible.

How to tie these properties together:

The definition of **A invertible** includes: there is a C so that $CA = I_n$

Suppose **there is a C so that $CA = I_n$** ; consider the system $A\vec{x} = \vec{0}$. Then $\vec{x} = I_n\vec{x} = CA\vec{x} = C\vec{0} = \vec{0}$ so $\vec{0}$ is the only solution.

Suppose **the only solution of the equation $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$** .

Then there are no free variables in the system of equations given by $A\vec{x} = \vec{0}$, so all n positions are pivot positions.

Suppose **when put in echelon form, A has n pivot positions**.

Then the pivots occupy the whole diagonal, so when put in reduced echelon form, the matrix becomes I_n .

Suppose **when put in reduced echelon form, the matrix becomes I_n** .

We have seen that the same row operations applied to I_n will give A^{-1} , so A is invertible.

We have just shown that these are equivalent:

- 1 A is invertible.
- 2 When put in echelon form, A has n pivot positions.
- 3 When put in **reduced** echelon form, A becomes I_n
- 4 The only solution of the equation $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.
- 5 For each $\vec{b} \in \mathbb{R}^n$ there is a solution of the equation $A\vec{x} = \vec{b}$.
- 6 Linear transformation $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is one-to-one (injective).
- 7 Linear transformation $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is onto (surjective).
- 8 The columns of A are linearly independent.
- 9 The columns of A span $\mathbb{R}^n$.
- 10 There is an $n \times n$ matrix C so that $CA = I_n$.
- 11 There is an $n \times n$ matrix D so that $AD = I_n$.
- 12 A^T is invertible

iClicker question

Is the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 2 & 5 \end{bmatrix}$$

invertible?

- A Yes, the theorem says it is.
- B It's easy to see the theorem says it is not.
- C You can't easily say without doing a long calculation.

Discussion

You may have noticed that

$$-2 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + 0 \cdot \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or equivalently

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

So 4) is false. Since 4) is equivalent to A being invertible, A is not invertible.

We can tie in the last statement via:

Theorem

A is invertible if and only if A^T is invertible. Indeed

$$(A^T)^{-1} = (A^{-1})^T$$

Proof: Suppose A is invertible and C is its inverse. Then $AC = I_n = CA$. Take the transpose:

$$C^T A^T = (AC)^T = I_n^T = (CA)^T = A^T C^T.$$

Now substitute $I_n^T = I_n$ in the middle to get

$$C^T A^T = I_n = A^T C^T.$$

This shows that C^T is an inverse for A^T .

Tie in: A invertible $\iff$ there is a D so that $AD = I_n$

Apply the equivalence of 1) & 11) to the **matrix** A^T :

A^T invertible $\iff$ there is a matrix C so that $CA^T = I_n$

$$\iff (CA^T)^T = (A^T)^T C^T = AC^T = I_n$$

Set $D = C^T$ and then $AC^T = I_n \iff AD = I_n$.

Putting it all together:

A invertible $\iff A^T$ invertible $\iff$

there is a matrix D (namely C^T) so that $AD = I_n$.

We now have that these are equivalent:

- 1 A is invertible.
- 2 When put in echelon form, A has n pivot positions.
- 3 When put in **reduced** echelon form, A becomes I_n
- 4 The only solution of the equation $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.
- 5 For each $\vec{b} \in \mathbb{R}^n$ there is a solution of the equation $A\vec{x} = \vec{b}$.
- 6 Linear transformation $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is one-to-one (injective)
- 7 Linear transformation $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is onto (surjective).
- 8 The columns of A are linearly independent.
- 9 The columns of A span $\mathbb{R}^n$.
- 10 There is an $n \times n$ matrix C so that $CA = I_n$.
- 11 There is an $n \times n$ matrix D so that $AD = I_n$.
- 12 A^T is invertible

iClicker question

Is the matrix

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$$

invertible?

- A Yes, the theorem says it is.
- B It's easy to see the theorem says it is not.
- C You can't easily say without doing a long calculation.

Discussion

Observe that

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 2 & 5 \end{bmatrix}^T = A^T.$$

Since we know that A is not invertible, it follows from 10) that $A^T = B$ is not invertible.

Recall: the product $A\vec{x}$ is a vector that linearly combines (coefficients the entries in $\vec{x}$) the column vectors in A .

Hence these two statements are equivalent:

- The only solution of the equation $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.
- The columns of A are linearly independent.

Since the first is black, we can color the second black.

By the same argument, these three statements are equivalent (but none is yet black)

- For each $\vec{b} \in \mathbb{R}^n$ there is a solution of the equation $A\vec{x} = \vec{b}$.
- The columns of A span $\mathbb{R}^n$.
- Linear transformation $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is onto (surjective).

This leaves us at:

- ① A is invertible.
- ② When put in echelon form, A has n pivot positions.
- ③ When put in reduced echelon form, A becomes I_n
- ④ The only solution of the equation $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.
- ⑤ For each $\vec{b} \in \mathbb{R}^n$ there is a solution of the equation $A\vec{x} = \vec{b}$.
- ⑥ Linear transformation $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is one-to-one (injective)
- ⑦ Linear transformation $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is onto (surjective).
- ⑧ The columns of A are linearly independent.
- ⑨ The columns of A span $\mathbb{R}^n$.
- ⑩ There is an $n \times n$ matrix C so that $CA = I_n$.
- ⑪ There is an $n \times n$ matrix D so that $AD = I_n$.
- ⑫ A^T is invertible

Tie green circle in with another equivalence, namely $5 \iff 11$:

Suppose there is a D so that $AD = I_n$. Then for any $\vec{b} \in \mathbb{R}^n$, let $\vec{x} = D\vec{b}$ and discover that $A\vec{x} = A(D\vec{b}) = (AD)\vec{b} = I_n\vec{b} = \vec{b}$.

To complete an equivalence, we need to show:

Theorem

Suppose for each $\vec{b} \in \mathbb{R}^n$ there is a solution of the equation $A\vec{x} = \vec{b}$. Then there is an $n \times n$ matrix D so that $AD = I_n$.

Theorem

Suppose for each $\vec{b} \in \mathbb{R}^n$ there is a solution of the equation $A\vec{x} = \vec{b}$. Then there is an $n \times n$ matrix D so that $AD = I_n$.

Proof: For each $\vec{e}_i$, $1 \leq i \leq n$, let $\vec{d}_i$ be the solution to the equation $A\vec{x} = \vec{e}_i$. That is:

$$A\vec{d}_i = \vec{e}_i$$

Now set

$$D = \begin{bmatrix} \vec{d}_1 & \vec{d}_2 & \dots & \vec{d}_n \end{bmatrix}$$

Then

$$\begin{aligned} AD &= A \begin{bmatrix} \vec{d}_1 & \vec{d}_2 & \dots & \vec{d}_n \end{bmatrix} = \begin{bmatrix} A\vec{d}_1 & A\vec{d}_2 & \dots & A\vec{d}_n \end{bmatrix} = \\ & \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \dots & \vec{e}_n \end{bmatrix} = I_n \end{aligned}$$

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Suppose I tell you the equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 3 & 4 & 5 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has only the solution $\vec{x} = \vec{0}$. What can you say about the equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 3 & 4 & 5 \end{bmatrix} \vec{x} = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix} ?$$

- A There is **at least one** solution $\vec{x}$.
- B There is **at most** one solution $\vec{x}$.
- C You can't easily say without doing a long calculation.

Discussion:

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 3 & 4 & 5 \end{bmatrix}.$$

Then the hypothesis is that the only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$. This is one of the black conditions, equivalent to A is invertible. We have just seen that these are all equivalent to the green conditions, including:

For each $\vec{b}$ there is a solution to the equation $A\vec{x} = \vec{b}$.

So option A), there is **at least** one solution to the equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 3 & 4 & 5 \end{bmatrix} \vec{x} = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix} \text{ is correct.}$$

Once we see that the last is also equivalent, option B) is also true: There is **at most** one solution $\vec{x}$.

The final item is brought into the equivalence via:

Theorem

These are equivalent:

- *The only solution to the equation $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.*
- *Linear transformation $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is one-to-one (injective)*

Proof: Suppose A is injective. Then whenever $A\vec{x} = \vec{0}$ we have $A\vec{x} = \vec{0} = A\vec{0}$. But injectivity then implies $\vec{x} = \vec{0}$.

Conversely, suppose the only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$. To see that A is then injective, suppose $A\vec{u} = A\vec{w}$. Then consider:

$$A(\vec{u} - \vec{w}) = A\vec{u} - A\vec{w} = \vec{0}.$$

The only solution to this equation is assumed to be $\vec{0}$.

So $\vec{u} - \vec{w} = \vec{0}$, which means $\vec{u} = \vec{w}$ and $\vec{u}, \vec{w}$ are **not** different.