

Determinants

Math 4A – Scharlemann

6 February 2015



CELL PHONES OFF

To any **square matrix** A , you can assign a real number, called the **determinant of A** (denoted $\det A$) that has nice properties:

- $\det A \neq 0 \iff A$ is **invertible**
- $\det AB = \det A \cdot \det B$
- $\det I_n = 1$
- Since $AA^{-1} = I_n$ we have

$$\det A \cdot \det A^{-1} = \det AA^{-1} = \det I_n = 1 \implies \det A^{-1} = \frac{1}{\det A}$$

Problem: The determinant is really complicated to define.

In particular: to calculate for an $n \times n$ matrix, must first calculate for lots of $(n-1) \times (n-1)$ submatrices.

For **each** of these, must calculate for many $(n-2) \times (n-2)$ submatrices...

Gets really complicated for n large.

(But for n large, there are easier ways.)

First example:

For 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is

$$\det A = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

This expression came up earlier:

Recall: For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

- A is invertible if and only if $ad - bc \neq 0$ ie $\det A \neq 0$.
- If A is invertible, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

iClicker question

What happens if we switch two rows of a 2 by 2 matrix?

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

How is $\det A$ related to $\det B$?

A $\det A = \det B$.

B $\det A = -\det B$.

C I can't see any relation between them.

D I'm confused.

In fact

$$\det B = cb - da = -(da - cb) = -\det A,$$

So B is true.

In general, need the idea of a **cofactor** of an $n \times n$ matrix A :

Let A_{ij} denote the $(n-1) \times (n-1)$ submatrix obtained from A by deleting the i -th row and j -th column.

In pictures:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix} \implies A_{ij} = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

The **cofactor** of a_{ij} is

$$C_{ij} = (-1)^{(i+j)} \det A_{ij}.$$

Another way of viewing the factor $(-1)^{(i+j)}$: The cofactor C_{ij} is $\det A_{ij}$ but with sign changed by ± 1 depending on the position of a_{ij} , $+$ in the top-left corner and then alternate $-$ and $+$ both across and down:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Now pick **any** row (say the 1^{st} row) or **any** column. Then

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

In other words, multiply each term in the row by its cofactor, and add them all up. Whew!

But notice: you only need cofactors for a single row (or single column). This can be useful.

Example: finding the determinant of a 3×3 matrix

$$A = \begin{bmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{bmatrix} \Rightarrow C_{22} = + \det \begin{bmatrix} 2 & -4 \\ 1 & 5 \end{bmatrix} \quad C_{23} = - \det \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$$

so

$$C_{22} = 2 \cdot 5 - (-4) \cdot 1 = 14 \quad C_{23} = -(2 \cdot (-1) - 3 \cdot 1) = 5$$

Then

$$\det A = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} = 0 \cdot ? + (-4) \cdot 14 + 2 \cdot 5 = -46$$

Warning: You may have learned a different way of calculating the determinant of a 3×3 matrix. It does not work for $n \geq 4$!

iClicker question

What happens if we switch two rows of a 3 by 3 matrix?

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad B = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

How is $\det A$ related to $\det B$?

- A $\det A = -(-\det B) = \det B$.
- B $\det A = -\det B$.
- C I can't see any relation between them.
- D I'm confused.

Discussion

B is true: Expand the two determinants via cofactors in a row that did **not** move. (Row 3 in this example.) All the cofactors are 2 by 2 matrices, and the previous question applies.

Theorem (Theorem 3 p. 169)

*For **any** size matrix, if you switch two rows, the determinant changes by a minus sign.*

Calculating determinant for an easy special case: Triangular matrix

Theorem

If A is *triangular* then $\det A = \text{product of the diagonal entries}$.

Example: when $n = 2$

$$A = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \implies \det A = ad - 0c = ad$$

When $n = 3$:

$$\det \begin{pmatrix} \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{pmatrix} = a_{11} \det \begin{pmatrix} \begin{bmatrix} a_{22} & 0 \\ a_{32} & a_{33} \end{bmatrix} \end{pmatrix} = a_{11} a_{22} a_{33}$$

Reason formula works:

Suppose you know it's true for any **smaller** triangular matrix. Then

$$\det \begin{pmatrix} \begin{bmatrix} \mathbf{a_{11}} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & \cdots & \cdots & a_{nn} \end{bmatrix} \end{pmatrix} =$$
$$\mathbf{a_{11}} \det \begin{pmatrix} \begin{bmatrix} a_{22} & 0 & 0 & \cdots & 0 \\ a_{32} & a_{33} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n2} & \cdots & \cdots & \cdots & a_{nn} \end{bmatrix} \end{pmatrix} = \mathbf{a_{11}} a_{22} a_{33} \cdots a_{nn}$$

iClicker question

What is the determinant of the matrix:

$$\det \begin{pmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_{21} & 2 & 0 & \cdots & 0 \\ a_{31} & a_{32} & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & \cdots & n \end{bmatrix} \end{pmatrix} =$$

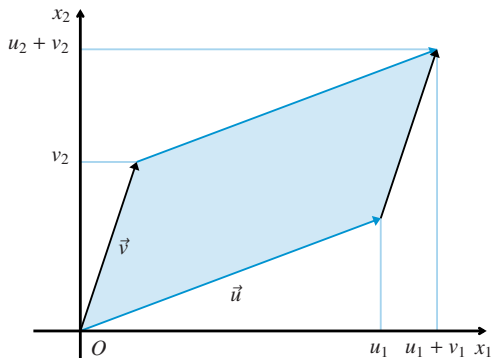
- A) n .
- B) $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$.
- C) $1 \cdot 2 \cdot 3 \cdot \dots \cdot n = n!$
- D) I'm confused.

Why does the determinant have great properties?

Answer: Because it's connected to **geometry**

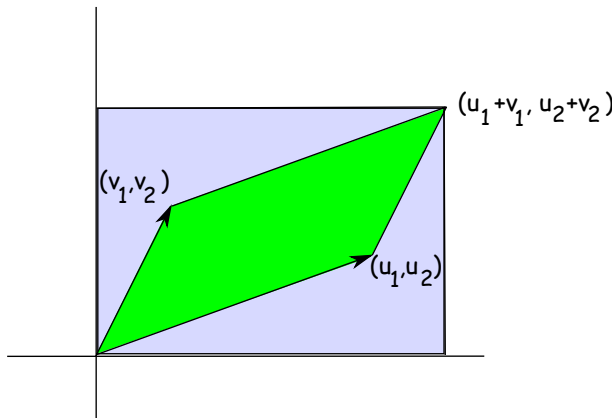
"God always geometrizes" – Plato

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. What is the area?



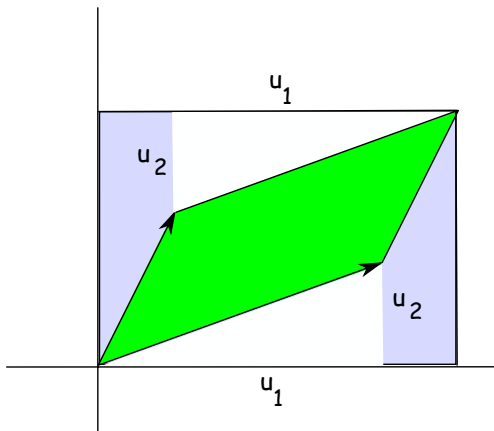
Answer:

$$(u_1 + v_1)(u_2 + v_2) -$$



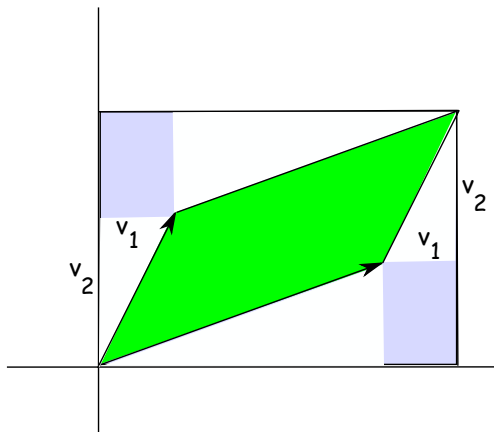
Answer:

$$(u_1 + v_1)(u_2 + v_2) - u_1 u_2$$



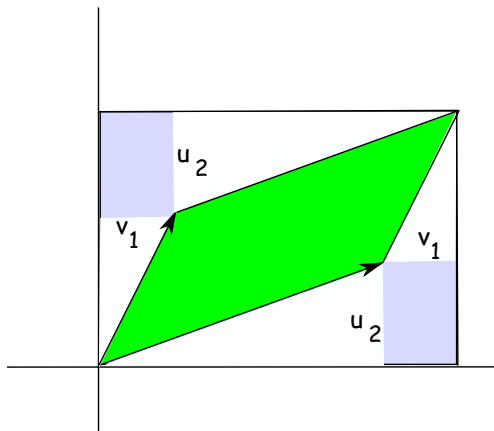
Answer:

$$(u_1 + v_1)(u_2 + v_2) - u_1 u_2 - v_1 v_2$$



Answer:

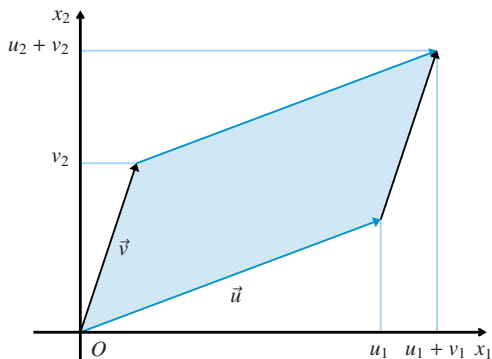
$$(u_1 + v_1)(u_2 + v_2) - u_1 u_2 - v_1 v_2 - 2v_1 u_2$$



$$= (u_1 u_2 + u_1 v_2 + v_1 u_2 + v_1 v_2) - u_1 u_2 - v_1 v_2 - 2v_1 u_2 = u_1 v_2 - v_1 u_2$$

Answer:

$$\text{Area} = u_1 v_2 - v_1 u_2 = \left| \det \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \right|.$$

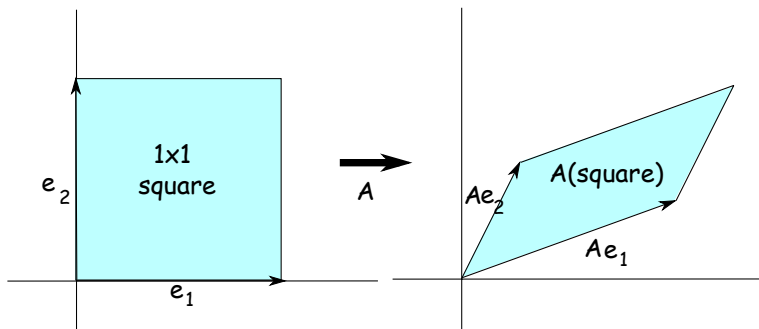


Another point of view:

Let

$$A = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \implies \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = A\vec{e}_1; \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = A\vec{e}_2$$

So parallelogram is $A(\text{unit square})$:

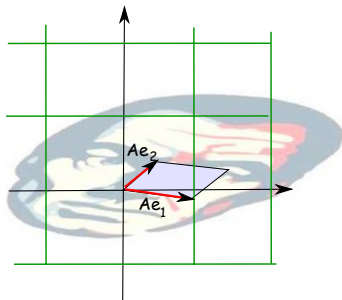
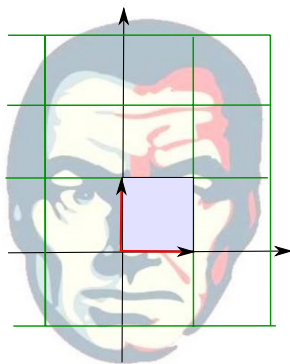


Then A multiplies area of a square by $|\det A|$.

But (think integral calculus), any figure is limit of little squares, so:

Theorem

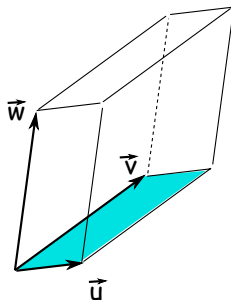
Under a linear transformation with matrix A , the area of *any figure* is *multiplied by $|\det A|$* .



Technical note: $\det A$ negative $\iff$ figure “flipped”.

More difficult to check: The volume of the parallelepiped induced by three vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ is

$$\text{Volume}(\vec{u}, \vec{v}, \vec{w}) = \left| \det \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} \right|.$$



Then a similar argument shows:

A 3×3 matrix A will multiply the volume of any solid by $|\det A|$.
And onward to the n^{th} dimension...!

First slide revisited:

- $\det A \neq 0 \iff A$ is invertible
- $\det AB = \det A \cdot \det B$
- $\det I_n = 1$

⋮

But for n large, there are easier ways to calculate $\det A$.

To be continued...